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INSURANCE FOR FACE AMOUNT OR PAID-UP INSURANCE AMOUNT IF GREATER

CECIL J. NESBITT AND MARJORIE L. VAN EENAM

PREVIOUS papers have pointed out several ways to construct an endowment plan which provides for a maturity value greater than the face amount of insurance. The major difference in construction is the manner in which the amounts of insurance are graded into the maturity value. As the increased amount of insurance Fassel uses the net level premium reserve [1]*; Budinger, the Illinois Standard modified reserve [1, discussion, page 72]; Rosser, the Commissioners Reserve Valuation Method reserve [8, discussion, page 535]; Espie [4] and Hahn [5], cash values permitted by the Standard Non-Forfeiture Law. Other authors and discussers have added to the subject (see the References). This paper presents a different approach; the paid-up insurance amount is taken as the sum insured whenever it exceeds the face amount. We will show that during such period the increment in the amount of insurance in any year is the portion of paid-up insurance which can be purchased by the cash value premium at net attained-age rates. Unlike the other plans mentioned, this plan, then, has no "interest only" feature.

Since paid-up amounts exceed the cash values on which they are based (and usually the corresponding reserves as well), the cross-over point occurs earlier on this plan than on the other plans mentioned, and thereafter its amounts of insurance are higher than theirs, though increasing by smaller increments. Some of the earlier authors experienced difficulty in choosing a form for their paid-up and extended term options; and company practice differs in this regard. No similar difficulty arises in this instance, for the natural amounts of paid-up or extended term insurance would on no occasion exceed the amount provided under the regular operation of the plan.

THE GENERAL THEORY

The insurance plan

Let us consider an m-payment, n-year endowment insurance on the life of (x) with face amount 1 and maturity value $1 + k$. The amount of

* Numbers in brackets apply to the list of References at the end of the paper.

insurance in any year is to be the face amount or terminal paid-up insurance amount, whichever is larger. If ${}_t\mathrm{W}$ is the paid-up insurance amount at the end of the tth year, and if b denotes the greatest integer for which ${}_b\mathrm{W} \leq 1$, the amount of insurance is 1 for the first b years, and thereafter is ${}_t\mathrm{W}(t > b)$. Corresponding to ${}_t\mathrm{W}$ is the tth year cash value ${}_t\mathrm{CV}$, which is based upon the generalized cash value annual premium ${}^C\mathrm{P}$. In addition to providing the benefits, ${}^C\mathrm{P}$ over its m-year payment term is sufficient to provide an extra initial expense of E. In case of death during an insurance year, the amount of insurance is assumed payable at the end of the year.

On any such plan the paid-up insurance amount is level after premiums cease. Therefore, in order that the amount of insurance be built up from 1 to $1 + k$, it is essential that the inequality $b < m$ hold. Since no further increase in the amount of insurance will occur after m years, the amount after duration m is constant and equal to the maturity value $1 + k$. The requirement $b < m$ excludes the possibility of varying amounts of insurance under a single premium plan of this form; the single premium plan would necessarily be an ordinary level n-year endowment insurance with face amount $1 + k$. This indicates one of the characteristic differences of the present plan from the face amount or cash value if greater plan.

Premium-analysis equations

The cash value premium for our plan might be expressed in the usual way as

$$ {}^C\mathrm{P} = \left[\mathrm{A}^{1}_{x:\overline{b}|} + \left(\sum_{h=b}^{n-1} {}_{h+1}\mathrm{W} \cdot \mathrm{C}_{x+h} \right) \Big/ \mathrm{D}_x + (1+k)\mathrm{A}_{x:\overline{n}|}^{\;\;\,1} + E \right] \Big/ \ddot{a}_{x:\overline{m}|} . \quad (1) $$

However, this is not an explicit formula as ${}^C\mathrm{P}$ is involved in ${}_{h+1}\mathrm{W}$ and possibly also in E. To obtain a more convenient formula we shall make use of a premium-analysis equation which relates ${}^C\mathrm{P}$ to the paid-up insurance amounts. Let ${}_{t+1}S$ denote the amount of insurance in insurance year $t + 1$ and assume that the rth insurance year is the first year in which the cash value is positive. The general form of this equation, which applies to other plans of insurance as well as to the plan considered in this paper, is

$$ {}^C\mathrm{P} = ({}_{t+1}S - {}_{t+1}\mathrm{W})\, v q_{x+t} + ({}_{t+1}\mathrm{W} - {}_t\mathrm{W})\, \mathrm{A}_{x+t:\overline{n-t}|} , \qquad r \leq t < m . \quad (2) $$

In our plan ${}_{t+1}S = 1$ for $t < b$, and ${}_{t+1}S = {}_{t+1}\mathrm{W}$ for $t \geq b$, so we have the two cases

$$ {}^C\mathrm{P} = (1 - {}_{t+1}\mathrm{W})\, v q_{x+t} + ({}_{t+1}\mathrm{W} - {}_t\mathrm{W})\, \mathrm{A}_{x+t:\overline{n-t}|} , \qquad r \leq t < b \quad (2a) $$

and

$$ {}^C\mathrm{P} = ({}_{t+1}\mathrm{W} - {}_t\mathrm{W})\, \mathrm{A}_{x+t:\overline{n-t}|} , \qquad b \leq t < m . \quad (2b) $$

Equation (2) states that the cash value premium for insurance year $t+1$, $t \geq r$, must be exactly sufficient to provide at the end of that year (i) in case of death during the year, the excess, if any, of the amount of insurance over the paid-up amount that is available at the end of the year, and (ii) in case of either death or survival, an increment of paid-up insurance such that the total amount of paid-up insurance will become ${}_{t+1}\mathrm{W}$. Whenever ${}_{t+1}S = {}_{t+1}\mathrm{W}$, the entire cash value premium is used to purchase increments of paid-up insurance. For the plan under consideration such a situation holds after b years.

While this verbal reasoning may serve to justify equation (2), an algebraic proof may be more convincing. For this purpose we start with the retrospective formulas

$$ {}_{t}\mathrm{CV} \cdot \mathrm{D}_{x+t} = {}^{C}\mathrm{P}\,(\mathrm{N}_x - \mathrm{N}_{x+t}) - \sum_{h=0}^{t-1} {}_{h+1}S \cdot \mathrm{C}_{x+h} - E \cdot \mathrm{D}_x $$

$$ {}_{t+1}\mathrm{CV} \cdot \mathrm{D}_{x+t+1} = {}^{C}\mathrm{P}\,(\mathrm{N}_x - \mathrm{N}_{x+t+1}) - \sum_{h=0}^{t} {}_{h+1}S \cdot \mathrm{C}_{x+h} - E \cdot \mathrm{D}_x \,, $$

from which we obtain

$$ {}_{t+1}\mathrm{CV} \cdot \mathrm{D}_{x+t+1} - {}_{t}\mathrm{CV} \cdot \mathrm{D}_{x+t} = {}^{C}\mathrm{P} \cdot \mathrm{D}_{x+t} - {}_{t+1}S \cdot \mathrm{C}_{x+t} \,, $$

or

$$ {}^{C}\mathrm{P} = {}_{t+1}S \cdot v q_{x+t} + {}_{t+1}\mathrm{CV} \cdot v p_{x+t} - {}_{t}\mathrm{CV} \,. \tag{3} $$

Equation (3) is itself a premium-analysis equation for ${}^{C}\mathrm{P}$ of a different form than equation (2). To obtain the latter equation, we use the relations

$$ {}_{t}\mathrm{CV} = {}_{t}\mathrm{W} \cdot \mathrm{A}_{x+t:\overline{n-t}|} \,, $$

$$ {}_{t+1}\mathrm{CV} \cdot v p_{x+t} = {}_{t+1}\mathrm{W} \cdot \mathrm{A}_{x+t+1:\overline{n-t-1}|} \cdot v p_{x+t} $$

$$ = {}_{t+1}\mathrm{W}\,[\mathrm{A}_{x+t:\overline{n-t}|} - v q_{x+t}] $$

in equation (3), and the result is equation (2).

Turning now to the special equations (2*a*) and (2*b*) one notes that the first of these holds for the usual plans with level amount of insurance. Equation (2*b*), however, is characteristic of our plan and from this equation will stem the special formulas for the plan.

Cash value premium

The equation (2*b*) furnishes a start towards a formula for ${}^{C}\mathrm{P}$. It may be rewritten as

$$ {}^{C}\mathrm{P}\,(\mathrm{A}_{x+t:\overline{n-t}|})^{-1} = \Delta\, {}_{t}\mathrm{W} \,, \qquad b \leq t < m \,, \tag{4} $$

which, when summed from $t = b$ to $t = m - 1$, yields the prospective formula

$$ {}_bW = 1 + k - {}^cP \sum_{t=b}^{m-1} (A_{x+t:\overline{n-t}|})^{-1} , \tag{5}$$

since ${}_mW = 1 + k$. The retrospective formula results from substituting the retrospective expression for ${}_bCV$ in

$$ {}_bW = {}_bCV\,(A_{x+b:\overline{n-b}|})^{-1} ,$$

so that we also have

$$ {}_bW = \frac{{}^cP \cdot {}_bu_x - {}_bk_x - E\,(A_{x:\overline{b}|}^{\;\;1})^{-1}}{A_{x+b:\overline{n-b}|}} . \tag{6}$$

After equating (5) and (6), and solving for ${}^cP$, one obtains

$$ {}^cP = \frac{{}_bk_x + (1+k)\,A_{x+b:\overline{n-b}|} + E\,(A_{x:\overline{b}|}^{\;\;1})^{-1}}{{}_bu_x + A_{x+b:\overline{n-b}|}\sum_{t=b}^{m-1} (A_{x+t:\overline{n-t}|})^{-1}} ,$$

or

$$ {}^cP = \frac{A_{x:\overline{b}|}^{1} + (1+k)\,A_{x:\overline{b}|}^{\;\;1}\,A_{x+b:\overline{n-b}|} + E}{\ddot{a}_{x:\overline{b}|} + A_{x:\overline{b}|}^{\;\;1}\,A_{x+b:\overline{n-b}|}\sum_{t=b}^{m-1} (A_{x+t:\overline{n-t}|})^{-1}} . \tag{7}$$

Other ways of obtaining formula (7) may suggest themselves to the reader. One alternative proof results from reasoning that the cash value premiums paid for b years must be sufficient to provide b-year term insurance, a b-year cash value to survivors, and the extra initial expense, *i.e.*,

$$ {}^cP\ddot{a}_{x:\overline{b}|} = A_{x:\overline{b}|}^{1} + {}_bCV \cdot A_{x:\overline{b}|}^{\;\;1} + E $$

$$ = A_{x:\overline{b}|}^{1} + {}_bW \cdot A_{x:\overline{b}|}^{\;\;1} \cdot A_{x+b:\overline{n-b}|} + E . $$

This reduces to (7) when substitution for ${}_bW$ is made from formula (5). Equating prospective and retrospective formulas for ${}_bCV$ would result in a similar proof.

For computational purposes it is useful to introduce the function

$$ {}_yA_x = \sum_{z=x}^{y-1} (A_{z:\overline{y-z}|})^{-1} , \tag{8}$$

where $y = x + n$ is the age at maturity. Then (7) may be written as

$$^{C}\mathrm{P} = \frac{\mathrm{A}^{1}_{x:\overline{b|}} + (1+k)\,\mathrm{A}_{x:\overline{b|}}^{\;\;1}\,\mathrm{A}_{x+b:\overline{n-b|}} + E}{\ddot{a}_{x:\overline{b|}} + \mathrm{A}_{x:\overline{b|}}^{\;\;1}\,\mathrm{A}_{x+b:\overline{n-b|}}\,({}_y\mathrm{A}_{x+b} - {}_y\mathrm{A}_{x+m})}\,. \tag{9}$$

Formula (9) is not too unwieldy in computation, provided:

i) It is possible to obtain b quickly.

ii) The expression for E is not complicated.

iii) Tables of the function ${}_y\mathrm{A}_x$ are made available. If the required single premiums are available, preparation of these tables of ${}_y\mathrm{A}_x$, for the usual choices of age at maturity y, would be a simple matter.

Criterion for b

Let us choose as basis for comparison with our plan special b-payment and $(b+1)$-payment n-year endowment insurances of 1 on the life of (x), each with an extra initial expense of the same amount E as for our plan. Let ${}_{b:b}\mathrm{W}^{E}_{x:\overline{n|}}$ be the b-year paid-up insurance amount and ${}^{C}_{b}\mathrm{P}^{E}_{x:\overline{n|}}$ the cash value premium for this special b-payment endowment insurance, and assume corresponding notations for the $(b+1)$-payment insurance. Both the b-payment endowment and our plan provide 1 on death during the first b years, but as the former has ${}_{b:b}\mathrm{W}^{E}_{x:\overline{n|}} = 1$ and the latter, ${}_b\mathrm{W} \le 1$, it follows that

$${}^{C}_{b}\mathrm{P}^{E}_{x:\overline{n|}} \ge {}^{C}\mathrm{P}\,.$$

On the other hand, because the $(b+1)$-payment endowment provides only 1 on death during year $b+1$ and ${}_{b+1:b+1}\mathrm{W}^{E}_{x:\overline{n|}} = 1$, whereas corresponding amounts on our policy exceed 1, earlier amounts of insurance on the two plans being identical,

$${}^{C}\mathrm{P} > {}^{C}_{b+1}\mathrm{P}^{E}_{x:\overline{n|}}\,.$$

Putting the two inequalities together we obtain the bracketing relation

$${}^{C}_{b}\mathrm{P}^{E}_{x:\overline{n|}} \ge {}^{C}\mathrm{P} > {}^{C}_{b+1}\mathrm{P}^{E}_{x:\overline{n|}}\,. \tag{10}$$

Algebraically, the relation ${}^{C}_{b}\mathrm{P}^{E}_{x:\overline{n|}} \ge {}^{C}\mathrm{P}$ may be established by comparing retrospective formulas [see formula (6)] for ${}_{b:b}\mathrm{W}^{E}_{x:\overline{n|}}$ and ${}_b\mathrm{W}$; and the relation ${}^{C}\mathrm{P} > {}^{C}_{b+1}\mathrm{P}^{E}_{x:\overline{n|}}$ by comparing the formulas for ${}_b\mathrm{W}$ and

$${}_{b+1:b+1}\mathrm{W}^{E}_{x:\overline{n|}}\,.$$

Because of (10), we know that b is the greatest integer such that

$${}^{C}_{b}\mathrm{P}^{E}_{x:\overline{n|}} = \frac{\mathrm{A}_{x:\overline{n|}} + E}{\ddot{a}_{x:\overline{b|}}} \ge \frac{\mathrm{A}^{1}_{x:\overline{b|}} + (1+k)\,\mathrm{A}_{x:\overline{b|}}^{\;\;1}\mathrm{A}_{x+b:\overline{n-b|}} + E}{\ddot{a}_{x:\overline{b|}} + \mathrm{A}_{x:\overline{b|}}^{\;\;1}\mathrm{A}_{x+b:\overline{n-b|}}({}_y\mathrm{A}_{x+b} - {}_y\mathrm{A}_{x+m})} = {}^{C}\mathrm{P}\,.$$

Hence, b is the greatest integer such that

$$ {}^{C}_{b}P^{E}_{x:\overline{n}|}[\ddot{a}_{x:\overline{b}|}+A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|}({}_{y}A_{x+b}-{}_{y}A_{x+m})]\geq A^{\;1}_{x:\overline{b}|}+A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} $$

$$ +E+kA_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} $$

and, since ${}^{C}_{b}P^{E}_{x:\overline{n}|}\ddot{a}_{x:\overline{b}|}$ equals the first three terms of the right member, there is obtained the criterion that b is the greatest integer such that

$$ {}^{C}_{b}P^{E}_{x:\overline{n}|}({}_{y}A_{x+b}-{}_{y}A_{x+m})\geq k\;. \tag{11} $$

This criterion appears easy to apply. However, in practice E at times would be a function of ${}^{C}P$, and hence E and ${}^{C}_{b}P^{E}_{x:\overline{n}|}$ cannot be determined exactly until ${}^{C}P$, which in turn depends upon b, is determined. In such instances a simplified trial and error method seems warranted. One possible procedure in these cases is to apply (11) using an estimated value of E.

Amount of insurance

After b years, the amount of insurance ${}_{h}S$ begins to rise from 1 to $1+k$ and is equal to the paid-up insurance amount. To obtain an expression for ${}_{h}S$, we sum (4) over the range $t=h$ to $t=m-1$ and replace ${}_{m}W$ by $1+k$, whereupon

$$ {}_{h}S={}_{h}W=1+k-{}^{C}P({}_{y}A_{x+h}-{}_{y}A_{x+m})\,,\qquad b\leq h<m\;. \tag{12a} $$

We also know that

$$ {}_{h}S={}_{h}W=1+k\;,\qquad m\leq h\leq n\;. \tag{12b} $$

Equations (12a) and (12b) show that the amount of insurance in any insurance year after duration b is the full maturity value reduced by the paid-up amounts, if any, to be purchased by future cash value premiums.

Nonforfeiture benefits

After ${}^{C}P$, b, and the amounts of insurance are determined, cash values come easily. Three cases arise:

i) When $h\leq b$, the retrospective formula, namely

$$ {}_{h}CV={}^{C}P\cdot{}_{h}u_{x}-{}_{h}k_{x}-E\,(A_{x:\overline{h}|}^{\;\;\;1})^{-1} \tag{13a} $$

is most direct.

ii) When $b\leq h<m$ use of (12a) gives

$$ {}_{h}CV=[1+k-{}^{C}P({}_{y}A_{x+h}-{}_{y}A_{x+m})]A_{x+h:\overline{n-h}|}\,. \tag{13b} $$

iii) When $m\leq h\leq n$, we have

$$ {}_{h}CV=(1+k)A_{x+h:\overline{n-h}|}\,. \tag{13c} $$

One may also use the Fackler accumulation formula to obtain the cash values, introducing the varying amounts of insurance provided by the plan after b years.

Paid-up insurance amounts for durations of at least b years are given by (12a) and (12b). All other paid-up insurance amounts and all extended insurance benefits may be obtained in the usual manner from the cash values. After b years, the tabular paid-up and extended insurance benefits are identical.

Reserves

Let us assume that the reserve basis provides for extra initial expense of amount E'. For example, in the net level case $E' = 0$, and in the Commissioners Reserve Valuation Method [6] (to be abbreviated hereafter to CRVM) $E' = (a) - (b)$ as defined by the Standard Valuation Law. Assuming further that if net premiums are modified the modification extends for the whole premium payment term, and that reserves and cash values are on the same mortality and interest basis, we find the following relation between terminal reserves and cash values:

$$_{h}\mathrm{V}' = {}_{h}\mathrm{CV} + \frac{E - E'}{\ddot{a}_{x:\overline{m}|}}\, \ddot{a}_{x+h:\overline{m-h}|}\,, \tag{14}$$

where $_{h}\mathrm{V}'$ denotes the terminal reserve at duration h [3, page 344]. Note that to have $_{h}\mathrm{V}' \geq {}_{h}\mathrm{CV}$, we must have $E \geq E'$.

Remarks

It is interesting to compare (7) with the premium formula stated by Espie [4, page 47] which specializes by proper choice of E to any of the plans providing insurance for face amount or cash value (or reserve) if greater. Espie's formula, translated into present value functions, and adjusted to an m-year premium period, is

$$\frac{\mathrm{A}^{1}_{x:\overline{a}|} + (1+k)\,\mathrm{A}_{x:\overline{a}|}^{\;\;1}\, v^{n-a} + E}{\ddot{a}_{x:\overline{a}|} + \mathrm{A}_{x:\overline{a}|}^{\;\;1}\, \ddot{a}_{\overline{m-a}|}}\,,$$

where the amount of insurance exceeds 1 after the first a years and the premium payment period is greater than a years. For the two types of plan, the difference in terms which contain pure endowment factors arises because of the different expressions for prospective cash values at the end of the period in which the amount of insurance is 1, with the one cash value free of mortality and the other cash value involving increments of paid-up insurance which, of course, depend upon mortality. Also, if the two types of plan are based on comparable assumptions, a will normally exceed b, since the paid-up amount will not be less than the comparable cash value.

The formulas developed above may readily be specialized to the case where $m = n$, that is, where premiums are payable until maturity. In that case, the amount of insurance is $1 + k$ only in the year immediately preceding maturity; in other words, there is no second period of level death benefits such as occurs in the limited-payment case. In the applications which follow, the premium payment period and endowment term will coincide.

APPLICATIONS

Any attempt to specialize the general theory to fit a concrete plan requires that specific bases for nonforfeiture benefits and for reserves be chosen. Among the possible bases for nonforfeiture benefits are:

a) Full net level reserve
b) Modified reserve by the CRVM
c) Minimum (adjusted premium) cash value [3]
d) A basis intermediate to (*a*) and (*c*).

Possibilities for the reserve basis include net level and CRVM. The stated possibilities may be combined in eight ways. The combination of full net level nonforfeiture values and CRVM reserves may be eliminated as being impractical, but the other possibilities remain. Another source of variation is indicated by Nelson and Warren [7], who suggest that the amounts of insurance used in computing reserves need not necessarily coincide with the actual amounts under the plan, provided a conservative choice of such amounts is taken for reserve purposes. Thus, for example, paid-up amounts determined from CRVM reserves may be the assumed amounts of insurance after b years for valuation purposes, even though actual amounts of insurance after b years are the minimum paid-up insurance amounts.

For the plans to be considered in this section it is assumed that premiums are paid during the whole n-year insurance term and that reserves and paid-up amounts are on the same interest and mortality basis. We have chosen to discuss the following plans:

Plan	Nonforfeiture Basis	Reserve Basis	Basis for Amounts of Insurance Used in Computing Reserves
Net level	Full net level	Full net level	Full net level
CRVM	CRVM	CRVM	CRVM
Adjusted premium	Minimum	CRVM	Minimum

For each plan it is assumed that the basis for the amounts of insurance used in computing reserves is the same as the nonforfeiture basis; that is, the actual amounts of insurance are used. If the suggestion of Nelson and Warren were followed, the reserves for the CRVM plan might be used for the adjusted premium plan.

To this point, no mention has been made of the equivalent uniform amount of insurance which may require consideration when minimum nonforfeiture benefits or CRVM reserves are being used. The Standard Non-Forfeiture Law clearly defines the equivalent uniform amount, which for our general insurance plan may be determined from

$$ {}^{C}\mathrm{P}\ddot{a}_{x:\overline{m}|} = E + (1+h)\,\mathrm{A}^{1}_{x:\overline{n}|} + (1+k)\,\mathrm{A}_{x:\overline{n}|}^{\;\;\;1}, $$

where $1 + h$ is the equivalent uniform amount of insurance. We thus obtain

$$ 1+h = \frac{{}^{C}\mathrm{P}\ddot{a}_{x:\overline{m}|} - E - (1+k)\,\mathrm{A}_{x:\overline{n}|}^{\;\;\;1}}{\mathrm{A}^{1}_{x:\overline{n}|}}. \tag{15} $$

The Standard Valuation Law permits the actuary considerable freedom in regard to plans with varying amounts of insurance. Thus, if an equivalent uniform amount is to be used in determining the extra initial expense provided by the reserve modification, it might be taken as the amount determined by formula (15); the amount resulting if first year benefits were excluded; or the amount resulting if to the actual death benefits were added a hypothetical insurance benefit of $1 + k$ from maturity date, for life. For illustrative purposes, we are following Espie by using (15). Actually, in some of our plans, calculation of $1 + h$ is unnecessary as, for instance, in the net level plan below.

For the three plans which will be presented, formulas will be stated without detailed proof. In every case, they follow from the General Theory section by specializing E and E'.

The net level plan: insurance for face amount or paid-up amount equivalent to net level reserve, if greater; reserves based on net level method and actual amounts of insurance

The formulas of the General Theory section apply immediately upon setting $m = n$ and $E = E' = 0$. In particular,

$$ \mathrm{P} = \frac{\mathrm{A}^{1}_{x:\overline{b}|} + (1+k)\,\mathrm{A}_{x:\overline{b}|}^{\;\;\;1}\mathrm{A}_{x+b:\overline{n-b}|}}{\ddot{a}_{x:\overline{b}|} + \mathrm{A}_{x:\overline{b}|}^{\;\;\;1}\mathrm{A}_{x+b:\overline{n-b}|} \cdot {}_{y}\mathrm{A}_{x+b}}. \tag{16} $$

We note that the criterion for b simplifies to

$$ {}_bP_{x:\overline{n}|} \cdot {}_yA_{x+b} \geq k \tag{17} $$

and that ${}_hCV = {}_hV$.

CRVM plan: insurance for face amount or paid-up amount equivalent to CRVM reserve, if greater; reserves based on CRVM and actual amounts of insurance

We now have

$$ E = E' = \left\{ \begin{matrix} (1+h)\,{}_{19}P_{x+1} \\ \beta^F \end{matrix} \right\} - c_x , \tag{18} $$

where β^F is the full preliminary term renewal net premium for the plan and $\{\quad\}$ denotes that the lesser quantity is to be chosen.

Case I. Assume $(1+h)_{19}P_{x+1} < \beta^F$, that is,

$$ E = (1+h)\,{}_{19}P_{x+1} - c_x . \tag{18a} $$

On substituting from (18*a*) into (15) with $m = n$ and solving for $1 + h$, we obtain

$$ 1 + h = \frac{{}^CP\ddot{a}_{x:\overline{n}|} + c_x - (1+k)\,A_{x:\overline{n}|}^{\;\;1}}{A^1_{x:\overline{n}|} + {}_{19}P_{x+1}} . \tag{19} $$

Further, from (9) with $m = n$

$$ {}^CP = \beta^C = \frac{A^1_{x:\overline{b}|} + (1+k)\,A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} + (1+h)\,{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{b}|} + A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} \cdot {}_yA_{x+b}} , \tag{20} $$

where β^C denotes the modified premium by the CRVM. Substitution from (19) into (20) and solution for ${}^CP$ yields

$$ {}^CP = \beta^C = \frac{A^1_{x:\overline{b}|} + (1+k)\,A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} - f_1(x, n, k)}{\ddot{a}_{x:\overline{b}|} + A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} \cdot {}_yA_{x+b} - g_1(x, n)} , \tag{21} $$

where

$$ f_1(x, n, k) = \frac{c_x A^1_{x:\overline{n}|} + (1+k)\,A_{x:\overline{n}|}^{\;\;1} \cdot {}_{19}P_{x+1}}{A^1_{x:\overline{n}|} + {}_{19}P_{x+1}} $$

and

$$ g_1(x, n) = \frac{{}_{19}P_{x+1}\,\ddot{a}_{x:\overline{n}|}}{A^1_{x:\overline{n}|} + {}_{19}P_{x+1}} . $$

The criterion for b is not readily usable in exact form, since E depends upon $1 + h$ which in turn depends upon ${}^{C}\mathrm{P}$. A suggested procedure for calculations is as follows:

i) Choose a trial b, either directly, or indirectly by first estimating either $1 + h$ or E and then applying the criterion.

ii) Compute a trial ${}^{C}\mathrm{P}$ from (21).

iii) Using this ${}^{C}\mathrm{P}$, check to see whether ${}_{b}\mathrm{W} \leq 1 < {}_{b+1}\mathrm{W}$ using (12a). If necessary repeat these steps until a proper ${}^{C}\mathrm{P}$ is found.

iv) Use (19) to compute $1 + h$.

v) In case of doubt that E was properly chosen, compute β^{F} by the formula

$$\beta^{F} = \frac{(1+h)\,\mathrm{A}^{1}_{x:\overline{n}|} + (1+k)\,\mathrm{A}_{x:\overline{n}|}^{\;\;1} - c_x}{a_{x:\overline{n-1}|}}. \tag{22}$$

If $(1 + h)_{19}\mathrm{P}_{x+1} < \beta^{F}$ proceed; otherwise, start again, assuming Case II below.

vi) Determine E from (18a). A check on the preceding computations is furnished by $E = {}^{C}\mathrm{P}\ddot{a}_{x:\overline{n}|} - (1 + h)A^{1}_{x:\overline{n}|} - (1 + k)A_{x:\overline{n}|}^{\;\;1}$.

vii) Compute ${}_{h}\mathrm{W}$ and ${}_{h}\mathrm{CV} = {}_{h}\mathrm{V}'$, using the appropriate formulas from the General Theory section.

Case II. Assume $(1 + h)_{19}\mathrm{P}_{x+1} \geq \beta^{F}$, that is

$$E = E' = \beta^{F} - c_x\,. \tag{18b}$$

The modified renewal net premium is

$${}^{C}\mathrm{P} = \beta^{F} = \frac{\mathrm{A}^{1}_{x+1:\overline{b-1}|} + (1+k)\,\mathrm{A}_{x+1:\overline{b-1}|}^{\;\;\;\;1}\mathrm{A}_{x+b:\overline{n-b}|}}{\ddot{a}_{x+1:\overline{b-1}|} + \mathrm{A}_{x+1:\overline{b-1}|}^{\;\;\;\;1}\mathrm{A}_{x+b:\overline{n-b}|}\cdot{}_{y}\mathrm{A}_{x+b}}, \tag{23}$$

which is identical with the net level annual premium for an insurance of our type on the life of $(x + 1)$, with cross-over age $x + b$ and maturity age $x + n$. Consequently, b may be determined by the use of the criterion (17) adjusted to the form

$${}_{b-1}\mathrm{P}_{x+1:\overline{n-1}|}\cdot{}_{y}\mathrm{A}_{x+b} \geq k\,.$$

Remaining procedures would be as for the net level plan adjusted to age $x + 1$.

The adjusted premium plan: insurance for face amount or minimum paid-up amount if greater; reserves based on CRVM and actual amounts of insurance

The first year expense allowance is

$$E = (1+h)\left(.02 + .25\left\{\begin{matrix}{}^{A}\mathrm{P}\\ \mathrm{OL}\\ .04\end{matrix}\right\}\right) + .4\left\{\begin{matrix}{}^{A}\mathrm{P}\\ .04\,(1+h)\end{matrix}\right\}, \tag{24}$$

where ${}^{A}_{OL}P$ is the adjusted premium for an ordinary life insurance of level amount 1 on the life of (x) and ${}^{A}P$ is the adjusted premium for the given plan. Once the age of the insured and the mortality and interest assumptions have been specified, the term $\left(.02 + .25\left\{\begin{matrix}{}^{A}_{OL}P\\ .04\end{matrix}\right\}\right)$ is determined and may for convenience be denoted by $c(x)$; however, the term $.4\left\{\begin{matrix}{}^{A}P\\ .04(1+h)\end{matrix}\right\}$ affords a choice; so again two cases arise. In addition, reserves are subdivided into two categories since E' is of the form (18) with β^F and $1 + h$ determined in regard to the present plan.

Case I. Assume ${}^{A}P < .04\,(1 + h)$. This implies that

$$E = (1 + h)\, c\,(x) + .4\,{}^{A}P\,. \tag{24a}$$

Then, using (24*a*) and (15) with $m = n$, we obtain

$$1 + h = \frac{{}^{A}P\,(\ddot{a}_{x:\overline{n}|} - .4) - (1 + k)\,A_{x:\overline{n}|}^{\;1}}{A_{x:\overline{n}|}^{1} + c\,(x)}. \tag{25}$$

Next steps are to use (24*a*) for the term E in (9), to substitute for $1 + h$ the right member of (25), and to solve for ${}^{C}P = {}^{A}P$, the result being

$${}^{C}P = {}^{A}P = \frac{A_{x:\overline{b}|}^{1} + (1 + k)\,A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|} - f_2\,(x,\,n,\,k)}{\ddot{a}_{x:\overline{b}|} + A_{x:\overline{b}|}^{\;\;1}A_{x+b:\overline{n-b}|}\cdot {}_{y}A_{x+b} - g_2\,(x,\,n)}\,, \tag{26}$$

where

$$f_2\,(x,\,n,\,k) = c\,(x)\,\frac{(1 + k)\,A_{x:\overline{n}|}^{\;1}}{A_{x:\overline{n}|}^{1} + c\,(x)}\,,$$

$$g_2\,(x,\,n) = c\,(x)\,\frac{\ddot{a}_{x:\overline{n}|} - .4}{A_{x:\overline{n}|}^{1} + c\,(x)} + .4$$

and, as above noted,

$$c\,(x) = .02 + .25\left\{\begin{matrix}{}^{A}_{OL}P\\ .04\end{matrix}\right\}.$$

The criterion for b should not be applied on an exact basis since that would involve laborious calculations. Instead, we suggest the use of steps similar to those listed for Case I of the CRVM plan. For the adjusted premium plan, reserves would not be equal to the cash values; however, they could be calculated by the use of (14) after the relative size of $(1 + h)_{19}P_{x+1}$ and β^F has been established. In cases where β^F is, or is suspected to be, the modified renewal net premium, it must be computed. Since the present value of the modified net premiums equals the present value of the benefits which, in turn, equals the present value of the cash value premiums less E, we may write

$$c_x + \beta^F a_{x:\overline{n-1}|} = {}^C\mathrm{P}\ddot{a}_{x:\overline{n}|} - E\,,$$

which gives

$$\beta^F = {}^C\mathrm{P} + \frac{{}^C\mathrm{P} - E - c_x}{a_{x:\overline{n-1}|}}. \tag{27}$$

Alternatively, formula (22) might be used. It would not be correct to use formula (23) because, for the present plan, paid-up amounts are not based upon the reserves.

Case II. Assume ${}^A\mathrm{P} \geq .04\,(1 + h)$. From (24) it follows that

$$E = (1+h)\,[\,c\,(x) + .016\,]. \tag{24b}$$

Further, from (24*b*) and (15), we have

$$1 + h = \frac{{}^A\mathrm{P}\ddot{a}_{x:\overline{n}|} - (1+k)\,\mathrm{A}_{x:\overline{n}|}^{\;\;1}}{\mathrm{A}_{x:\overline{n}|}^{1} + c\,(x) + .016}. \tag{28}$$

Then (9), (24*b*) and (28) yield

$${}^C\mathrm{P} = {}^A\mathrm{P} = \frac{\mathrm{A}_{x:\overline{b}|}^{1} + (1+k)\,\mathrm{A}_{x:\overline{b}|}^{\;\;1}\mathrm{A}_{x+b:\overline{n-b}|} - f_3\,(x,\, n,\, k)}{\ddot{a}_{x:\overline{b}|} + \mathrm{A}_{x:\overline{b}|}^{\;\;1}\mathrm{A}_{x+b:\overline{n-b}|}\cdot{}_y\mathrm{A}_{x+b} - g_3\,(x,\, n)}, \tag{29}$$

where

$$f_3\,(x,\, n,\, k) = [\,c\,(x) + .016\,]\,\frac{(1+k)\,\mathrm{A}_{x:\overline{n}|}^{\;\;1}}{\mathrm{A}_{x:\overline{n}|}^{1} + c\,(x) + .016}$$

and

$$g_3\,(x,\, n) = [\,c\,(x) + .016\,]\,\frac{\ddot{a}_{x:\overline{n}|}}{\mathrm{A}_{x:\overline{n}|}^{1} + c\,(x) + .016}.$$

Once more, the criterion for b proves difficult to apply on an exact basis but may be used to give an approximate result if an approximate $1 + h$ is first chosen. As pointed out by Lang [4, discussion, page 374], for the face amount or cash value if greater plan the value of $1 + h$ does not vary much with age at issue, and for our plan the variation of $1 + h$ appears to be limited although somewhat larger. Steps in the complete calculation again are similar to those given for Case I of the CRVM plan. Step (v), if necessary, would require the computation of ${}^A\mathrm{P}$ using (26) and, as a start, the value of b determined for the present Case II assumption. The discussion of reserves in connection with the adjusted premium plan, Case I, also fits this case.

NUMERICAL ILLUSTRATIONS

For purpose of comparison, the plan providing insurance for face amount or net level reserve if greater, which will be labeled "the Fassel plan," will be added to the three plans described in the preceding section. In the numerical illustrations below, the following assumptions apply to all four plans:

Face amount: $1,000.

Age at issue, x: 35.

Age at maturity, $y = x + n$: 65.

Premium payment term, m: 30.

Maturity value, $1,000 $(1 + k)$: present value of an annuity due of $10 per month for 10 years certain and life. Computed according to the a–1949(m) Table, with $2\frac{1}{2}\%$ interest; $1,000 $(1 + k)$ = $1,582.

Mortality prior to maturity: CSO Table.

Interest prior to maturity: $2\frac{1}{2}$ percent.

Table I lists the basic values for each of the four specific plans, Table II

TABLE I

BASIC VALUES*

Value	Fassel Plan	Net Level Plan	CRVM Plan	Adj. Prem. Plan
Cash value premium, ${}^{c}P$	$ 38.35827	$ 39.12795	$ 40.80771	$ 41.46515
Face amount period, b (years)	21	17	17	17
Equiv. unif. amount, $1,000 $(1+k)$†	$1,125.5037	$1,190.4816	$1,184.6444	$1,182.3598
Extra initial expense provided by cash value premiums, E	0	0	$ 33.59722	$ 46.74684
Extra initial expense provided by reserve method, E'	0	0	$ 33.59722	$ 33.52379

* Here ${}^{c}P$, E and E' denote values for an insurance with face amount $1,000.

† The eighth figure is not significant.

offers a comparison of amounts of insurance during certain insurance years, and Table III presents a comparison of terminal reserves at various durations.

Table I indicates that the equivalent uniform amount of insurance is approximately the same for the three plans for face amount or paid-up insurance amount if greater. Bearing this out, Table II shows that on these plans the amounts of insurance are also approximately the same in any insurance year, but are considerably larger than those provided by the

Fassel plan, with the maximum increase (21 percent) occurring in the 22d insurance year when the amount under the Fassel plan is just beginning to increase. However, it may be noted that the premium for the net level plan is only 2.0 percent greater than the premium for the Fassel plan, whereas the premium for the adjusted premium plan is 6.0 percent greater than that of the net level plan. Table III indicates, as one would expect, that reserves on the net level plan exceed those on the Fassel plan except at the very late durations, and that the reserves for the CRVM plan, with minor exceptions, are slightly larger than those for the adjusted premium plan.

TABLE II

COMPARISON OF AMOUNTS OF INSURANCE

Insurance Year	Amounts of Insurance				(2) As Percentage of (1)
	Fassel Plan (1)	Net Level Plan (2)	CRVM Plan (3)	Adjusted Prem. Plan (4)	
1–17........	$1,000	$1,000	$1,000	$1,000	100.0%
18...........	1,000	1,037	1,013	1,004	103.7
20...........	1,000	1,137	1,118	1,111	113.7
21...........	1,000	1,186	1,169	1,163	118.6
22...........	1,017	1,234	1,219	1,213	121.3
23...........	1,081	1,281	1,268	1,263	118.5
25...........	1,216	1,372	1,363	1,359	112.8
29...........	1,505	1,542	1,540	1,539	102.5
30...........	1,582	1,582	1,582	1,582	100.0

TABLE III

COMPARISON OF TERMINAL RESERVES

End of Insurance Year	Terminal Reserve			
	Fassel Plan	Net Level Plan	CRVM Plan	Adjusted Premium Plan
1.......	$ 34.89	$ 35.68	$ 2.81	$ 2.86
2.......	70.56	72.17	40.05	40.06
5.......	182.54	186.76	156.94	156.86
10.......	387.52	396.70	371.01	370.75
15.......	620.53	635.73	614.52	614.03
20.......	892.57	911.07	895.79	895.42
25.......	1,215.60	1,221.21	1,213.18	1,213.13
29.......	1,505.06	1,504.29	1,502.61	1,502.64
30.......	1,582.00	1,582.00	1,582.00	1,582.00

CONCLUSION

In regard to plans where the maturity value exceeds the face amount of insurance, as occurs in connection with the currently popular insurance-annuity plans, there are many ways in which the amounts of insurance could be graded into the maturity value. The usual solution is the face amount or cash value if greater plan which has a natural and meaningful mathematical basis, and a certain amount of simplicity during the period of grading.

Another solution which also has these qualities has been presented in this paper. This second solution is more of an "insurance" plan and less of a "savings" plan than the other. Its theory appears to be a little more complicated than the theory for the other plan but that may be partly due to the unfamiliar formulas which occur. However, it avoids the difficulty of the other plan in regard to nonforfeiture benefits at the later durations.

We hope that this new plan may prove to be of both theoretical and practical interest.

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