Pricing Perpetual Fund Protection with Withdrawal Option

Hans U. Gerber

Ecole des hautes études commerciales
Université de Lausanne
CH-1015 Lausanne, Switzerland
Phone: 41 21 692 3371   Fax: 41 21 692 3305
E-mail: hgerber@hec.unil.ch

Elias S.W. Shiu

Department of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242-1409, U.S.A.
Phone: 319 335 2580   Fax: 319 335 3017
and
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hung Hom, Hong Kong, China
Phone: 852 2766 6925   Fax: 852 2362 9045
E-mail: eshiu@stat.uiowa.edu

ABSTRACT

Consider an American option that provides the amount

$$F(t) = S_2(t) \max\{1, \max_{0 \le \tau \le t} \frac{S_1(\tau)}{S_2(\tau)}\},$$

if it is exercised at time t, $t \ge 0$. For simplicity of language, we interpret $S_1(t)$ and $S_2(t)$ as the prices of two stocks. The price of stock 2 is provided with a dynamic protection that is defined in terms of the prices of stock 1: if $F(t) > S_1(t)$, the instantaneous rates of return of $F(t)$ and $S_2(t)$ are identical. And if $F(t)$ threatens to fall below $S_1(t)$, just enough funds are provided to prevent this from happening. For the two stock prices, the bivariate Black-Scholes model with constant dividend-yield rates is assumed. In the case of a perpetual option, closed form expressions for the optimal exercise strategy and the price of this option are given. Furthermore, this price is compared to the price of the perpetual maximum option, and it is shown that the optimal exercise of the maximum option occurs before that of the dynamic fund protection. Two general concepts in the theory of option pricing are illustrated: the smooth pasting condition and the construction of the replicating portfolio. The general result can be applied to two special cases. One is where the guaranteed level $S_1(t)$ is a deterministic exponential or constant function. The other is where $S_2(t)$ is an exponential or constant function; in this case, known results concerning the pricing of Russian options are retrieved. Finally, we consider a generalization of the perpetual lookback put option which has payoff $[F(t) - \kappa S_1(t)]$, if it is exercised at time t. This option can be priced with the same technique.

1. Introduction

Equity-indexed annuities (EIAs) can be viewed as mutual funds wrapped around with various guarantees. An overview of EIAs can be found in the Society of Actuaries study note by Mitchell and Slater (1996), a 1997 Task Force Report of the American Academy of Actuaries, and the book by Streiff and DiBiase (1999). In the context of the classical Black-Scholes (1973) model, Tiong (2000a, 2000b) and Lee (2002) have discussed the pricing of some such guarantees.

In a recent paper in this journal, Gerber and Pafumi (2000) have proposed the concept of a dynamic guarantee or protection, which is applicable to EIA products. The primary (or "naked") fund is replaced by a protected (or "upgraded") fund. The guarantee is that the value of the protected fund does not fall below a guaranteed level or floor at all times before the maturity date of the contract. Assuming a geometric Brownian motion for the primary fund, Gerber and Pafumi (2000) derived a closed form formula for pricing this dynamic guarantee if early withdrawal from the fund (before maturity) is not permitted. In a subsequent paper in this journal, Imai and Boyle (2001, Section 3) suppose that an early withdrawal from the fund is possible, and then show that it is optimal to not exercise this early withdrawal option and to cash in the fund accumulation only at maturity. However, their conclusion is based on a crucial assumption. It is assumed that all dividends are reinvested in the primary fund, so that the discounted value of a unit of the primary fund is a martingale with respect to the risk-neutral probability measure (which is the probability measure used for pricing purposes). Without this assumption, it may very well be optimal to exercise the withdrawal option early, i.e., to cash in the fund accumulation before maturity. In Section 6 we shall give an explicit illustration of this in the case of a perpetual option.

This paper studies the pricing of dynamic protection without maturity date. The investor chooses the date to cash in the fund accumulation. Furthermore, the guaranteed level is not necessarily constant or exponential, but can be stochastic, such as that given by a stock price or stock index. For simplicity of language, let us explain the general problem considered in this paper in terms of stocks. For $t \geq 0$, let $S_1(t)$ and $S_2(t)$ be the time-t prices of two stocks. Assuming $S_1(0) \leq S_2(0)$, we consider a security that gives its owner the amount

$$F(t) = S_2(t) \max\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\} \tag{1.1}$$

at a time t of his choice. Hence the security can be viewed as a *perpetual* option, i.e., an American option without a maturity date. Note also that its payoff is path-dependent. In the context of dynamic protection, $S_2(t)$ is the time-t value of one unit of the primary fund, $S_1(t)$ is the guaranteed level at time t, and the quantity (1.1) is the time-t value of the protected fund. What is the price of this option, and what is the optimal early exercise strategy? Explicit answers are given in Section 2, with the proof given in Section 3. Furthermore, detailed explanation for (1.1) is given in Section 2.

Russian options were introduced by Shepp and Shiryaev (1993); they are discussed in Section 10.11 of the actuarial textbook Panjer et al. (1998). A Russian option can be viewed as a special case of the general option considered in the last paragraph. Hence the main results concerning Russian options can be easily retrieved from the results in Section 2; this is done in Section 7. Shepp and Shiryaev's (1993) result created a surprise because it gives a closed form formula for pricing an American option with a *path-dependent* payoff.

Section 8 considers a generalization of the perpetual *lookback put option* whose payoff is $[F(t) - \kappa S_1(t)]$, if it is exercised at time t. Here, κ is a constant between 0 and 1. This option can be priced with the same technique, generalizing the result in Section 10.12 of Panjer et al. (1998).

Note that the *dynamic fund protection option* should be distinguished from the *maximum option* (also called *alternative option* or *greater-of option*), whose payoff is

$$\max\{S_1(t), S_2(t)\} = S_2(t) \max\{1, \frac{S_1(t)}{S_2(t)}\}, \tag{1.2}$$

if it is exercised at time t. A comparison of (1.2) with (1.1) reveals that the maximum option is a less expensive security. In Section 4, an explicit comparison of the prices of the *perpetual* dynamic fund protection and the *perpetual* maximum option is provided. It is shown that the optimal exercise of the maximum option occurs before that of the dynamic fund protection.

2. The General Problem and Its Solution

For $t \geq 0$, let $S_1(t)$ and $S_2(t)$ be the time-t prices of two stocks. Consider using the price of stock 1 as a dynamic protection for stock 2. With this guarantee, $S_2(t)$ is replaced by

$$F(t) = S_2(t) \max\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\}, \qquad t \geq 0. \tag{2.1}$$

We assume $S_1(0) \leq S_2(0)$, so that $F(0) = S_2(0)$. The interpretation of (2.1) is as follows. Whenever $F(t) > S_1(t)$, the instantaneous rates of return of $\{F(t)\}$ and $\{S_2(t)\}$ are identical,

$$\frac{dF(t)}{F(t)} = \frac{dS_2(t)}{S_2(t)}. \tag{2.2}$$

Whenever $F(t)$ threatens to fall below $S_1(t)$, just enough funds are added to prevent this from happening.

There is an alternative way to motivate (2.1). Consider a contract that provides a sufficient number of units of stock 2 so that the total value of these units is at least the value of one unit of stock 1 at any time. Let $n(\tau)$ denote the aggregate number of units of stock 2 at time τ, $\tau \geq 0$. The following three conditions must be satisfied:

(i) $n(0) = 1$;

(ii) $n(\tau)$ is a nondecreasing function of τ;

(iii) $n(\tau)S_2(\tau) \geq S_1(\tau)$ for all τ.

Condition (i) merely states that we start with one unit of stock 2. Condition (ii) means that additional units can be credited, but those units can never be taken away afterwards. From conditions (ii) and (iii), it follows that

$$n(t) \geq n(\tau) \geq \frac{S_1(\tau)}{S_2(\tau)} \qquad \text{for } 0 \leq \tau \leq t$$

and hence

$$n(t) \geq \max_{0 \le \tau \le t} \frac{S_1(\tau)}{S_2(\tau)}.$$

Because of (i), we have

$$n(t) \geq \max\{1, \max_{0 \le \tau \le t} \frac{S_1(\tau)}{S_2(\tau)}\}. \tag{2.3}$$

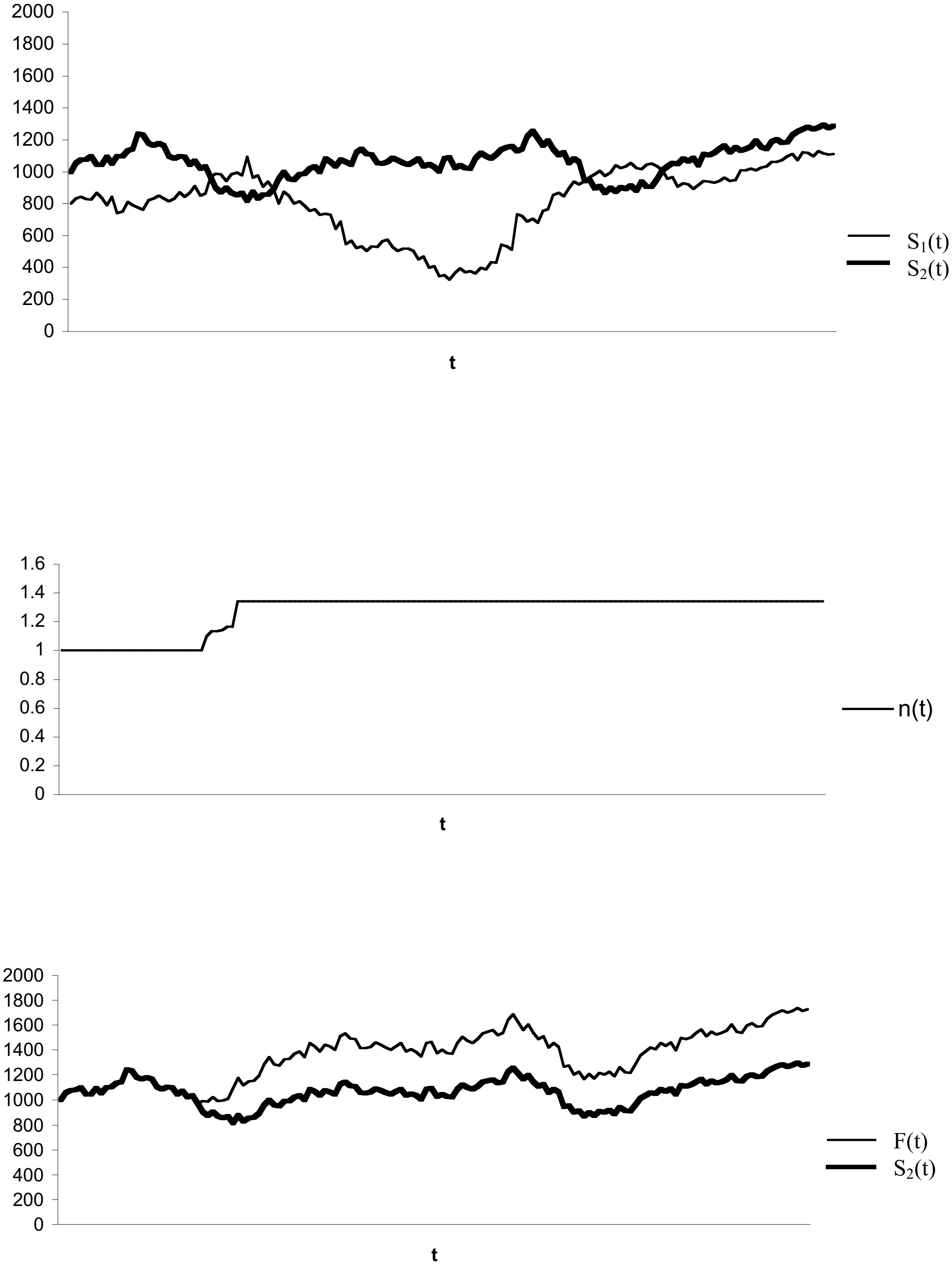
Evidently, there is an infinite number of functions n(t) that satisfy these three conditions. To obtain the guarantee with the least cost, we choose the smallest such function, that is the one with equality in (2.3):

$$n(t) = \max\{1, \max_{0 \le \tau \le t} \frac{S_1(\tau)}{S_2(\tau)}\}. \tag{2.4}$$

Then the value of the aggregate units at time t, $n(t)\, S_2(t)$, is identical to F(t), see (2.1). Formulas (2.4) and (2.1) are illustrated in the second and third panels of Figure 1.

Figure 1

The definitions of n(t) and F(t)



A main goal in this paper is to price the perpetual option with payoff F(t) given by (2.1). We follow the classical Black-Scholes type model. We assume a constant risk-free force of interest $r > 0$. For $j = 1, 2$, let

$$S_j(t) = S_j(0)\, e^{X_j(t)}, \qquad t \geq 0. \tag{2.5}$$

It is assumed that $\{X_1(t), X_2(t)\}$ is a bivariate Wiener process in the Q-measure (the probability measure that is used for pricing derivatives), with correlation ρ, instantaneous variances σ_1^2 and σ_2^2, and drift parameters

$$\mu_j = r - \frac{\sigma_j^2}{2} - \zeta_j, \qquad j = 1, 2. \tag{2.6}$$

We assume that ζ_1 and ζ_2 are positive. The constant ζ_j can be interpreted as the dividend-yield rate of stock j; that is, one may consider that dividends of amount $\zeta_j S_j(t)dt$ are paid between time t and time t+dt. However, this interpretation of the constants ζ_1 and ζ_2 is not needed in the analysis below, except for Section 5 where replicating portfolios are discussed.

The option is perpetual. Thus its time-0 price is

$$V(s_1, s_2) = \sup_T E[e^{-rT} F(T)], \tag{2.7}$$

where T is any exercise time, F(T) is given by (2.1), and $s_j = S_j(0)$, $j = 1, 2$. The *stopping time* T for which the maximum is attained is called the *optimal exercise strategy.* As explained in the Appendix, the optimal exercise strategy is one of the form

$$T_\varphi = \min\{t \mid S_1(t) = \varphi F(t)\} \tag{2.8}$$

with $0 < \varphi < 1$. That is, the option is exercised as soon as the ratio $S_1(t)/F(t)$ falls to the level φ.

Let

$$V(s_1, s_2; \varphi) = E[e^{-rT_\varphi} F(T_\varphi)], \qquad \varphi s_2 \le s_1 \le s_2, \tag{2.9}$$

denote the value of such a strategy, with $s_j = S_j(0)$, $j = 1, 2$. In Section 3, we shall show that

$$V(s_1, s_2; \varphi) = \frac{h(s_1/s_2)}{h(\varphi)} s_2, \tag{2.10}$$

where

$$h(z) = (\theta_2 - 1)z^{\theta_1} + (1 - \theta_1)z^{\theta_2}, \quad z > 0. \tag{2.11}$$

Here, θ_1 and θ_2 are the solutions of the quadratic equation,

$$\begin{aligned} 0 &= -r + E[\theta X_1(1) + (1-\theta)X_2(1)] + \frac{1}{2}\mathrm{Var}[\theta X_1(1) + (1-\theta)X_2(1)] \\ &= -r + \mu_1\theta + \mu_2(1-\theta) + \frac{\sigma_1^2}{2}\theta^2 + \frac{\sigma_2^2}{2}(1-\theta)^2 + \rho\sigma_1\sigma_2\theta(1-\theta), \end{aligned} \tag{2.12}$$

with μ_1 and μ_2 given by (2.6). Note that, because of (2.6), the expression on right-hand side of (2.12) equals $-\zeta_2$ for $\theta = 0$ and equals $-\zeta_1$ for $\theta = 1$. Hence, one solution of (2.12) is negative, and the other is greater than one, say $\theta_1 < 0$, $\theta_2 > 1$. Note that $h(z) \to \infty$ for $z \to \infty$ and for $z \to 0$. Furthermore,

$$h''(z) = (\theta_2 - 1)(1 - \theta_1)(-\theta_1 z^{\theta_1 - 2} + \theta_2 z^{\theta_2 - 2}) > 0$$

Hence the graph of the function $h(z)$, $z > 0$, is U-shaped, and the function $h(z)$ has a unique minimum.

Let $\breve{\varphi}$ denote the optimal value of φ, i.e., the value that maximizes the expression (2.10), or, equivalently, that minimizes its denominator $h(\varphi)$. This leads to

$$h'(\breve{\varphi}) = 0, \tag{2.13}$$

or

$$\breve{\varphi} = \left(\frac{-\theta_1(\theta_2 - 1)}{\theta_2(1 - \theta_1)} \right)^{\frac{1}{\theta_2 - \theta_1}}. \tag{2.14}$$

The number $\breve{\varphi}$ is between 0 and 1 because both $-\theta_1/(1 - \theta_1)$ and $(\theta_2 - 1)/\theta_2$ are between 0 and 1. The optimal strategy is to exercise the option the first time when $S_1(t) = \breve{\varphi}F(t)$, if $s_1 > \breve{\varphi}s_2$, and to exercise it immediately, if $s_1 \le \breve{\varphi}s_2$. Hence the *price* of the option is

$$V(s_1, s_2) = \begin{cases} \dfrac{h(s_1/s_2)}{h(\breve{\varphi})} s_2 & \text{if } \breve{\varphi} < \dfrac{s_1}{s_2} \le 1 \\ s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \le \breve{\varphi} \end{cases}, \tag{2.15}$$

with h(.) given by (2.11). In Section 4 we shall indicate an alternative expression for the price of the option

The payoff of the option is path dependent, but with a simple structure: if the option has not been exercised by time t, $t \ge 0$, then $\breve{\varphi} < S_1(t)/F(t) \le 1$, and the only relevant information for the future is the values of $S_1(t)$ and $F(t)$. The time-t option price follows immediately from (2.15). It is

$$\frac{h(S_1(t)/F(t))}{h(\breve{\varphi})} F(t). \tag{2.16}$$

Furthermore, the amount

$$\left[\frac{h(S_1(t)/F(t))}{h(\breve{\varphi})} - 1\right] F(t) \tag{2.17}$$

is the time-t value for having the possibility to cash in the funds at a future date of the investor's choice.

Remarks: We have assumed that $\zeta_1 > 0$ and $\zeta_2 > 0$. Consider what happens when one of these conditions is violated.

(a) If $\zeta_1 > 0$ and $\zeta_2 = 0$, it follows that $\theta_1 = 0$ and $\theta_2 > 1$. Hence, from (2.10) and (2.11),

$$V(s_1, s_2; \varphi) = \frac{\theta_2 - 1 + (s_1/s_2)^{\theta_2}}{\theta_2 - 1 + \varphi^{\theta_2}} s_2. \tag{2.18}$$

The supremum is obtained for $\varphi = 0$. It is

$$\begin{aligned} V(s_1, s_2; 0) &= s_2 + \frac{s_2}{\theta_2 - 1}\left(\frac{s_1}{s_2}\right)^{\theta_2} \\ &= s_2 + \frac{s_1}{R}\left(\frac{s_1}{s_2}\right)^{R}, \end{aligned} \tag{2.19}$$

with

$$\begin{aligned} R &= \theta_2 - 1 \\ &= \frac{2\zeta_1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \end{aligned} \tag{2.20}$$

In this case there is no optimal withdrawal strategy. This phenomenon can be explained as follows. Because stock 2 does not pay any dividends, there is no incentive for early withdrawal (cf. Imai and Boyle 2000, Section 3). On the other hand, in order to take advantage of the guarantee, the option is exercised as late as possible, that is … never. Ingersoll (1987, page 373) calls this type of situation the problem of "infinities." Expression (2.19) should be interpreted as the sum of the price for one share of stock 2 and the price for perpetual dynamic protection. In fact, formula (2.19) generalizes formula (2.13) of Gerber and Pafumi (2000). Note that the denominator of (2.20) is $\mathrm{Var}[X_1(1) - X_2(1)]$.

(b) If $\zeta_1 = 0$ and $\zeta_2 \geq 0$, it follows that $\theta_1 \leq 0$ and $\theta_2 = 1$. Hence, from (2.10) and (2.11),

$$V(s_1, s_2; \varphi) = \frac{s_1}{\varphi}. \tag{2.21}$$

In this case, the supremum is obviously infinite. Formula (2.21) can be directly obtained from (2.9), which, by (2.8), is

$$V(s_1, s_2; \varphi) = \frac{E[e^{-rT_\varphi} S_1(T_\varphi)]}{\varphi}. \tag{2.22}$$

With $\zeta_1 = 0$, $\{e^{-rt} S_1(t)\}$ is a martingale. Hence the expectation in (2.22) is s_1 by the *optional sampling theorem*; this proves (2.21).

3. Derivation of Formula (2.10)

The function $V(s_1, s_2; \varphi)$ can be evaluated by means of martingales as in Gerber and Shiu (1994, 1996a). First note that the function $V(s_1, s_2; \varphi)$, defined by (2.9), is homogeneous of degree 1, i.e., for each $z > 0$,

$$V(zs_1, zs_2; \varphi) = z\, V(s_1, s_2; \varphi), \qquad \varphi s_2 \le s_1 \le s_2. \tag{3.1}$$

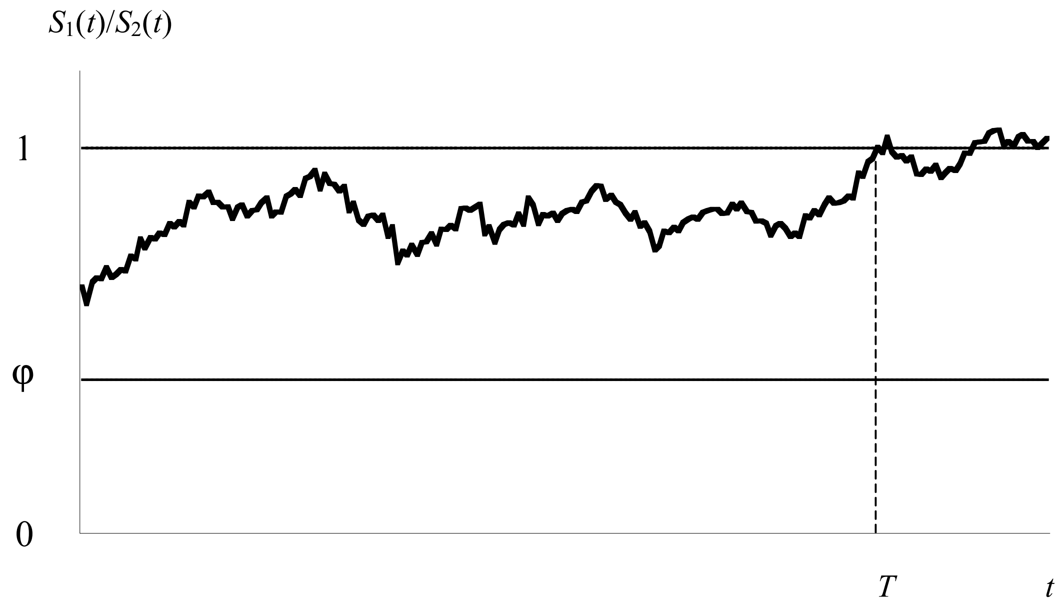
Consider

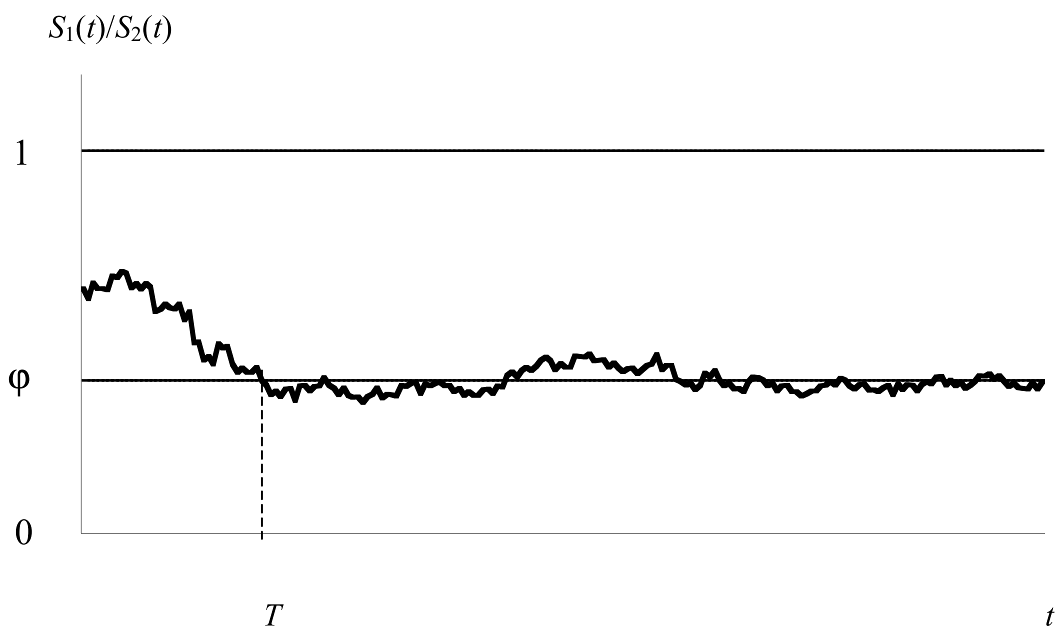
$$T = \min\{t \mid S_1(t) = \varphi S_2(t) \text{ or } S_1(t) = S_2(t)\}, \tag{3.2}$$

the first time when the ratio of prices, $S_1(t)/S_2(t)$, attains the value φ or 1. See Figure 2.

Figure 2

The Definition of T in Formula (3.2)





By conditioning on when the ratio first attains the value φ or 1, we can rewrite the expectation (2.9) as

$$
\begin{aligned}
V(s_1, s_2; \varphi) &= E[e^{-rT} S_2(T)\, I(S_1(T) = \varphi\, S_2(T))] \\
&\quad + E[e^{-rT} V(S_2(T), S_2(T); \varphi)\, I(S_1(T) = S_2(T))] \\
&= E[e^{-rT} S_2(T)\, I(S_1(T) = \varphi S_2(T))] \\
&\quad + E[e^{-rT} S_2(T)\, I(S_1(T) = S_2(T))]\, V(1, 1; \varphi) \qquad (3.3)
\end{aligned}
$$

by (3.1) with $z = S_2(T)$. Here, I(.) denotes the indicator function of an event. This suggests the following definitions:

$$A(s_1, s_2; \varphi) = E[e^{-rT} S_2(T)\, I(S_1(T) = \varphi S_2(T))], \qquad (3.4)$$

and

$$B(s_1, s_2; \varphi) = E[e^{-rT} S_2(T)\, I(S_1(T) = S_2(T))]. \qquad (3.5)$$

Then (3.3) becomes

$$V(s_1, s_2; \varphi) = A(s_1, s_2; \varphi) + B(s_1, s_2; \varphi)\, V(1, 1; \varphi). \qquad (3.6)$$

To get closed form expressions for $A(s_1, s_2; \varphi)$ and $B(s_1, s_2; \varphi)$, we seek θ so that the stochastic process

$$\{e^{-rt} S_2(t)\, [S_1(t)/S_2(t)]^{\theta}\} \qquad (3.7)$$

becomes a martingale. Equivalently, we seek θ so that

$$\{e^{-rt + \theta X_1(t) + (1-\theta)X_2(t)}\} \qquad (3.8)$$

is a martingale, which is the case if

$$e^{-r} E[e^{\theta X_1(1) + (1-\theta)X_2(1)}] = 1. \qquad (3.9)$$

This, in turn, leads to the quadratic equation (2.12), which has two solutions, $\theta_1 < 0$, and $\theta_2 > 1$. With $\theta = \theta_j$, the stochastic process (3.7) is a martingale; if we stop it at time T and apply the optional sampling theorem, we obtain

$$s_1^{\theta_j} s_2^{1-\theta_j} = E[e^{-rT} S_2(T) [S_1(T)/S_2(T)]^{\theta_j}]$$

$$= A(s_1, s_2; \varphi)\varphi^{\theta_j} + B(s_1, s_2; \varphi), \qquad j = 1, 2. \tag{3.10}$$

These are two linear equations for A and B. Their solution is

$$A(s_1, s_2; \varphi) = \frac{s_1^{\theta_1} s_2^{1-\theta_1} - s_1^{\theta_2} s_2^{1-\theta_2}}{\varphi^{\theta_1} - \varphi^{\theta_2}}, \tag{3.11}$$

$$B(s_1, s_2; \varphi) = \frac{s_1^{\theta_2} s_2^{1-\theta_2}\varphi^{\theta_1} - s_1^{\theta_1} s_2^{1-\theta_1}\varphi^{\theta_2}}{\varphi^{\theta_1} - \varphi^{\theta_2}}. \tag{3.12}$$

Substituting (3.11) and (3.12) in the right-hand side of (3.6) yields

$$V(s_1, s_2; \varphi) = \frac{(s_1^{\theta_1} s_2^{1-\theta_1} - s_1^{\theta_2} s_2^{1-\theta_2}) + (s_1^{\theta_2} s_2^{1-\theta_2}\varphi^{\theta_1} - s_1^{\theta_1} s_2^{1-\theta_1}\varphi^{\theta_2})V(1,1;\varphi)}{\varphi^{\theta_1} - \varphi^{\theta_2}}. \tag{3.13}$$

To determine $V(1, 1; \varphi)$, we use the condition

$$\left.\frac{\partial V(s_1, s_2; \varphi)}{\partial s_2}\right|_{s_1=s_2} = 0. \tag{3.14}$$

(An intuitive explanation of this condition is that when s_2 is "close" to s_1, the guarantee will be used instantaneously, and so the value of V is unaffected by marginal changes in s_2. For a rigorous derivation of a similar condition, see Goldman, Sosin and Gatto (1979).) Differentiating (3.13) with respect to s_2 and applying (3.14), we obtain the equation

$$0 = [(1-\theta_1) - (1-\theta_2)] + [(1-\theta_2)\varphi^{\theta_1} - (1-\theta_1)\varphi^{\theta_2}]V(1, 1; \varphi)$$

$$= \theta_2 - \theta_1 - h(\varphi)V(1, 1; \varphi),$$

where h(j) is defined by (2.11). Hence

$$V(1, 1; j) = \frac{\theta_2 - \theta_1}{h(\varphi)}. \tag{3.15}$$

Finally, we substitute (3.15) in (3.13) to obtain (2.10) after some simplifications.

4. Comparison with the Perpetual Maximum Option

By comparing (1.1) with (1.2), we have seen that the maximum option is cheaper than the dynamic fund protection with withdrawal option. If the options are perpetual, an analytical comparison of the prices is possible.

According to Section 8 in Gerber and Shiu (1996a), the price of the perpetual maximum options is

$$W(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \le \breve{b} \\ s_2\, g\left(\dfrac{s_1}{\breve{b} s_2}\right) & \text{if } \breve{b} < \dfrac{s_1}{s_2} < \breve{c}, \\ s_1 & \text{if } \dfrac{s_1}{s_2} \ge \breve{c} \end{cases} \tag{4.1}$$

where

$$g(x) = \frac{\theta_2 x^{\theta_1} - \theta_1 x^{\theta_2}}{\theta_2 - \theta_1}, \qquad x > 0, \tag{4.2}$$

and the endpoints of the optimal continuation (non-exercise) interval are

$$\breve{b} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{(\theta_2-1)/(\theta_2-\theta_1)}, \tag{4.3}$$

$$\breve{c} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{\theta_2/(\theta_2-\theta_1)}. \tag{4.4}$$

Note that

$$\frac{\breve{b}}{\breve{c}} = \breve{\varphi} \tag{4.5}$$

by (2.14), and that

$$0 < \breve{b} < 1 < \breve{c}. \tag{4.6}$$

Thus

$$0 < \breve{\varphi} < \breve{b} < 1. \tag{4.7}$$

Formula (4.1) cannot be compared immediately with (2.15). Therefore, we now derive an alternative expression for $V(s_1, s_2)$. The first-order condition (2.13) is the same as

$$(1-\theta_1)\theta_2\breve{\varphi}^{\theta_2} = -(\theta_2-1)\theta_1\breve{\varphi}^{\theta_1}, \tag{4.8}$$

applying which to (2.11) yields the formulas

$$h(\breve{\varphi}) = \frac{\theta_2-\theta_1}{\theta_2}(\theta_2-1)\breve{\varphi}^{\theta_1} \tag{4.9}$$

and

$$h(\breve{\varphi}) = -\frac{\theta_2-\theta_1}{\theta_1}(1-\theta_1)\breve{\varphi}^{\theta_2}. \tag{4.10}$$

It follows from (4.9) that

$$\frac{(\theta_2 - 1)z^{\theta_1}}{h(\breve{\varphi})} = \frac{\theta_2}{\theta_2-\theta_1}\left(\frac{z}{\breve{\varphi}}\right)^{\theta_1}.$$

Similarly, it follows from (4.10) that

$$\frac{(1 - \theta_1)z^{\theta_2}}{h(\breve{\varphi})} = \frac{-\theta_1}{\theta_2-\theta_1}\left(\frac{z}{\breve{\varphi}}\right)^{\theta_2}.$$

By (2.11) the sum of the left-hand sides of the last two formulas is $h(z)/h(\breve{\varphi})$, while by (4.2) the sum of the right-hand sides is $g(z/\breve{\varphi})$. Thus we have

$$\frac{h(z)}{h(\breve{\varphi})} = g(\frac{z}{\breve{\varphi}}), \qquad z > 0. \tag{4.11}$$

From (4.11) and (2.15) we obtain the alternative expression for the price of the perpetual dynamic protection with optimal withdrawal,

$$V(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \le \breve{\varphi} \\ s_2\, g(\dfrac{s_1}{\breve{\varphi} s_2}) & \text{if } \dfrac{s_1}{s_2} > \breve{\varphi} \end{cases}. \tag{4.12}$$

This formula can be compared directly with (4.1). We see that the differences of the prices is

$$V(s_1, s_2) - W(s_1, s_2) = \begin{cases} 0 & \text{if } \dfrac{s_1}{s_2} \le \breve{\varphi} \\ s_2 \left[g(\dfrac{s_1}{\breve{\varphi} s_2}) - 1 \right] & \text{if } \breve{\varphi} < \dfrac{s_1}{s_2} \le \breve{b} \\ s_2 \left[g(\dfrac{s_1}{\breve{\varphi} s_2}) - g(\dfrac{s_1}{\breve{b} s_2}) \right] & \text{if } \breve{b} < \dfrac{s_1}{s_2} \le 1 \end{cases} \tag{4.13}$$

Both differences in the second and third lines are strictly positive. To verify this, observe that $g(x)$, $x > 0$, as defined by (4.2), is a positive convex function, with minimum value 1 attained for $x = 1$.

There is an ordering between the optimal exercise times: if the dynamic fund protection option and the maximum option are exercised optimally, the former is always exercised after the latter. To show this, suppose that it has not been optimal to exercise the maximum option by time t, that is,

$$\breve{b} < \frac{S_1(\tau)}{S_2(\tau)} < \breve{c} \qquad \text{for } 0 \le \tau \le t. \tag{4.14}$$

It follows from this, (1.1) and $\breve{c} > 1$ that

$$F(\tau) < \breve{c} S_2(\tau) \qquad \text{for } 0 \le \tau \le t, \tag{4.15}$$

and hence

$$\frac{S_1(\tau)}{F(\tau)} > \frac{S_1(\tau)}{\breve{c} S_2(\tau)} > \frac{\breve{b}}{\breve{c}} = \breve{\varphi} \qquad \text{for } 0 \le \tau \le t \tag{4.16}$$

by (4.14) and (4.5). This shows that the dynamic fund protection option has indeed not been exercised by time t.

5. Smooth Pasting Condition and Replicating Portfolio

Because an explicit expression for the price of the option is available, it can be used to illustrate two general concepts in the theory of option pricing. The first concept is the *smooth pasting condition* or the *principle of smooth fit*. In the finance literature, it is also called the *high contact condition*, a term coined by the Nobel laureate Paul Samuelson (1965). Brekke and Øksendal (1991) have proved that, under weak conditions, a solution proposal to an optimal stopping problem satisfying the smooth pasting condition is in fact an optimal solution to the problem. Shepp and Shiryaev (1993, p. 636) have pointed out that the smooth pasting condition was discovered by the great Russian mathematician A.N. Kolmogorov in the 1950's, and it was later independently found by H. Chernoff in the United States and by D.V. Lindley in Great Britain. A brief history of the smooth pasting condition can be found in Dubins, Shepp and Shiryaev (1993, p. 238).

For each φ, $0 < \varphi < 1$, the exercise strategy T_φ gives rise to a value function

$$\begin{cases} V(s_1,s_2;\varphi) & \text{if } \varphi < \dfrac{s_1}{s_2} \le 1 \\ s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \le \varphi \end{cases}, \tag{5.1}$$

which is illustrated in Figure 3. The price function (2.15) is (5.1) with $\varphi = \breve{\varphi}$. It follows from (2.10) that the value function (5.1) is continuous along the junction $s_1 = \varphi s_2$. The smooth pasting condition states the optimal $\breve{\varphi}$ is such that the value function (5.1) is continuously differentiable. In other words, $\breve{\varphi}$ is such that the gradient of the price function (2.15) is continuous along the junction $s_1 = \breve{\varphi} s_2$:

$$\left.\frac{\partial V(s_1,\ s_2;\ \breve{\varphi})}{\partial s_1}\right|_{s_1=\breve{\varphi}s_2} = 0, \tag{5.2}$$

$$\left.\frac{\partial V(s_1,\ s_2;\ \breve{\varphi})}{\partial s_2}\right|_{s_1=\breve{\varphi}s_2} = 1. \tag{5.3}$$

To verify this, we differentiate (2.10) to obtain

$$\frac{\partial V(s_1,\ s_2;\ \varphi)}{\partial s_1} = \frac{h'(s_1/s_2)}{h(\varphi)} \tag{5.4}$$

and

$$\frac{\partial V(s_1,\ s_2;\ \varphi)}{\partial s_2} = \frac{h(s_1/s_2) - (s_1/s_2)h'(s_1/s_2)}{h(\varphi)}. \tag{5.5}$$

Hence

$$\left.\frac{\partial V(s_1,\ s_2;\ \varphi)}{\partial s_1}\right|_{s_1=\varphi s_2} = \frac{h'(\varphi)}{h(\varphi)} \tag{5.6}$$

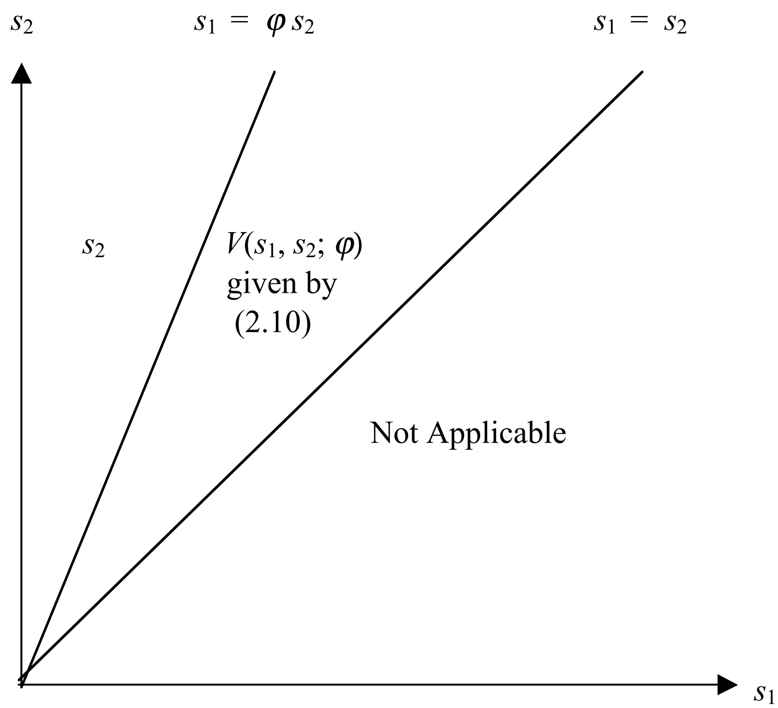
and

$$\left.\frac{\partial V(s_1,\ s_2;\ \varphi)}{\partial s_2}\right|_{s_1=\varphi s_2} = 1 - \frac{\varphi h'(\varphi)}{h(\varphi)}. \tag{5.7}$$

Now it is obvious that (5.6) has the value 0, and (5.7) the value 1, if and only if $h'(\varphi)=0$, i.e., $\varphi = \breve{\varphi}$ as given by (2.14). In other words, the four conditions (5.2), (5.3), (2.13) and (2.14) are equivalent.

Figure 3

The Value Function of Exercise Strategy T_φ



The second concept we want to illustrate is the concept of the *replicating portfolio*. For exercise strategy T_φ, the payoff is $F(T_\varphi)$ at time T_φ. At least in theory, it is possible to replicate this payoff with a self-financing portfolio that has value $V(S_1(t), S_2(t); \varphi)$ at time t, $0 \le t \le T_\varphi$, by continuously adjusting the allocation of the assets (including the stock dividends) to stock 1, stock 2, and to the riskfree investment. For ease of notation, let us discuss the composition of the replicating portfolio at time t = 0. According to formula (8.35) of Gerber and Shiu (1996b), the amount of stock j invested in the replicating portfolio must be

$$s_j \frac{\partial V(s_1, s_2; \varphi)}{\partial s_j}, \qquad j = 1, 2, \tag{5.8}$$

and the complement is invested in the riskfree asset. It follows from (5.4), (5.5) and (2.10) that

$$s_1 \frac{\partial V(s_1,\, s_2;\, \varphi)}{\partial s_1} \;+\; s_2 \frac{\partial V(s_1,\, s_2;\, \varphi)}{\partial s_2} \;=\; V(s_1, s_2; \varphi). \tag{5.9}$$

(Note that this formula is also an immediate consequence of Euler's theorem for homogeneous functions of degree 1.) Formula (5.9) shows that the assets are exclusively invested in stocks of types 1 and 2, and nothing is invested in the riskfree asset. This suggests that $V(s_1, s_1; \varphi)$ does not depend on the value of the riskfree force of interest r. At first sight, this comes as a surprise, because r shows up in equation (2.12). However, if we apply (2.6) to (2.12), then r disappears in the quadratic equation that defines θ_1 and θ_2. Hence $V(s_1, s_1; \varphi)$ is indeed independent of r.

The replicating portfolio is particularly simple when $s_1 = s_2$. Then, according to (3.14), no funds are invested in stock 2, hence the total amount $V(s_1, s_1; \varphi)$ is invested in stock 1 in this situation. (The same conclusion can be reached by setting $s_1 = s_2$ in (5.4) and (5.5), and observing that $h'(1) = h(1)$.) These findings are plausible: when $s_1 = s_2$, the guarantee, which is in terms of $\{S_1(t)\}$, is used immediately.

For $\varphi = \breve{\varphi}$, the initial capital $V(s_1, s_2; \breve{\varphi})$ is the price of the option. Here the replicating portfolio is also very simple, if $s_1 = \breve{\varphi} s_2$: then, according to formulas (5.2) and (5.3), all the funds must be invested in stocks of type 2.

The preceding formulas are readily adapted to future times t. If the option has not been exercised by time t, it suffices to replace s_1 by $S_1(t)$ and s_2 by $F(t)$.

6. Dynamic Protection with an Exponential Guaranteed Level

In this and the following section, we consider two special cases. In each, the price of one of the two "stocks" follows a geometric Brownian motion, while the other is deterministic (exponential or constant).

Let $\{S(t)\}$ be the price process of a stock,

$$S(t) = S(0)\, e^{X(t)}, \qquad t \geq 0. \tag{6.1}$$

Here $\{X(t)\}$ is a Brownian motion (Wiener process), with instantaneous variance σ^2 and drift

$$\mu = r - \frac{\sigma^2}{2} - \zeta, \tag{6.2}$$

where the positive constant, ζ, can be interpreted as the dividend-yield rate. We consider the dynamic protection that was introduced by Gerber and Pafumi (2000), with a deterministic guaranteed level $Ke^{\gamma t}$ at time t ($0 < K \leq S(0)$, $-\infty < \gamma < r$). However, there is now a withdrawal feature built in this protection: if the option owner chooses to exercise the option at time t, he obtains the amount

$$F(t) = S(t) \max\{1, \max_{0 \leq \tau \leq t} \frac{Ke^{\gamma\tau}}{S(\tau)}\}. \tag{6.3}$$

An example for the guaranteed level function is

$$Ke^{\gamma t} = S(0)\, 0.9\, (1.03)^t,$$

which may arise from nonforfeiture requirements.

What is the price of this perpetual option, and what is the optimal exercise strategy? This problem can be viewed as a special case of the general problem that has been treated in Sections 2, 3 and 5. To see this, it suffices to set

$$S_1(t) \;=\; Ke^{\gamma t}, \quad t \geq 0, \tag{6.4a}$$

$$S_2(t) \;=\; S(t), \quad t \geq 0, \tag{6.4b}$$

Hence $\sigma_1 = 0$, $\mu_1 = \gamma$, $\zeta_1 = r - \gamma$, and $\sigma_2 = \sigma$, $\mu_2 = \mu$, $\zeta_2 = \zeta$. It follows from (2.15) that the price of the perpetual option is

$$V(s, K) \;=\; \begin{cases} \dfrac{h(K/s)}{h(\breve{\varphi})} s & \text{if } \breve{\varphi} < \dfrac{K}{s} \leq 1 \\[2ex] s & \text{if } 0 < \dfrac{K}{s} \leq \breve{\varphi} \end{cases}, \tag{6.5}$$

where $s = S(0)$. According to (2.12), θ_1 and θ_2 are now the solution of the quadratic equation

$$-r \;+\; \gamma\theta \;+\; \mu(1-\theta) \;+\; \frac{\sigma^2}{2}(1-\theta)^2 \;=\; 0. \tag{6.6}$$

The optimal withdrawal takes place at the first time when the ratio $Ke^{\gamma t}/F(t)$ falls to the level $\breve{\varphi}$, with $\breve{\varphi}$ given by formula (2.14). By (6.2) we can rewrite (6.6) as

$$\frac{\sigma^2}{2}\theta^2 \;-\; (r + \frac{\sigma^2}{2} - \gamma - \zeta)\theta \;-\; \zeta \;=\; 0. \tag{6.7}$$

It follows that θ_1 and θ_2 — and with that the price of the perpetual option — depend on r and γ only through their difference.

7. Indexed Russian Options

In Section 6, we looked at the special case where $S_1(t)$ is an exponential function. In this section, we reverse the roles: let

$$S_1(t) \;=\; S(t), \quad t \geq 0, \tag{7.1a}$$

$$S_2(t) \;=\; me^{\gamma t}, \quad t \geq 0, \tag{7.1b}$$

where $\{S(t)\}$ is the price process of a stock as in Section 6, $m \geq S(0)$, and $-\infty < \gamma \leq r$. Here $\sigma_1 = \sigma$, $\mu_1 = \mu$, $\zeta_1 = \zeta$, $\sigma_2 = 0$, $\mu_2 = \gamma$, $\zeta_2 = r - \gamma$. Formula (2.1) boils down to

$$F(t) \;=\; \max\{me^{\gamma t},\ \max_{0\leq\tau\leq t} e^{\gamma(t-\tau)}S(\tau)\}. \tag{7.2}$$

An option, whose owner can get this amount at a time t of his or her choice, is called an *indexed Russian option.* Shepp and Shiryaev (1993) studied the case $\gamma = 0$; with m interpreted as the maximal stock price of the past, F(t) is the observed maximal stock price up to time t. They coined the term "Russian option" in honor of A.N. Kolmogorov who first enunciated the smooth pasting condition. Other papers with discussions on Russian options are Duffie and Harrison (1993), Gerber, Michaud and Shiu (1995), Gerber and Shiu (1994, 1996a), Guo (2002), Guo and Shepp (2001), Kallianpur (1998), Kramkov and Shiryaev (1994), and Shepp and Shiryaev (1994). Expositions on Russian options can also be found in books such as Kwok (1998), Panjer et al. (1998), and Shiryaev (1999).

From (2.15) we see that, with $s = S(0)$, the price of the indexed Russian option is

$$V(s, m) \;=\; \begin{cases} \dfrac{h(s/m)}{h(\breve{\varphi})}m & \text{if } \breve{\varphi}m < s \leq m \\ m & \text{if } s \leq \breve{\varphi}m \end{cases}. \tag{7.3}$$

According to (2.12), θ_1 and θ_2 are now the solution of the quadratic equation

$$-r \;+\; \mu\theta \;+\; \gamma(1-\theta) \;+\; \frac{\sigma^2}{2}\theta^2 \;=\; 0. \tag{7.4}$$

The optimal strategy is to exercise the indexed Russian option at the first time when the ratio S(t)/F(t) falls to the level $\breve{\varphi}$, with $\breve{\varphi}$ given by formula (2.14). By (6.2) we can rewrite (7.4) as

$$\frac{\sigma^2}{2}\theta^2 \;-\; (r-\gamma-\frac{\sigma^2}{2}-\zeta)\theta \;-\; (r-\gamma) \;=\; 0. \tag{7.5}$$

This shows that the price of the indexed Russian option depends on r and γ only through their difference.

The formula given by Shepp and Shiryaev (1993) is

$$V(s, m) \;=\; \begin{cases} m\, g(\frac{s}{\breve{\varphi} m}) & \text{if } \breve{\varphi} m < s \le m \\ m & \text{if } s \le \breve{\varphi} m \end{cases}, \tag{7.6}$$

with g(x) defined by (4.2). It is obtained from (4.11).

8. Generalized Perpetual Lookback Put Options

As mentioned in the last section, with $\gamma = 0$ and with m interpreted as the maximal stock price of the past, F(t) defined by (7.2) is the observed maximal stock price up to time t, $t \ge 0$. A *lookback put* option is an option with payoff

$$F(t) \;-\; S(t), \tag{8.1}$$

with t being the exercise time. Here we consider a generalization of the perpetual lookback put option, a security that pays its holder

$$F(t) \;-\; \kappa S_1(t), \tag{8.2}$$

if the holder chooses to exercise it at time t. The function F(t) in (8.2) is defined by (2.1) and κ is a constant between 0 and 1.

Again, we only have to consider exercise strategies of the form (2.8). For each strategy T_φ, let

$$V(s_1, s_2; \varphi, \kappa) \;=\; E(e^{-rT_\varphi}[F(T_\varphi) \;-\; \kappa S_1(T_\varphi)]), \quad \varphi s_2 \le s_1 \le s_2, \tag{8.3}$$

denote its value, where $s_j = S_j(0)$, j = 1, 2. Since

$$F(T_\varphi) - \kappa S_1(T_\varphi) = (1 - \kappa\varphi)F(T_\varphi),$$

we have

$$\begin{aligned} V(s_1, s_2; \varphi, \kappa) &= (1 - \kappa\varphi)V(s_1, s_2; \varphi) \\ &= (1 - \kappa\varphi)\frac{h(s_1/s_2)}{h(\varphi)}s_2 \end{aligned} \tag{8.4}$$

by (2.10). Let $\hat{\varphi}$ be the number that maximizes the expression

$$\frac{1-\kappa\varphi}{h(\varphi)}. \tag{8.5}$$

Then the time-0 price of the generalized perpetual put option is

$$\begin{cases} \dfrac{h(s_1/s_2)}{h(\hat{\varphi})}(1-\kappa\hat{\varphi})s_2 & \text{if } \hat{\varphi} < \dfrac{s_1}{s_2} \le 1 \\ (1-\kappa\hat{\varphi})s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \le \hat{\varphi} \end{cases}. \tag{8.6}$$

The above is a generalization of Section 10.12 of Panjer et al. (1998). To understand the smooth pasting condition in this case, one will find Exercise 10.31 and Figure 10.6 of Panjer et al. (1998) useful. Also, $\hat{\varphi}$ satisfies the equation

$$\kappa = \frac{h'(\hat{\varphi})}{(\theta_1 - 1)(\theta_2 - 1)(\hat{\varphi}^{\theta_1} - \hat{\varphi}^{\theta_2})}, \tag{8.7}$$

generalizing Exercise 10.32 of Panjer et al. (1998).

Acknowledgment

We thank the anonymous referees for their insightful comments, which contributed to an improvement of the paper. Elias Shiu gratefully acknowledges the generous support from the Principal Financial Group Foundation and Robert J. Myers, FCA, FCAS, FSA.

References

American Academy of Actuaries. 1997. *Final Report of the Equity Indexed Products Task Force.* Part of it has been reprinted as *Society of Actuaries Study Note* 8V-312-01.

Black, F., and Scholes, M. 1973. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81: 637-654.

Brekke, K.A., and Øksendal, B. 1991. "The High Contact Principle as a Sufficiency Condition for Optimal Stopping," in *Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems,* edited by D. Lund and B. Øksendal, 187-208. Amsterdam: North-Holland.

Dubins, L.E., Shepp, L.A., and Shiryaev, A.N. 1993. "Optimal Stopping Rules and Maximal Inequalities for Bessel Process," *Theory of Probability and Its Applications* 38: 226-261.

Duffie, J.D., and Harrison, M.J. 1993. "Arbitrage Pricing of Russian Options and Perpetual Lookback Options," *Annals of Applied Probability* 3: 641-651.

Gerber, H.U., Michaud, F., and Shiu, E.S.W. 1995. "Pricing Russian Options with the Compound Poisson Process," *Transactions of the XXV International Congress of Actuaries* (3): 243-264.

Gerber, H.U., and Pafumi, G. 2000. "Pricing Dynamic Investment Fund Protection," *North American Actuarial Journal* 4 (2): 28-37; Discussion 5 (1): 153-157.

Gerber, H.U., and Shiu, E.S.W. 1994. "Pricing Financial Contracts with Indexed Homogeneous Payoff," *Bulletin of the Swiss Association of Actuaries* 94: 143-166.

Gerber, H.U., and Shiu, E.S.W. 1996a. "Martingale Approach to Pricing Perpetual American Options on Two Stocks," *Mathematical Finance* 6: 303-322.

Gerber, H.U., and Shiu, E.S.W. 1996b. "Actuarial Bridges to Dynamic Hedging and Option Pricing," *Insurance: Mathematics and Economics* 18: 183-218.

Gerber, H.U., and Shiu, E.S.W. 2003. "Pricing Lookback Options and Dynamic Guarantees," *North American Actuarial Journal* 7 (1): to appear.

Goldman, M.B., Sosin, H.B., and Gatto, M.A. 1979. "Path Dependent Options: 'Buy at the Low, Sell at the High'," *Journal of Finance* 34: 1111-1127.

Guo, X. 2002. "Some Lookback Option Pricing Problems," in *Recent Development in Mathematical Finance*, edited by J. Yong, 39-48. Singapore: World Scientific.

Guo, X. and Shepp, L. 2001. "Some Optimal Stopping Problems with Nontrivial Boundaries for Pricing Exotic Options," *Journal of Applied Probability* 38: 647-658.

Imai, J., and Boyle, P.P. 2001. "Dynamic Fund Protection," *North American Actuarial Journal* 5 (3): 31-49; Discussion 49-51.

Ingersoll, J.E., Jr. 1987. *Theory of Financial Decision Making.* Totowa, N.J.: Rowan and Littlefield.

Kallianpur, G. 1998. "The Russian Options," in *Stochastic Processes and Related Topics: In Memory of Stamatis Cambanis 1943-1995*, edited by I. Karatzas, B.S. Rajput, and M.S. Taqqu, 231-252. Boston: Birkhäuser.

Kramkov, D.O., and Shiryaev, A.N. 1994. "On the Rational Pricing of the 'Russian Option' for the Symmetric Binomial Model of a (B, S)-Market," *Theory of Probability and Its Applications* 39: 153-161.

Kwok, Y.-K. 1998. *Mathematical Models of Financial Derivatives.* Singapore: Springer-Verlag.

Lee, H. 2002. *Pricing Exotic Options with Applications to Equity-Indexed Annuities.* Doctoral Thesis, The University of Iowa.

Mitchell, G.T., and Slater, J. Jr., 1996. "Equity-Indexed Annuities: New Territory on the Efficient Frontier," *Society of Actuaries Study Note* #441-99-96.

Panjer, H.H. (editor), Boyle, P.P., Cox, S.H., Dufresne, D., Gerber, H.U., Mueller, H.H., Pedersen, H.W., Pliska, S.R., Sherris, M., Shiu, E.S., and Tan, K.S. 1998. *Financial Economics: With Applications to Investments, Insurance, and Pensions.* Schaumburg, IL: The Actuarial Foundation.

Samuelson, P.A. 1965. "Rational Theory of Warrant Pricing," *Industrial Management Review,* 6 (2): 13-32.

Shepp, L., and Shiryaev, A.N. 1993. "The Russian Option: Reduced Regret," *Annals of Applied Probability* 3: 631-640.

Shepp, L., and Shiryaev, A.N. 1994. "A New Look at the 'Russian Option'," *Theory of Probability and Its Applications* 39: 103-119.

Shiryaev, A. N. 1999. *Essentials of Stochastic Finance: Facts, Models, Theory.* Singapore: World Scientific.

Streiff, T.F., and DiBiase, C.A. 1999. *Equity Indexed Annuities.* Chicago: Dearborn Financial.

Tiong, S. 2000a. *Equity Indexed Annuities in the Black-Scholes Environment.* Doctoral Thesis, The University of Iowa.

Tiong, S. 2000b. "Valuing Equity-Indexed Annuities," *North American Actuarial Journal* 4 (4): 149-163; Discussions 163-170; 5(3): 128-136.

Appendix

The purpose of this Appendix is to explain why it suffices to consider exercise strategies of the form (2.8). Mathematically, the maximization problem is an *optimal stopping* problem. The optimal continuation region is

$$C = \{(s_1, s_2) \mid V(s_1, s_2) > s_2\}.$$

Then the optimal strategy is to exercise the option at the first time t when $(S_1(t), F(t))$ is not in the region C. Note that $V(s_1, s_2)$ is a homogeneous function of degree 1. Thus

$$V(s_1, s_2) = s_2 V(s_1/s_2, 1).$$

It follows that

$$C = \{(s_1, s_2) \mid V(s_1/s_2, 1) > 1\}.$$

Because V(z, 1) is a nondecreasing function of z, we conclude that

$$C = \{(s_1, s_2) \mid \frac{s_1}{s_2} > \varphi\}$$

for some number φ. Hence the optimal strategy is indeed of the form (2.8).