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FUNDAMENTALS OF PENSION FUNDING

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I. INTRODUCTION

AMONG the tools of the pension actuary are a variety of techniques which for want of better terminology will here be called *funding methods*. By funding method is meant the budgeting scheme or the payment plan under which the benefits are to be financed. The choice of funding method in no way affects true over-all costs, which are a function of the benefits to be provided and certain other factors such as rates of mortality, interest, and employee withdrawal. The funding method is, however, the controlling factor in determining how much of the eventual cost is to be paid at any particular point of time. Funding method, as employed in this paper, should not be confused with funding *medium*, *i.e.*, the vehicle (such as Deposit Administration of Self-Administered Trust) by means of which the funding arrangements are carried out.

The funding methods commonly used in the pension field are perhaps fairly well understood by the actuaries who use them, but the actuarial literature on this subject is extremely sparse. The classic British papers on pensions devote themselves largely to the techniques of valuing complicated benefits. They put little or no emphasis on the possible variations in funding method, relying almost entirely on what is essentially individual level premium funding. Perhaps the best description of the various funding methods will be found in the "Bulletin on Section 23(p)" put out by the U.S. Treasury Department. Even this is only a very sketchy and superficial treatment, and the beginner in the pension field pretty much has to dig the ideas out for himself. This paper attempts, in some measure, to get at least the fundamentals of pension funding into actuarial literature.

Part II following introduces certain fundamental concepts, among them the "mature population" and "mature fund" concepts. By means of the "Equation of Maturity" a logical classification system for the various funding methods is devised. Assumptions and notation necessary for actuarial analysis are set forth.

Part III describes and classifies various methods which are thought to include most of those in common use among actuaries active in the pension field. The rather simple algebra is developed for each method of funding (under the rigid conditions of an initially stationary population) as a sort of theoretical base on which to build a more practical understanding.

Part IV looks into the characteristics of these methods under less idealistic conditions. Certain seeming inconsistencies which arise in practice are explained.

Part V introduces the rather treacherous subject of "adjustment for gains and losses," and describes various methods of making such adjustment.

II. FUNDAMENTAL CONCEPTS, ASSUMPTIONS, NOTATION

MATURE POPULATION CONCEPT

All actuaries are familiar with the "service" table derived from estimates of rates of death, withdrawal, and new hirings. The l_x^s column of this table represents approximately the age distribution of the employee group *after* the group reaches what we call a "stationary" condition.

Most employee groups today are immature; *i.e.*, they contain more younger members and fewer pensioners than the l_x^s column of the underlying service table would indicate. Yet most of us accept the idea that any employee group of sufficient size can be assumed (for want of better information) to approach a mature or stationary condition eventually. It seems logical, therefore, to employ this mature population concept in the classification of funding methods.

EQUATION OF MATURITY

It is apparent that a pension fund, like any other fund, grows or shrinks as income exceeds outgo, or vice versa. Contributions and interest make up income. Benefits paid are outgo. Thus if benefits (B) and contributions (C) are both assumed payable at the beginning of a year, and if the fund (F) is measured at the beginning of the year (prior to either contributions or benefits then due), the following relationship holds.

$$v\Delta F = C + dF - B \qquad (1)$$

where ΔF is the change in F over the year and d is the rate of discount.

It is the essence of the mature population concept that benefits (B) eventually become stationary. Moreover, it is characteristic of all of the funding methods described in this paper that at or after the time when the employee population becomes stationary, the contribution (C) and the fund (F) reach (or approach) a constant. ΔF therefore becomes zero and equation (1) becomes

$$C + dF = B \qquad (2)$$

where C, F, and B are all constants. Equation (2) can be thought of as an Equation of Maturity.

Note that this equation does not necessarily hold as soon as the population reaches maturity. Sufficient time must have elapsed so that C and F have reached their ultimate levels as well. In point of time the concept of a *mature fund* may therefore be one step beyond the idea of a *mature population.*

CLASSIFICATION OF FUNDING METHODS

In the Equation of Maturity, B and d are entirely independent of the funding method. Therefore, in the ultimate situation, the various funding methods differ only as to the relative sizes of F and C. At one extreme $F = 0$ and $C = B$; at the other $C = 0$ and $F = B/d$. Between these two extremes lie the funding methods commonly employed.

It is logical to classify these funding methods in ascending order of F (or descending order of C, which is the same thing). This scheme of classification will be used throughout this paper.

ASSUMPTIONS

The actuarial analysis of the ultimate situation to which a given funding method leads is materially simplified if a mature population is assumed, not after many years, but right from the inauguration of the plan. The concept of an *initially* mature population (both as to active and retired lives) is therefore employed as a starting point and as a base on which to build. The unreality of the assumption that the employee population is stationary *from the beginning* is nonetheless recognized, and observations as to the more realistic situation follow in Parts IV and V.

Moreover, since this paper concerns itself only with fundamentals, complications arising from benefit increases, death benefits, etc., are avoided by assuming the simplest benefits possible. Unless otherwise indicated, the algebraic statements and demonstrations found in this paper are based on the following assumptions.

Assume a population, stationary from the moment the pension plan is established, such that the number attaining age x in a given year is l_x. It is immaterial to this discussion whether the table is of the single or multiple decrement type, so long as l_{x+1} represents the survivors one year hence of the group l_x. It is likewise immaterial whether l_x represents numbers of lives, or whether it be thought of as dollars of salary; *i.e.*, the l_x used in this paper can be thought of as meaning $s_x l_x$ in cases where a salary scale (a function of age only) is introduced.

Further assume a single retirement age r, and that the pension benefit for each life (or each \$1 of salary) reaching retirement age is \$1.00 payable annually in advance. Assume that the plan provides no death or withdrawal benefits of any description.

NOTATION

Let a be youngest age in the service table, so that the stationary population is supported by l_a new entrants yearly.

Let ω be limiting age of service table.

Let C_t represent the tth annual contribution to the pension plan, payable annually in advance. Superscripts to the left indicate the funding method under consideration. For example ${}^{EAN}C_1$ represents the first contribution under entry age normal, and ${}^{A}C_\infty$ represents the ultimate contribution if aggregate funding is used.

Let F_t represent the fund (or reserve) built up after t years (before contribution or benefits then due). Again superscripts indicate funding method.

III. DESCRIPTION AND CLASSIFICATION OF FUNDING METHODS

CLASS I FUNDING

Under the scheme of classification previously described, Class I is logically assigned to what is commonly known as "pay as you go" funding. No contributions are made to the plan beyond those immediately necessary to meet benefit payments falling due. Contributions (${}^{P}C_t$) are exactly equal to benefits for all values of t, and ${}^{P}F_t$ is zero for all values of t.

Since the initially mature population previously described produces constant benefit payments, pay-as-you-go funding for such a group produces level contributions equal to $\sum_{r}^{\omega} l_x$.

CLASS II FUNDING

If no funding whatsoever is contemplated for active lives, but if the present value of future pension benefits is contributed for each life as it reaches retirement, we have what has come to be known as "terminal" funding. Since this method produces higher eventual contributions and lower eventual reserves than any of the other common methods except Class I, terminal funding is assigned to Class II.

When terminal funding is applied to an initially mature population, all contributions except the first are equal and can be quantitatively expressed as $l_r\ddot{a}_r$. The principle of full funding for all retired lives requires, however, that the first contribution be considerably greater to fund the benefits of those already beyond retirement age at the time the plan is inaugurated. The initial contribution is in fact ${}^{T}C_1 = \sum_{r}^{\omega} l_x \ddot{a}_x$ and exceeds the ultimate level contribution ${}^{T}C_\infty = l_r \ddot{a}_r$ by $\sum_{r+1}^{\omega} l_x \ddot{a}_x$.

This extra contribution in the first year arises because the plan was not always in existence but came into being after certain individuals had already retired. Here we find the first suggestion of "normal cost" and "accrued liability," two concepts frequently employed in the pension business.

Normal Cost is commonly understood to mean the level of contribution which a funding method would currently produce, were it not for a late start in paying for benefits. *Accrued Liability*, measured at any time, represents the difference between the then present value of future benefits and the present value of future normal costs. The portion of the accrued liability not offset by assets is called the *unfunded accrued liability*. The accrued liability, when measured at the establishment of the plan, is commonly referred to as the *initial accrued liability*.

Under Class II or terminal funding applied to an initially mature group we have seen that normal cost is represented by $l_r \ddot{a}_r$, and the initial accrued liability by $\sum_{r+1}^{\omega} l_x \ddot{a}_x$. The accrued liability does not change with the passage of time if the group is mature from the beginning. Once the accrued liability has been paid off,

$${}^{T}C_\infty = l_r \ddot{a}_r$$

$${}^{T}F_\infty = \sum_{r+1}^{\omega} l_x \ddot{a}_x$$

and the fundamental Equation of Maturity can be checked out by the identity

$$l_r \ddot{a}_r + d \sum_{r+1}^{\omega} l_x \ddot{a}_x \equiv \sum_{r}^{\omega} l_x .$$

Note that ${}^{T}F_\infty$, the ultimate reserve built up, and the accrued liability are, as we might expect, algebraically identical.

CLASS III FUNDING

The so-called "unit credit" or "single premium" method of funding is the first method here considered that funds in any respect for employees not yet retired. Since this method builds up lower reserves than methods yet to be considered, it is here classified as Class III.

Unit credit funding is based on the principle that the pension to be provided at retirement age will be divided into as many "units" as there are active membership years, with one unit assigned to each year. The normal cost as to any individual pension in any year becomes the cost to fully fund on a single premium basis the unit assigned to that year. The accrued liability at any time is the present value of all units of pension assigned to prior years. Under this method of funding particularly the accrued liability is often referred to as the "past service" liability.

To the extent practicable the units assigned to various years are equal in amount. For any individual, therefore, the "normal" cost rises each year, since the value of a deferred annuity commencing at age r is an increasing function of attained age. For the group as a whole, however, the normal cost remains level under the rigid conditions previously imposed.

Algebraically the normal cost is

$$\frac{1}{r-a}\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x .$$

The accrued liability is

$$\frac{1}{r-a}\sum_{a}^{r-1}(x-a)\, l_{x\ r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x .$$

Under this method of funding the initial accrued liability can be paid off in a variety of ways. A common method is to amortize the liability by means of an annuity certain over a period of n years, the accrued liability payment becoming $k\%$ of the initial accrued liability, where $k = 100/\ddot{a}_{\overline{n}|}$. A requirement in some plans using unit credit funding is that the accrued liability as to any individual will be funded by the time said individual retires. In any case, once the accrued liability is fully funded

$${}^{U}C_\infty = \frac{1}{r-a}\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x$$

and

$${}^{U}F_\infty = \frac{1}{r-a}\sum_{a}^{r-1}(x-a)\, l_{x\ r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x .$$

Once again the ultimate fund and the accrued liability are equal under the rigid conditions imposed.

The algebraic identity

$$\frac{1}{r-a}\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x + d\left[\frac{1}{r-a}\sum_{a}^{r-1}(x-a)l_{x\ r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x\right] \equiv \sum_{r}^{\omega} l_x$$

is, of course, an expression of the Equation of Maturity applied to Class III funding. Note that it is also an algebraic statement that if the accrued liability is not paid off, but instead is amortized in perpetuity by paying interest alone, unit credit funding for an initially mature population degenerates into pay as you go.

CLASS IV FUNDING

Four of the better known funding methods are logically classed together, because we will see that once the ultimate condition has been reached these methods produce identical contributions and build up identical reserves.

1. *Entry Age Normal Method*

This method, as its title implies, visualizes the normal cost for any given employee as the level payment (or level percentage of pay) necessary to fund the benefit over the working lifetime of such employee. The normal cost for a unit benefit for any individual entering at age a is therefore

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}.$$

The accrued liability as to any individual age x ($x < r$) is

$${}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}.$$

If we look at the group instead of the individual, we find the accrued liability is

$$\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_{a}^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}.$$

When this last expression is written in the form

$$\sum_{r}^{\omega} l_x\ddot{a}_x + \sum_{a}^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}\right)$$

it is apparent that the initial accrued liability is simply viewed as the full net single premium for benefits for retired lives, plus the sum of the individual full net level premium reserves for each unit of benefit for active lives, where such reserves are calculated as of ages when accrued liability is being computed and as if funding began (and therefore net level premium was computed) at age a. The normal cost,

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}$$

for each active life, is of course

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x$$

for the group as a whole.

As in the unit credit method, the initial accrued liability can be funded in a variety of ways, commonly by level payments for a fixed number of years. There may be a requirement that accrued liability be funded with sufficient rapidity that benefits for all retired lives are completely funded. Once the accrued liability has been completely liquidated, ${}^{\text{EAN}}C_\infty$ is the normal cost

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x$$

and

$${}^{\text{EAN}}F_\infty = \sum_r^\omega l_x\ddot{a}_x + \sum_a^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}\right).$$

Once again the ultimate fund, under the rigid conditions imposed, becomes identical with the unchanging accrued liability. Once again an algebraic identity

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x + d\left[\sum_r^\omega l_x\ddot{a}_x + \sum_a^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}\right)\right] \equiv \sum_r^\omega l_x$$

proves out the Equation of Maturity, and at the same time shows us that if accrued liability payments are reduced to interest only, the contribution equals the benefits, and accordingly no funds are built up.

2. *Individual Level Premium Funding*

A second Class IV method funds the benefits as to any individual from date of entry (or date plan is established, if later) to retirement date as a level amount (or as a level percentage of pay). As to individuals who enter the group *after* the establishment of the plan, it is apparent that this method and entry age normal are identical. For the original staff, however, the individual level premium method of funding has the effect of funding the accrued liability (as to any individual) over his future working lifetime, or in exactly the same manner as the normal cost.

For an individual age x when the plan is inaugurated, individual level premium funding requires a payment of

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}$$

for each year that such individual remains in active service. But note that since

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}$$

can be expressed as

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}+\left(\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}-\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right),$$

the contribution under level premium funding can be viewed as the normal cost (*i.e.*, the cost for new entrants) plus an accrued liability payment of

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}-\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}.$$

Extending this concept to the entire population, we see that the initial contribution to the plan is simply

$$\begin{aligned}{}^{\mathrm{ILP}}C_1 &= \sum_a^{r-1} l_x \frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}+\sum_r^{\omega} l_x\ddot{a}_x \\ &= \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x+\sum_a^{r-1}\left(\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}-\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right) l_x+\sum_r^{\omega} l_x\ddot{a}_x\end{aligned}$$

where the first term of the second form can be thought of as a normal cost, and the last two terms can be considered payment toward the accrued liability.

We find the situation t years after the inauguration of the plan to be as follows:

$$\begin{aligned}{}^{\mathrm{ILP}}C_{t+1} &= \sum_{a+t+1}^{r-1} l_x \frac{{}_{r-x+t|}\ddot{a}_{x-t}}{\ddot{a}_{x-t:\overline{r-x+t}|}}+\sum_a^{a+t} l_x \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} \\ &= \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x+\sum_{a+t+1}^{r-1} l_x\left(\frac{{}_{r-x+t|}\ddot{a}_{x-t}}{\ddot{a}_{x-t:\overline{r-x+t}|}}-\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right).\end{aligned}$$

The normal cost remains level but the accrued liability payment decreases each year as t increases, until after $r - a$ years accrued liability is all paid off.

It can thus be seen that individual level premium funding is really a special case of entry age normal, where accrued liability is funded over $r - a$ years by high initial but decreasing payments. The initial payment toward the accrued liability is especially high since, among other things, it completely funds for the initial pensioners. It can of course be demonstrated that the present value, as of date of plan, of these accrued liability payments is identical to the entry age normal initial accrued liability.

3. *Aggregate Funding*

The principle behind the aggregate method is that of equating present value of unfunded future benefits to present value of future contributions, where the contribution per active life (or per dollar of salary) per year is assumed constant. It may seem at first thought that the resulting contributions should remain level from year to year for an initially stable population, since the very principle implies spreading the value of total benefits levelly over future life years.

This supposition regarding the aggregate method is absolutely correct *provided* future new entrants are taken into account, both in valuing present value of future benefits and in calculating present value of future active life years. Demonstration I in the Appendix shows us that in the first year the so-computed aggregate contribution under our rigid conditions is exactly $\sum_{r}^{\omega} l_x$ which we recognize as the pay-as-you-go payment. Since the contribution just equals the benefits, no reserves build up and contributions continue to duplicate the level Class I contribution.

The common use of the aggregate method, however, ignores new entrants. The effect, of course, is to subtract $v/d\, l_a\, {}_{r-a|}\ddot{a}_a$ from the numerator and $v/d\, l_a\, \ddot{a}_{a:\overline{r-a}|}$ from the denominator of equation (1) of Demonstration I. Since, where A, B, C, and D are positive constants, if

$$\frac{A}{B} > \frac{C}{D},$$

then

$$\frac{A}{B} > \frac{A+C}{B+D},$$

it follows that ${}^{A}C_1$ (new entrants disregarded) is greater than the level pay-as-you-go payment if

$$\frac{\sum_{a}^{r-1} l_x\, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x \ddot{a}_x}{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|}} > \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}.$$

This latter inequality is proven by the same algebraic principle.

The ignoring of new entrants therefore produces, in the first year, a contribution in excess of benefits, and starts the accumulation of a reserve.

In any year thereafter

$$^{A}C_t = \frac{\sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x \ddot{a}_x - {}^{A}F_{t-1}}{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|}} \sum_{a}^{r-1} l_x .$$

As F_t increases, $^{A}C_t$ decreases. It can be shown that as $^{A}C_t$ decreases, the increment to $^{A}F_t$,

$$\Delta^{A}F_t \equiv \left[d^{A}F_t + {}^{A}C_{t+1} - \sum_{r}^{\omega} l_x \right] (1+i) ,$$

decreases. The fund continues to increase, but at a slower and slower rate, so long as $\Delta^{A}F_t$ is positive, *i.e.*, so long as

$$^{A}C_t > \sum_{r}^{\omega} l_x - d^{A}F_t .$$

It is shown in Demonstration II that under this process $^{A}C_t$ approaches asymptotically its limit

$$^{A}C_\infty = \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} \sum_{a}^{r-1} l_x$$

which we recognize as the normal cost under other Class IV methods. Similarly $^{A}F_t$ approaches, but never reaches, a limit identical to $^{EAN}F_\infty$ and $^{ILP}F_\infty$. The aggregate method of funding can therefore be considered another special case of entry age normal, where the accrued liability is paid off rather rapidly at the beginning, but at a slower and slower rate, such that the accrued liability is completely paid off only at infinity.

If for instance the average temporary annuity y (see Demonstration II) is $100/k$, the first payment toward the accrued liability is $k\%$ of the accrued liability. Later payments are, however, $k\%$ of the decreasing *unfunded* accrued liability. Compare the foregoing with $k\%$ funding of the accrued liability under entry age normal, where the $k\%$ applies to the full accrued liability rather than to the unfunded portion only.*

* A peculiarity of the aggregate method is that the assumption of heavier death or withdrawal rates sometimes leads to a higher initial contribution. The higher decrements reduce the average temporary annuity, thereby increasing the percentage k. The increase in k may be enough to offset the decrease in normal cost and accrued liability.

It can be demonstrated that the initial contribution under the aggregate method is generally lower than that under individual level premium funding. A temporary annuity $\ddot{a}_{x:\overline{r-x}|}$ which decreases with advancing age is a sufficient, but not necessary, condition for ${}^{A}C_1 < {}^{ILP}C_1$. If, due to heavy withdrawal assumptions at young ages, $\ddot{a}_{x:\overline{r-x}|}$ increases through a significant portion of its range, there may be rare exceptions to the general relationship.

4. *Attained Age Normal*

There may be some confusion in respect to the "attained age normal" method, arising from certain Class III characteristics in what is essentially a Class IV method.

Total benefits are divided into past service and future service benefits exactly as under unit credit funding, and as under Class III funding there is complete freedom as to the manner in which the past service liability shall be paid off. For future service benefits, however, the aggregate method is adopted.

The first year contribution toward future service becomes

$$\frac{\frac{1}{r-a}\sum_{a}^{r-1}(r-x)\,l_{x\ r-x|}\ddot{a}_x}{\sum_{a}^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}\sum_{a}^{r-1} l_x\,.$$

Since this amount is somewhat higher than the Class III normal cost

$$\frac{1}{r-a}\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x$$

(which is level under our initially mature population assumptions), it is apparent that future service contributions under attained age normal are of a decreasing nature.

Future service costs after the first year are commonly calculated in the form

$$\frac{\sum_{a}^{r-1} l_{x\ r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x - \begin{array}{c}\text{Unfunded past service}\\ \text{liability}\end{array} - {}^{AAN}F_{t-1}}{\sum_{a}^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}\sum_{a}^{r-1} l_x\,.$$

We perhaps get a better idea of the essential characteristics of attained age normal, however, if we express the *t*th future service contribution in the identical form

$$\frac{\frac{1}{r-a}\sum_{a}^{r-1}(r-x)\,l_{x\;r-x|}\ddot{a}_x - f_{t-1}}{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x|}}}\sum_{a}^{r-1} l_x\,,$$

where f_t is that portion of ${}^{AAN}F_t$ built up by the accumulated excess (with interest) of the attained age normal future service contribution over the unit credit one.

As f_t grows the attained age normal future contribution decreases. It can be shown that f_t approaches as a limit the amount by which Class IV accrued liability exceeds the Class III accrued liability, and that if the initial past service liability is completely liquidated ${}^{AAN}C_t$ and ${}^{AAN}F_t$ have as limits ${}^{EAN}C_\infty$ and ${}^{EAN}F_\infty$ respectively.

Attained age normal is therefore a true Class IV method. Its accrued liability is actually as great as under the other Class IV methods, but attained age normal looks at the accrued liability in two parts. The method imposes no restrictions as to how the "past service" part, equal in magnitude to the Class III accrued liability, shall be funded. The second portion is liquidated by the decreasing accrued liability payments, which are the excess of the future service contribution over the ultimate future service contribution. Similarity with the aggregate method is of course noted, but whereas under the aggregate method all accrued liability is liquidated by rigid decreasing payments, under attained age normal only a portion of the accrued liability is so funded and the funding as to the remaining accrued liability is unspecified.

CLASS V FUNDING

Beyond the various variations of Class IV funding previously discussed, there is nothing of a practical nature, but funding methods which produce higher eventual reserves and lower eventual contributions than any of the methods so far discussed are, of course, theoretically possible. Perhaps the simplest of these is *initial* funding, where an employee's benefits are fully funded as soon as he is hired.

Here normal cost is $l_{a\;r-a|}\ddot{a}_a$, accrued liability and ${}^{I}F_\infty$ are both

$$\sum_{a+1}^{r-1} l_{x\;r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x\,,$$

and Equation of Maturity is expressed by

$$l_{a\;r-a|}\ddot{a}_a + d\left[\sum_{a+1}^{r-1} l_{x\;r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x\right] \equiv \sum_{r}^{\omega} l_x\,.$$

CLASS VI FUNDING

Even less practical than Class V, but included here only to illustrate the extreme in heavy funding, is what might be called *complete* funding. If by one means or another an accrued liability of $1/d\sum_r^\omega l_x$ is fully paid off, interest on the funds built up will exactly meet the benefit payments $\sum_r^\omega l_x$.

ILLUSTRATION OF INITIALLY MATURE SITUATION

It may be enlightening to illustrate the foregoing discussion of the operation of the various funding methods under the assumption of an initially mature population by means of a numerical example. Table I shows the l_x column of a hypothetical stationary population, made up of exactly 1,000 active and 150 retired lives, maintained by 100 new entrants each year all age 30. Each year 1% of the active lives retire and 9% die

TABLE I

x	l_x	x	l_x	x	l_x
30	100	55	14	80	5
31	84	56	13	81	4
32	71	57	13	82	4
33	60	58	12	83	4
34	51	59	12	84	3
35	44	60	11	85	3
36	40	61	11	86	2
37	36	62	11	87	2
38	34	63	10	88	1
39	32	64	10	89	1
40	30	65	10	90	1
41	28	66	10	91	1
42	27	67	9	92	1
43	26	68	9	93	1
44	25	69	9	94	1
45	24	70	8	95	1
46	23	71	8		
47	22	72	8		
48	21	73	7		
49	20	74	7		
50	19	75	7		
51	18	76	6		
52	17	77	6		
53	16	78	6		
54	15	79	5		

or withdraw. The combined rate of death and withdrawal is 16% at age 30, and approximately equivalent to the Standard Annuity Table q_x at retired ages.

Table II illustrates the yearly contribution and the build up of funds under each of the several funding methods, assuming $2\frac{1}{2}\%$ interest, a benefit of $420 annually, and the stationary population of Table I. Twenty years has been chosen as the period of amortization of the initial accrued liability for those funding methods permitting such treatment.

IV. MODIFICATIONS FOR INITIALLY IMMATURE FUND

Let us at this point abandon one of the rigid assumptions previously imposed and look into the common situation where the group is *not* initially mature, but is to a greater or less extent immature. For the present we will continue to assume that all actuarial assumptions are realized, leaving the question of actuarial gains and losses to Part V. As we abandon the assumption that the group is initially mature (though we retain the concept that the population will eventually approach a stationary condition), we replace the l_x of the stationary population by the l'_x of the immature population. As we should expect, the identities expressing the Equation of Maturity do not hold after this substitution until such time as the population has become mature and the l'_x's approach the l_x's. Moreover, we find that the conclusions previously reached for the initially mature fund must be modified in several other respects.

NORMAL COSTS NO LONGER LEVEL

If the initial group is immature it follows that Class I funding will produce contributions which are initially very low, but which increase rather rapidly, eventually leveling off when maturity of the group is attained.

Class II funding requires contributions which tend to fluctuate rather widely as number of retirements vary from year to year. Moreover, beneath this erraticism of contributions is an underlying tendency for costs to increase, since as the group matures the number retiring each year tends to grow, even if the size of the staff as a whole remains stationary.

The normal cost for Class III or unit credit funding (for a given staff and benefits) remains constant *if* actuarial assumptions are realized, and *if* the average age of the active staff does not change. The average age here meant is not the simple arithmetic mean, but the age corresponding to the weighted average single premium deferred annuity, where the single premium at each age is weighted by units being funded at such age. If the group is initially immature, however, it is axiomatic that this average age will slowly increase and normal costs will slowly rise before eventually

TABLE II

	Class I	Class II	Class III	Class IV				Class V	Class VI
	Pay As You Go	Terminal Funding	Unit Credit 20 Yr.	Entry Age Normal 20 Yr.	Individual Level Prem.	Aggregate	Attained Age Normal 20 Yr.	Initial Funding 20 Yr.	Complete Funding 20 Yr.
	Initial Accrued Liability								
	None	$502,104	$1,206,924	$1,471,873	$1,471,873	$1,471,873	$1,471,873	$1,706,173	$2,583,000
	Normal Cost								
	$63,000	$ 50,753	$ 33,563	$ 27,101	$ 27,101	$ 27,101	$ 27,101	$ 21,386	None
Beg. of Year	**Contributions**								
1	$63,000	$552,857	$ 109,095	$ 119,214	$ 772,667	$ 183,109	$ 130,716	$ 128,163	$ 161,651
2	"	50,753	"	"	164,606	170,060	128,367	"	"
3	"	"	"	"	135,627	158,103	126,215	"	"
4	"	"	"	"	116,377	147,145	124,242	"	"
5	"	"	"	"	102,007	137,104	122,437	"	"
10	"	"	"	"	61,631	98,178	115,427	"	"
15	"	"	"	"	43,398	73,026	110,900	"	"
20	"	"	"	"	34,348	56,775	107,975	"	"
21	"	"	33,563	27,101	33,177	54,293	31,995	21,386	None
25	"	"	"	"	29,855	46,274	30,552	"	"
30	"	"	"	"	27,760	39,489	29,331	"	"
35	"	"	"	"	27,101	35,105	28,542	"	"
40	"	"	"	"	"	32,273	28,032	"	"
50	"	"	"	"	"	29,260	27,489	"	"
Limit	63,000	50,753	33,563	27,101	27,101	27,101	27,101	21,386	None
End of Year	**Funds**								
1	None	$502,104	$ 47,248	$ 57,620	$ 727,409	$ 123,112	$ 69,409	$ 66,792	$ 101,117
2	"	"	95,677	116,680	849,740	235,926	138,145	135,253	204,762
3	"	"	145,316	177,217	945,426	339,304	206,394	205,427	310,998
4	"	"	196,197	239,267	1,023,774	434,036	274,327	277,354	419,890
5	"	"	248,350	302,868	1,089,350	520,844	342,108	351,080	531,504
10	"	"	529,335	645,536	1,298,517	857,380	683,671	748,294	1,132,853
15	"	"	849,244	1,033,233	1,398,142	1,074,828	1,040,721	1,197,706	1,813,223
20	"	"	1,206,924	1,471,873	1,444,780	1,215,329	1,425,696	1,706,173	2,583,000
21	"	"	"	"	1,450,331	1,236,788	1,429,559	"	"
25	"	"	"	"	1,464,588	1,306,112	1,442,038	"	"
30	"	"	"	"	1,471,046	1,364,770	1,452,596	"	"
35	"	"	"	"	1,471,873	1,402,671	1,459,419	"	"
40	"	"	"	"	"	1,427,160	1,463,827	"	"
50	"	"	"	"	"	1,453,208	1,468,516	"	"
Limit	None	502,104	1,206,924	1,471,873	1,471,873	1,471,873	1,471,873	1,706,173	2,583,000

leveling off. This rise may be pronounced if the group is unusually young at the establishment of the plan.

The possibility of increasing normal costs, even if all actuarial assumptions are realized, is not eliminated under Class IV funding. The expected increase in average age of the active life group will not, in itself, produce increasing normal costs. Level normal costs do, however, depend upon the average age of new entrants into the plan. If this average entry age remains constant and other actuarial assumptions are realized, normal costs will remain constant (assuming staff and benefit levels do not change). Again this average entry age is not a simple arithmetic mean, but the age corresponding to the weighted average level premium where the level premium for each entry age is weighted by benefits for those entering at such age.

ACCRUED LIABILITY NO LONGER CONSTANT

We have previously seen that under the assumption of an initially mature population the accrued liability produced by any of the funding methods discussed does not change with the passage of time. It takes no mathematical demonstration to convince us that, if the population is initially immature, the accrued liability will rise as the population grows older.

As a corollary we find that the funds will grow beyond the initial accrued liability up to the level of the ultimate accrued liability (assuming initial accrued liability is completely funded). The excess of the ultimate over the initial accrued liability is built up by the early year excess of normal costs plus interest on the initial accrued liability over benefit payments.

NORMAL COST PLUS INTEREST ON ACCRUED LIABILITY NO LONGER IDENTICAL TO PAY-AS-YOU-GO

We found earlier that for an initially mature group a contribution equal to normal cost plus interest on the initial accrued liability was exactly equal to benefit payments; accordingly no funds were built up and Class I funding resulted. This was true regardless of whether normal costs and accrued liability were those of Class II, III, IV, V, or VI.

If the original group is immature the payment of normal cost plus interest only on the initial accrued liability differs from pay-as-you-go in two respects: (1) the contributions are more nearly level instead of sharply increasing, and (2) a fund is built up, at any time t being equal in amount to the excess of the accrued liability at time t over the initial accrued liability. Despite these differences the author prefers to consider these methods contemplating no amortization of the initial accrued liability as Class I methods.

GENERAL RELATIONSHIP BETWEEN NORMAL COST AND ACCRUED LIABILITY MAY NOT HOLD

In Part III we found that, if the population was initially mature, the funding method producing the higher normal cost produces the lower accrued liability, and vice versa. In the initially immature case this general relationship may not hold immediately. For example, the Class I pay-as-you-go payment may be initially lower than first year terminal funding normal cost, even though Class II funding produces an accrued liability and Class I has none. The explanation is, of course, that the paradoxical situation is temporary.

Not quite so obvious is the situation we find if the unit credit method of funding, applied to a given group, produces a lower initial accrued liability than entry age normal (a result one would expect) and yet turns up a lower initial normal cost as well. Such a result is due to the immaturity of the group, which we have seen invariably leads to normal costs which rise under unit credit funding. The lower normal cost under unit credit is a temporary feature only, and the present value of all normal costs is higher under Class III funding, even though the normal cost in early years may be lower.

ILLUSTRATION OF INITIALLY IMMATURE SITUATION

By changing the preceding illustration somewhat we can make a good numerical representation of the course of contributions and the build up of funds in an initially immature situation. Table III represents an immature population of 1,000 active lives, with no retired lives initially. If this

TABLE III

x	l'_x	x	l'_x	x	l'_x
30	124	45	21	60	6
31	105	46	20	61	5
32	88	47	19	62	4
33	74	48	18	63	3
34	62	49	17	64	2
35	52	50	16	65 and up	0
36	44	51	15		
37	38	52	14		
38	33	53	13		
39	29	54	12		
40	27	55	11		
41	25	56	10		
42	24	57	9		
43	23	58	8		
44	22	59	7		

group experiences death and withdrawal exactly in accordance with the service table illustrated in Table I, and if sufficient new entrants come in at age 30 each year to keep active staff up to 1,000, the initially immature group will slowly approach the stationary population shown in Table I.

Table IV illustrates the effect of the several funding methods under these conditions. Two "Class I" methods besides pay-as-you-go are here illustrated, both of which in this particular example build up greater reserves than Class II. Because Class V and VI are practically unimportant these methods have been excluded from the illustration.

V. ADJUSTMENT FOR ACTUARIAL GAINS AND LOSSES

In Part III the operation of the various funding methods was described under conditions of (1) an initially mature population, and (2) experience strictly in accordance with the actuarial assumptions. The first of these ideal conditions was abandoned in the discussion of the initially immature population in Part IV. To complete the transition from the ideal to the realistic, we now abandon the second of the rigorous "ideal" conditions and look into the practical situation where the actuarial assumptions are never exactly realized.

ORIGIN OF ACTUARIAL GAINS AND LOSSES

The calculation of the contribution for any given year under any funding method is always based on a set of assumptions or estimates. As the actual experience unfolds it is found that each of these estimates is in error to a greater or less extent, and that these errors give rise to what have come to be known as actuarial gains or losses. The reader can undoubtedly enumerate many of the sources of gains or losses, the net effect of all of which is the total actuarial gain or loss for any particular period. Some of these sources may be overlooked in thinking through pension valuation problems, however. As an aid to clear thinking, a partial list of sources of actuarial gains is therefore here included. In each case the converse represents a source of actuarial loss. Under certain plan provisions or particular funding media any of the following may have no effect (or even the opposite effect). In general, however, an element of actuarial gain arises if:

1. Rates of employee mortality are higher than assumed.
2. Rates of employee withdrawal (especially nonvested withdrawal) are higher than assumed.
3. Rate of interest earned is higher than assumed.
4. Benefits which cannot be determined exactly are overestimated. This could arise, for example, by assuming too steep a salary scale for benefits based on

TABLE IV

	Class I			Class II	Class III	Class IV			
	Pay As You Go	Unit Credit Int. Only	Entry Age Normal Int. Only	Terminal Funding	Unit Credit 20 Yr.	Entry Age Normal 20 Yr.	Individual Level Prem.	Aggregate	Attained Age Normal 20 Yr.
	Initial Accrued Liability								
	None	$ 431,924	$ 661,315	None	$ 431,924	$ 661,315	$ 661,315	$ 661,315	$ 661,315
	Ultimate Accrued Liability								
	None	$1,206,924	$1,471,873	$ 502,104	$1,206,924	$1,471,873	$1,471,873	$1,471,873	$1,471,873
	Initial Normal Cost								
	None	$ 26,371	$ 27,101	None	$ 26,371	$ 27,101	$ 27,101	$ 27,101	$ 27,101
	Ultimate Normal Cost								
	$63,000	$ 33,563	$ 27,101	$ 50,753	$ 33,563	$ 27,101	$ 27,101	$ 27,101	$ 27,101
Beg. of Year	**Contributions**								
1	....	$ 36,906	$ 43,230	....	$ 53,402	$ 68,488	$ 126,488	$ 95,591	$ 77,889
2	$ 840	37,902	"	$ 10,151	54,398	"	112,387	89,867	75,903
3	2,100	38,771	"	15,226	55,267	"	101,472	84,685	74,106
4	3,543	39,562	"	18,456	56,058	"	92,778	79,995	72,479
5	5,326	40,234	"	23,070	56,731	"	85,061	75,728	70,999
10	17,270	42,324	"	39,041	58,821	"	57,235	59,233	65,277
15	30,006	43,437	"	42,295	59,933	"	42,032	43,331	61,484
20	40,582	44,367	"	44,134	60,863	"	34,060	37,730	58,947
21	42,356	44,543	"	44,409	34,008	27,101	33,002	36,858	31,521
25	48,158	45,229	"	45,316	34,694	"	29,971	34,015	30,233
30	54,443	45,468	"	55,829	34,934	"	27,900	31,568	29,125
35	62,999	44,014	"	63,442	33,480	"	27,101	29,949	28,391
40	65,559	43,612	"	50,369	33,077	"	"	28,930	27,929
50	64,249	43,923	"	49,227	33,388	"	"	27,867	27,448
Limit	63,000	44,098	43,230	50,753	33,563	27,101	27,101	27,101	27,101
End of Year	**Funds**								
1	None	$ 37,829	$ 44,311	None	$ 54,737	$ 70,200	$ 129,651	$ 97,981	$ 79,836
2	"	76,762	88,869	$ 9,543	111,002	141,293	247,228	191,683	158,772
3	"	116,269	133,249	23,236	168,273	212,873	355,265	281,125	236,547
4	"	156,094	177,260	39,103	226,307	284,763	455,613	366,515	313,120
5	"	195,777	220,543	58,267	284,655	356,622	548,731	447,840	388,263
10	"	380,564	417,303	178,161	569,997	707,342	918,561	794,067	737,424
15	"	528,079	570,864	288,992	831,283	1,035,096	1,160,817	1,090,139	1,039,107
20	"	638,136	682,428	364,714	1,070,060	1,343,743	1,315,868	1,251,691	1,302,039
21	"	656,331	700,385	375,937	1,088,255	1,361,700	1,339,177	1,277,347	1,323,484
25	"	719,556	760,785	410,149	1,151,480	1,422,100	1,413,822	1,362,729	1,395,201
30	"	781,305	815,616	454,999	1,213,230	1,476,931	1,475,905	1,438,799	1,459,655
35	"	803,511	834,594	528,172	1,235,435	1,495,910	1,495,910	1,471,378	1,484,796
40	"	793,576	826,569	536,121	1,225,500	1,487,884	1,487,884	1,472,026	1,480,699
50	"	770,124	806,286	501,002	1,202,048	1,467,601	1,467,601	1,460,955	1,464,588
Limit	None	775,000	810,558	502,104	1,206,924	1,471,873	1,471,873	1,471,873	1,471,873

salary, or underestimating Social Security benefits under a "$100 less Social Security" plan.
5. Retirements occur at a higher age than assumed.
6. The value of the pension fund assets appreciates.
7. Errors of various types, overstating the liabilities, are corrected.
8. Provision for expenses of administration is overly adequate.

DETERMINATION OF AMOUNT OF ACTUARIAL GAIN OR LOSS

It is seldom practical to determine the actuarial gain or loss for a given period by summing the various components. It is ordinarily not too difficult, however, to obtain the total gain or loss directly. The most convenient procedure for doing so depends somewhat on the method of funding.

An approach to the computation of gain or loss which has wide application is the comparison between (1) funds actually on hand at the end of the period, and (2) funds "expected" in accordance with the assumptions made. The latter is invariably the accrued liability at the end of the period, less the "expected" unamortized initial accrued liability, *i.e.*, unamortized initial liability at the beginning of the period, with interest to end of period, less payments within the period toward the initial accrued liability, with interest to end of period.

Under either unit credit (Class III) or entry age normal (Class IV) the desired result is obtained without difficulty by this general procedure.

Under Class II funding, where initial accrued liability is ordinarily paid off immediately, the gain or loss is measured by the excess of actual funds over present value of all benefits for retired lives.

Gains or losses under individual level premium funding *can* be obtained in exactly the generalized manner previously set forth. But since under this method payments toward the initial accrued liability are not immediately evident, and since "expected" funds are equal to the sum of the individual level premium reserves, it is more convenient to compare the actual funds with the level premium reserve. The calculation of the reserve item is somewhat arduous, and accordingly the adjustment for gains and losses is difficult under this method of funding, except under insured plans where no losses occur and dividends declared represent the gains.

The dollar amount of actuarial gains and losses involves some difficulty under the aggregate and attained age normal methods as well. The generalized procedure previously suggested is theoretically accurate, but is practically difficult because the calculations of the payments toward the initial accured liability consist of a year-by-year comparison of contributions made with the Class IV normal cost. The redeeming feature of both these methods, from a gain and loss viewpoint, is that adjustment for gain and loss can be easily made without previous determination of the absolute amount of such gain or loss.

TECHNIQUE OF GAIN OR LOSS ADJUSTMENT

The funding methods discussed in this paper employ one or either of two techniques in determining how much contributions will be adjusted to recognize previous actuarial gains or losses.

"Immediate" Method

The "immediate" method makes up any loss or offsets any gain, as soon as such gain or loss is evident, by addition to or deduction from the next contribution.

Pay-as-you-go funding invariably and automatically adjusts immediately for gain or loss. This is evident if we think of the contribution (actual benefit payments) as the sum of expected benefit payments plus adjustment for gain or loss. Class II or terminal funding also employs the immediate adjustment technique.

Fully insured plans (those employing conventional group annuities, group permanent, or individual insurance policies) in most cases apply any dividend against the next contribution. To the extent that the dividend immediately reflects actual experience, actuarial gain is recognized at once. The insurance company guarantees eliminate the possibility of actuarial loss.

Immediate adjustment is also theoretically possible under every other funding method considered, and is commonly used in several of them. Due to the difficulties of calculation it is seldom employed with the aggregate or attained age normal funding methods.

"Spread" Method

The "spread" method makes the gain or loss adjustment in easy stages, by spreading the adjustment into the future, such that the present value of future adjustments is equal to the dollar amount of the current gain or loss. Ordinarily the method of spreading follows the normal cost of the particular funding method employed, so that the adjustment for gain (or loss) becomes a deduction from (or addition to) future normal costs.

The spread adjustment method is the only convenient scheme under aggregate or attained age normal funding, has often been used with entry age normal, and is less commonly employed with unit credit funding. When gains and losses are spread under either of these last two methods the term "frozen initial liability" is frequently employed, to distinguish from the immediate adjustment forms of these same two methods.*

MECHANICS OF SPREAD ADJUSTMENT UNDER CLASS IV METHODS

The entry age normal—frozen initial liability—method relies on the equation

$$\text{Normal cost} = \frac{\text{Present Value All Benefits} - \text{Unamortized Initial Accrued Liability} - \text{Fund}}{\text{Weighted Average Temporary Annuity}}$$

which is an identity so long as "Fund" represents the funds which would have been built up if all actuarial assumptions were realized. By replacing expected fund by actual fund in the right side of this equation, we automatically compute normal cost adjusted for actuarial gain or loss. The adjustment becomes the dollar amount of such gain or loss divided by the weighted average temporary annuity. The same process, repeated in future valuations, respreads the unrecognized portion of the gain or loss over future life years of the then active group (at the same time spreading new gains or losses in the same fashion).

The amortization of the gain or loss by this method is identical to the amortization of the accrued liability under the aggregate method of funding. As might be expected the adjustment for the gain or loss of any period is never completed, but approaches zero as that period falls farther and farther into the past.

Under aggregate and attained age normal, the substitution of actual for expected fund has exactly the same effect as under entry age normal. Gains or losses are again spread in the decreasing asymptotic manner described above, and the adjustment is automatic, entailing no more work than if gain or loss did not exist.

Illustration. We can illustrate adjustment for gains and losses as far as it is discussed in this paper by going back to the illustration in Table IV. If at the end of the fourth year, for example, the fund suffers a loss of $10,000 through depreciation of securities, the immediate adjustment method calls for an extra contribution of $10,000 at the beginning of the 5th year. Table V following shows the amortization of this loss in future years when spread adjusting is employed.

* If it seems to the reader that "frozen initial liability" is something of a misnomer for a method under which funding of the accrued liability is contemplated, he may prefer the terminology suggested by Mr. Rae in *TSA* I, 274. "Frozen initial liability" might be better applied to the Class I methods described on page 33.

VI. CONCLUSION

In the Bureau of Internal Revenue's "Bulletin on Section 23(p)" and Reg. 111, Sec. 29.23(p) may be found the Treasury rules as to the maximum contribution for which full tax deduction can be claimed. A brief statement of the maximum contribution under the various funding methods here discussed will conclude this discussion.

Class I and Class II funding are not specifically recognized. Presumably contributions would be fully deductible if within maximum contributions established for one of the recognized funding methods.

Unit credit and entry age normal funding are lumped together as "Clause (iii)" methods. Provided the actuarial assumptions are satisfactory, contributions under both are fully deductible up to normal cost plus 10% of the initial accrued liability.

TABLE V

Year	Extra Contribution Added to Normal Cost
5	$1,030
6	948
7	872
8	803
9	739
10	681
20	294
30	124
40	51
50	21

Aggregate and individual level premium funding make up the "Clause (ii)" methods. Contributions under the aggregate method are fully deductible if average temporary annuity does not drop below 5. As to individual level premium funding the Treasury sets forth various tests, one of which must be satisfied in order to obtain full deduction. The effect of these tests is to limit maximum deduction to that under the "normal cost plus 10%" rule established for entry age normal.

Attained age normal is described as a "special" method with both Clause (ii) and Clause (iii) characteristics. Contributions are fully deductible if the past service liability payment is no greater than 10%.

The Treasury specifically requires periodic adjustment for actuarial gains. The immediate adjustment technique results in lowest possible contributions and is of course entirely acceptable as to gains. Although the Treasury position on spread adjustment is not too clear, the Bulletin description of aggregate, attained age normal, and the frozen initial liability form of entry age normal seems to imply approval of spread adjustment.

Actuarial losses can evidently be made up no faster than 10% per year, since for tax purposes they are considered additions to the initial accrued liability. Spread funding as previously described will ordinarily keep extra contributions for actuarial loss within the 10% maximum, and appears to be an acceptable technique for losses as well as gains.

APPENDIX

Demonstration I

$$^{A}C_1 = \frac{\text{Present value benefits}}{\text{Present value future active life years}} \times \text{Current active lives}$$

(where both present values include future new entrants)

$$= \frac{\sum_a^{r-1} l_{x\ r-x|}\ddot{a}_x + \sum_r^{\omega} l_x\ddot{a}_x + l_{a\ r-a|}\ddot{a}_a(v+v^2+\ldots.)}{\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|} + l_a\ddot{a}_{a:\overline{r-a}|}(v+v^2+\ldots.)} \sum_a^{r-1} l_x . \qquad (1)$$

But $$\sum_a^{r-1} l_{x\ r-x|}\ddot{a}_x \equiv v/d\ l_r\ddot{a}_r - v/d\ l_{a\ r-a|}\ddot{a}_a \qquad (2)$$

$$\sum_r^{\omega} l_x\ddot{a}_x \equiv 1/d \sum_r^{\omega} l_x - v/d\ l_r\ddot{a}_r \qquad (3)$$

$$\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|} \equiv 1/d \sum_a^{r-1} l_x - v/d\ l_a\ddot{a}_{a:\overline{r-a}|} \qquad (4)$$

$$v + v^2 + \ldots \equiv v/d\ . \qquad (5)$$

Substituting (2), (3), (4), and (5) in (1) $^{A}C_1 = \sum_r^{\omega} l_x .$

Demonstration II

Let

$$b = \sum_a^{r-1} l_{x\ r-x|}\ddot{a}_x + \sum_r^{\omega} l_x\ddot{a}_x \equiv 1/d \sum_r^{\omega} l_x - v/d\ l_{a\ r-a|}\ddot{a}_a$$

$$y = \frac{\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}{\sum_a^{r-1} l_x} \equiv \frac{1/d \sum_a^{r-1} l_x - v/d\ l_a\ddot{a}_{a:\overline{r-a}|}}{\sum_a^{r-1} l_x}$$

$$p = \sum_{r}^{\omega} l_x .$$

Now

$$^{A}C_t = \frac{b - {}^{A}F_{t-1}}{y} \tag{1}$$

and

$$\begin{aligned} {}^{A}F_t &= ({}^{A}F_{t-1} + {}^{A}C_t - p)(1+i) \\ &= \left[{}^{A}F_{t-1}\left(1 - \frac{1}{y}\right) + (b/y - p)\right](1+i) \end{aligned} \tag{2}$$

$$^{A}F_0 = 0$$

$$^{A}F_1 = (b/y - p)(1+i)$$

$$^{A}F_2 = (b/y - p)(1+i)\left[1 + (1+i)\left(1 - \frac{1}{y}\right)\right]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} {}^{A}F_t &= (b/y - p)(1+i)[1 + s + s^2 + \ldots + s^{t-1}] \\ &= (b/y - p)(1+i)\frac{1 - s^t}{1 - s} \end{aligned} \tag{3}$$

where

$$s = (1+i)(1 - 1/y) .$$

Now

$$0 \leq s \leq 1$$

as

$$1 \leq y \leq 1/d ;$$

but

$$v = \frac{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|}}{\sum_{a}^{r-1} l_x} > 1 , \quad \text{since} \quad \ddot{a}_{x:\overline{r-x}|} > 1$$

and

$$y = \frac{1/d \sum_{a}^{r-1} l_x - v/d \; l_a \ddot{a}_{a:\overline{r-a}|}}{\sum_{a}^{r-1} l_x} < \frac{1}{d}, \quad \text{since} \quad v/d \; l_a \ddot{a}_{a:\overline{r-a}|} > 1 .$$

$$\therefore \underset{t \to \infty}{\mathrm{L}} s^t = 0$$

and

$$^{A}F_{\infty} = (b/y - p)\left(\frac{1+i}{1-s}\right) = \frac{b - py}{1 - dy} . \tag{4}$$

From (1)

$$ {}^{A}C_{\infty} = \frac{b - {}^{A}F_{\infty}}{y} = \frac{p - bd}{1 - dy}. \tag{5} $$

Substituting the right hand forms of the definitions of y and b in (5) we obtain

$$ {}^{A}C_{\infty} = \frac{{}_{r-a|}\ddot{a}_{a}}{\ddot{a}_{a:\overline{r-a}|}} \sum_{a}^{r-1} l_{x}. \tag{6} $$

From (4)

$$ {}^{A}F_{\infty} = \frac{b - py}{1 - dy} = \frac{b(1 - dy) - y(p - bd)}{1 - dy} $$

$$ = b - y\,\frac{p - bd}{1 - dy} = b - {}^{A}C_{\infty} \cdot y $$

$$ = \sum_{a}^{r-1} l_{x\ r-x|}\,\ddot{a}_{x} + \sum_{r}^{\omega} l_{x}\ddot{a}_{x} - \frac{{}_{r-a|}\ddot{a}_{a}}{\ddot{a}_{a:\overline{r-a}|}} \sum_{a}^{r-1} l_{x}\ddot{a}_{x:\overline{r-x}|}. \tag{7} $$