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ACTUARIAL NOTE: VALUATION OF THE SHARES IN A SHARE-AND-SHARE-ALIKE LAST SURVIVOR ANNUITY

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IN VALUING an individual's share in various complicated types of trusts, one particular problem arises which is not dealt with in the general case in the standard actuarial textbooks. Frequently in such trusts, there is an arrangement whereby the income from the corpus is divided equally each year among the survivors of a given initial group. Accordingly, as time goes by, the income per surviving beneficiary increases, until finally the last survivor receives the entire income. Upon the death of the last survivor, the corpus would then pass on to a remainderman, such as a college, church, or other such institution. The value of the income payments to all the individuals is that of a joint and last survivor annuity, while the value to the remainderman is simply the excess of the corpus over the value of the income payments to the individuals.

In the aggregate, the value of a joint and last survivor annuity can be determined from well-known formulas, but a more complex problem arises in determining the value of the share of a particular individual. This aspect of the problem will be the subject of this note.

The value of an annuity to continue so long as at least one life out of m lives survives is written basically as follows:

$$a_{\overline{xyz\ldots(m)}} = \sum_{t=1}^{\infty} v^t \{1 - [(1-{}_tp_x)(1-{}_tp_y)(1-{}_tp_z)\ldots]\}. \quad (1)$$

From Spurgeon's *Life Contingencies* (1929 edition, p. 264) this may be expressed, using the operator Z, as follows:

$$a_{\overline{xyz\ldots(m)}} = Z - Z^2 + Z^3 - \ldots = \frac{Z}{1+Z}, \quad (2)$$

where Z^r denotes all the combinations of annuities for r joint lives that can be made out of the m lives (Z^r is zero for $r > m$).

Similarly, there is the following expression in terms of the operator Z

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for the value of an annuity payable only while exactly n lives out of m lives survive:

$$a_{\overline{xyz\ldots(m)}}^{[n]}=\frac{Z^n}{(1+Z)^{n+1}}. \qquad (3)$$

The value V_w of w's share in a share-and-share-alike last survivor annuity involving m other lives may be expressed as follows:

$$V_w=\sum_{r=0}^{m}\frac{1}{r+1}\cdot a_{w:\overline{xyz\ldots(m)}}^{[r]}. \qquad (4)$$

Using another operator Z, which is somewhat different from the one discussed previously, equation (4) may by analogy with (3) be expanded in powers of Z. Here Z^r signifies the sum of the values of annuities payable during the joint lifetime of the life w and each of the combinations of r lives that can be made out of the m lives. In equation (3), Z^0 has no meaning and is thus inapplicable, while here Z^0 by definition represents a_w. The expansion of the a's in equation (4) takes exactly the same form as (3) since the operator Z now used is the same as that in (3) except that in all cases it involves the independent life w, as does the annuity value in (4) as compared with that in (3). Thus we have:

$$V_w=\sum_{r=0}^{m}\frac{1}{r+1}\cdot\frac{Z^r}{(1+Z)^{r+1}}=\frac{Z^0}{(1+Z)}+\frac{Z}{2(1+Z)^2}+\frac{Z^2}{3(1+Z)^3}+\ldots. \qquad (5)$$

Thus

$$V_w=-\frac{1}{Z}\log_e\left(1-\frac{Z}{1+Z}\right)=\frac{1}{Z}\log_e(1+Z)$$
$$=Z^0-\frac{Z}{2}+\frac{Z^2}{3}-\frac{Z^3}{4}+\ldots+\frac{(-Z)^{m-1}}{m}. \qquad (6)$$

Then by substituting back for the operator Z,

$$V_w=a_w-\tfrac{1}{2}(a_{wx}+a_{wy}+a_{wz}+\ldots)+\tfrac{1}{3}(a_{wxy}+a_{wxz}+\ldots)-\ldots. \qquad (7)$$

Formula (7) can be checked by general reasoning. If w is still alive and all others are dead, then only the first term remains, thus providing the desired payment of 1 to w. If w and only one other life are alive, only the first two terms remain, and in the second one all the annuity values but one are eliminated so that the net payment to w is, as required, $\frac{1}{2}$. Likewise if only w and two others are alive, all terms but the first three vanish, and in the second term only two a's remain, while in the third term only one a remains; accordingly, the net result is a payment of one-third, which is as required. For the general case of w and r other lives being alive, the proportion payable to w is the sum of the coefficients, or

$$1-\frac{C_1^r}{2}+\frac{C_2^r}{3}-\ldots=\frac{1}{r+1}(C_1^{r+1}-C_2^{r+1}+C_3^{r+1}-\ldots) \tag{8}$$

$$=\frac{1}{r+1}[1-(1-1)^{r+1}]=\frac{1}{r+1},$$

which is as required.

Furthermore, it can readily be shown that if values of V are obtained for each of the other m lives in the same way, and if all the V's are summed, there is obtained equation (1), which is, of course, a necessary condition as to the correctness of equation (7).