Risk Sharing and Capital Allocation

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Abstract

In recent years a lot of time have been spent discussing a variety of different ways to measure the performance of an insurance company. One of the dominating concepts in the discussion of this has been "Economic Capital" and with this the question of allocation of the (Economic) Capital with the intention of introducing a risk related measure of performance. In this paper we view the performance measuring as a question of determining the appropriate distribution of a risk loading for participants in a risk pool having the overall risk priced by an actuarial premium principle. We show that the covariance principle can be found as the optimal solution to this risk sharing problem and show that the covariance principle fulfils some reasonable requirements. We furthermore suggest how the covariance principle can be at the basis of a target setting procedure for a multi line insurance company.

1. Introduction

In previous years there have been a growing interest in methods for managing the capital of banks and insurance companies. This is due to increasing focus on returns and shareholder value and an increasingly competitve market where the ability to manage risks can give a competitive advantage.

An important part of managing the risk of a company is to introduce measures of returns of different business areas based on the risk with which they contribute to the company's total risk. Broadly labelled Risk Adjusted Performance Measures (RAPM) a variety of methods are being pushed forward and consultants have found a new lucrative market.

In practice implementing RAPM starts with dividing the overall risk of the company into a number of subrisks. These are subsequently assessed either using statistical models or a subjective judgment and the individual risks are finally combined in order to quantify the overall risk of the company.

Since it is usually the case that the individual risks are not perfectly correlated some diversification effect occurs during the combination, implying that the overall risk is less than the sum of the individual risks.

The word risk can be interpreted in many ways, but in most cases it is, in line with traditional financial theory, used as a synonym for volatility. This is reflected in the definition of the probably most popular concepts in this area that of "Economic Capital" (EC).

The definition of EC can vary a bit but is based on a consideration of an amount needed to secure the company from "unxpected losses". This usually boils down to regarding a quantile in the distribution of the next year's profit, for instance the 1% quantile and denoting minus this as the EC. Sometimes the mean of the profit is added. In other words, if X denotes the profit of next years business then the EC on a 99% level is commonly defined as either:

$$EC = -\inf \{x | P(X < x) = 1\%\}$$

or

$$EC = -\inf \{x | P(X < x) = 1\%\} + EX$$

We see that the EC is closely connected to the Value at Risk (VaR) concept which is very popular in finance and the EC concept does in fact originate from the VaR. However the VaR is usually calculated on a very short time horizon where the expected profit is zero, which probably is the cause for the confusion about whether the mean is included or not.

It is furthermore seen that the EC ressembles a solvency margin as known from classical actuarial theory, but since the EC is (usually) calculated with a one-year time horizon, it is hardly appropriate to regard it as a theoretical "right" counterpart of the company's equity as is sometimes heard.

This interpretation of EC is of course also the reason for the interest in calculating return on EC and it has furthermore been natural to engage in allocating the overall EC back to the individual business units and using this allocated capital as a basis against which the individual returns can be measured.

Generally the EC can be regarded as a measure of the volatility of the business, and this allocation excercise is therefore an allocation of volatility and the measure of return on allocated EC becomes a measure of return relative to volatility. This idea of measuring returns relative to volatility is of course by no means new. Many of the RAPM's put forward are intimately related to the William F. Sharpe's (1966) ratio $\frac{r-r_0}{\sigma}$, where the return r in excess of the risk-free rate of return r_0 is measured relative to the volatility σ.

The acknowledging of EC being a volatility measure in a way frees the concept of the "Capital" interpretation and as we will show in section 3 it is natural to view a measure of performance of a sub-unit of the company as a question of whether the unit is able to collect an appropriate risk premium as calculated according to classical actuarial theory. The target for the individual risks is set by allocating the overall risk (volatility) and applying some common "market price of risk" on the allocated risk[1].

Though we focus on allocating risk and risk loading rather than capital, the problem of how to allocate is the same as the one extensively dealt with in the litterature. Despite the widespread interest in the problem it however still seems somewhat unclear what the natural requirements for a "good" (optimal) allocation would be. In section 3 we regard the company as a risk sharing arrangement like found in a mutual insurance company or a risk pool. We show that a risk allocation based on the covariances between the risks, the Covariance Principle, is the allocation optimal in the sense of reflecting the original risk composition in the best way. We set up some reasonable conditions for an allocation principle and show that these are fulfilled by the Covariance Principle, but not by a method of relative allocation often used in practice.

However before we get to that we briefly describe a framework for viewing an insurance company as a special kind of leveraged investment fund as thouroughly described in the Swiss Re publication "The Economics of Insurance" (Swiss Re 2001). This view opens for the combination of a finance theory approach to the performace measuring or target setting with the actuarial view that we present in

[1]This idea has previously been suggested by Stavros Christofides (1999) using the PH-premium principle

section 3.

2. Capital, return and risk

2.1. The insurance company

As Stavros Christofides once noted (Christofides 1999) one could argue that an insurance company don't need a capital base to do its business. In insurance the customer (the policyholder) pays in advance for a product that is to be delivered at some uncertain time in the future. If the premium paid in respect of the insurance contract exceeds the production price of the product then the company makes a profit just like in any other business, however (almost) without having to make initial investments in raw materials, production facilities etc.

The production price of the insurance contract could by a replicating portfolio argument be the price of a portfolio of zero coupon bonds with maturity at the times of expected payments in respect of the policy.

If the company only wrote one policy there would of course be some variability in the payments patterns, but by having enough policies in the company's insurance portfolio this variability would be reduced substantially.

Now policyholders would probably not be satisfied with a situation where an insurance company did not have a capital to absorb fluctuations in the payments in respect of the insurance contracts. Since this concern is naturally shared by governments, regulatory regimes are imposed calling insurance companies to hold capital in order to secure the policyholders claims.

This capital has to come from somewhere and usually the insurance company has a group of shareholders who equip the company with a capital base (Equity) with the purpose of earning a return.

The activities the insurance company undertakes in order to live up to this purpose can be therefore be divided into two categories:

1. Investing the capital into the financial markets

2. Using the capital as a support for writing insurance business

Under the first point the insurance company actually takes on the role as a portfolio manager for the shareholder investing the capital in a diversified portfolio just like any other investment fund. Under the second point the insurance company writes up to as much business as it is allowed due to regulatory requirements thereby having disposal of the prepaid premiums in what is in practice a loan from the policyholders.

Usually insurance companies do not distinguish between shareholders funds and policyholders funds when regarding their investment activity. This means that the insurance company actually functions as a leveraged investment fund where the shareholders funds are leveraged through the loaning arrangement with the policyholders. This again implies that the shareholders investment is exposed to more investment risk than it could have been had he just invested it himself besides being exposed to the risk stemming from the insurance business.

The question now is what return should the insurance company provide the shareholder with?

The first aim of the insurance company must be to provide the shareholder with a return on his invested capital comparable to what he could have achieved on his own if he went directly to the financial markets using the same investment profile. This return is usually referred to as the Base Cost of Capital and of course depends on the risk of the investment strategy i.e. what is the composition of the portfolio with regard to equity, bonds, property etc.

The Base Cost of Capital is independent of the insurance business and only reflects the market risk experienced everywhere in the financial markets. This type of risk is usually referred to as "systematic risk" since it influences the market as a whole and is not particular for some individual assets (see Myers & Brealey (2000) for an introduction to financial theory and different risk types).

Now besides letting the insurance company manage the investments of his funds the shareholder exposes his funds to risk of a different nature than the systematic risk of the investment that is insurance risk. Insurance risk is commonly regarded as "non-systematic" meaning that it as opposed to systematic risk is specific to individual assets.

In theory the non-systematic risk of the insurance business can be diversified away by the shareholder. This follows from the fact that non-systematic risk is assumed independent of the systematic risk which reflects the market fluctuations. Therefore just having enough (infinitely much) of the non-systematic risk means that it reduces to nothing which again implies that pricing this risk would lead to arbitrage possibilities.

Though it might not be entirely true for all insurance types that the insurance risk is non-systematic we will here accept this assumption. If the insurance risk does have a systematic part the shareholder will of course demand an additional return.

From the above it is clear that in theory the shareholder cannot demand return for taking on non-systematic risk such as insurance risk. However the theory is based on the assumption that capital is readily available in the markets and can be raised frictionless. In practice it will probably be hard to find someone who would expose their capital to non-systematic risk without asking for a return.

The reason for this is that there are some frictions inherent in the system, frictions that the shareholder can demand an additional return for. Using the definitions of Swiss Re (2001) these can be divided into four groups:

1. Agency Costs
2. Regulatory restrictions
3. Double taxation
4. Cost of financial distress

The first friction reflects the fact that the shareholder has renounced control of his investment. The second implies that the regulatory regime limits the liberty of action on behalf of the shareholder and the third is of course the fact that the shareholder in many countries is to be taxed twice on the return on his capital, firstly through the tax bill of the insurance company and secondly when receiving his return in form of dividends. The fourth friction reflects the fact that in a situation of financial distress the company might find it hard to raise new capital and will be imposed legal costs etc. The risk of financial distress is of course closely linked to the insurance business and this therefore makes the connection to the traditional actuarial view of requiring a risk premium for insurance risk.

We see that the return the shareholder can demand is:

1. A return comparable to a market index with the same investment profile
2. An additional return if the company uses the insurance business for leveraging
3. A return connected to the insurance risk and other frictions

The shareholder can set a loading to for example 3% additional return due to frictions. Since the non-systematic risk inherent in the insurance contracts can reasonably be regarded as the key driver for the loading it seems reasonable to interpret the loading as induced by the aggregation of the individual insurance risks.

In order to assess whether the insurance business is then delivering the required return one should therefore simply calculate the market value of a replicating portfolio of zero-coupon bonds, with the same term structure as the insurance liabilities. If the premium (less administrative costs) paid for the policies making up the liabilities exceeds the 3% then the insurance business is making a profit.

Having determined what overall profit the insurance portfolio is to provide the natural next question would therefore be how the individual risks, or group of risks, should contribute to the overall loading. This is the allocation problem and is the subject of the next section.

3. Allocation of risk in a risk sharing scheme

3.1. Risk Sharing

Risk sharing or pooling arrangements are at the base of insurance. Here a number of individuals, each carrying a certain risk Y_i, will agree to enter their risks into a pool with a total risk $Y. = \sum_i Y_i$, and the i'th individual will in exchange receive a share of the pool as illustrated in Figure 3.1[2]. If we assume that the pool transfers the risk to another insurer then the pool will pay a premium exceeding the expected value $E(Y.)$. We assume the (re)insurer calculates the premium according to a premium principle of the form

$$\pi = EY. + \alpha C(Y.) \tag{3.1}$$

where $C(\cdot)$ is some risk measure. The essence of this assumption is that the premium is calculated from the distribution of the total risk $Y.$.

Now the premium is to be paid by the individuals participating in the pool according to some allocation scheme. From the expression for (3.1) we see that since the expected value is linear, this comes down to finding a way to share the

[2] The risk Y_i comprise both risk from frequency and severity of claims and risk with regard to timing of the payments. The value of Y_i should therefore be determined as the value of a portfolio of zero-coupon bonds with matching maturities.

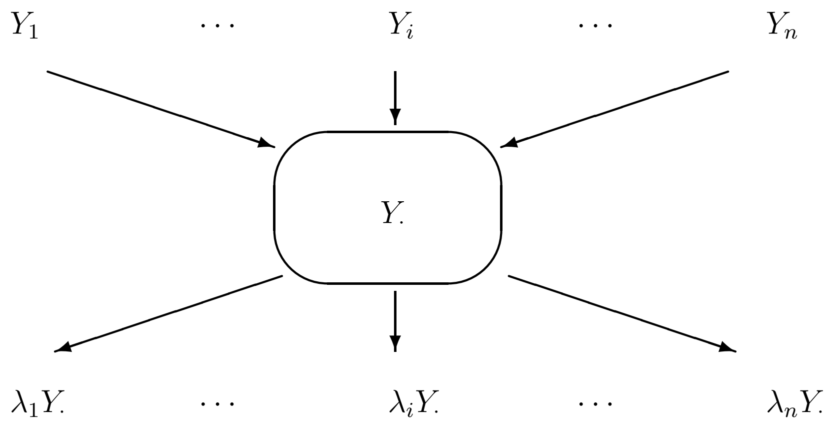

Figure 3.1: The mechanism of a risk pool

loading $\alpha C(Y.)$ between the participants in the pool. The individual premiums π_i will then be given by

$$\pi_i = EY_i + \alpha A(Y_i|Y_1, \ldots, Y_n), \tag{3.2}$$

where $A(Y_i|Y_1, \ldots, Y_n)$ is the risk allocated to individual i from the group consisting of risks $Y_1, \ldots, Y_n$.

From the above the analogy between the allocation of premium loading and the allocation of Economic Capital is obvious. In an EC framework the reinsurer is the owner of the company asking for some return proportional to the EC. The actuarial safety loading α is the "market price of risk" charged by the owners for taking on the risk of having capital in the company (e.g. the 3% from the previous paragraph). By calculating the contribution to the pool by (3.2) the individual risks are therefore charged with the same price per unit of allocated risk.

In making the connection between traditional actuarial methods and the EC framework it is worth noticing that the special case where $C(Y.)$ is the standard deviation. The corresponding premium principle is the classical standard deviation principle, whereas the EC is the VaR when one assumes a multivariate normal distribution.

Since the allocated capital $A(Y_i|Y_1,\ldots,Y_n)$ (and the premium loading) is constructed as a safeguard against fluctuations around the mean, we may without loss of generality in the sequel assume that all the Y_i's have zero mean.

We shall assume that the risk measure $C(\cdot)$ satisfies the following minimum requirements :

(a) $C(Y) \geq 0$ and $C(Y) = 0$ iff $Y \equiv 0$

(b) $C(aY) = |a|C(Y)$

(c) $C(Y+Z) \leq C(Y) + C(Z)$ (sub-additivity)

Item (a) states that any non-degenerate risk has a volatility. Item (b) states that the risk measure does not depend on the currency in which the risks are measured, and the fundamental property (c) states that there is always some diversification associated with the pooling of risks. We notice that the standard deviation fulfills (a) to (c).

Together (a)-(c) simply states that $C(\cdot)$ is required to be a norm on the linear space of zero-mean random variables.

3.2. Optimal allocation

When determining $A(Y_i|Y_1,\ldots,Y_n)$ for the individual units, a simple approach would be to use the stand-alone risk $C(Y_i)$ as a starting point. Because of (c), the sum of these will exceed the total risk $C(Y.)$, and one approach would be to scale the individual stand-alone risks to the appropriate overall level. That is one would use a relative (or pro-rate) allocation principle

$$A(Y_i|Y_1,\ldots,Y_n) = C(Y.)\frac{C(Y_i)}{C(Y_1)+\cdots+C(Y_n)}.$$

This principle for allocation of risk is very simple which is probably why it has become so popular for practical use. However as we shall demonstrate in section 4, this approach has some serious drawbacks, which actually makes it unsuited for practical purposes.

Another principle for allocation can be derived in the following way. The i'th individual transfers its risk to the pool in exchange of some share $\lambda_i Y.$ of the pool.

If the total risk $Y.$ is required a risk loading of c, say, then the individual shares $\{\lambda_i Y.\}_i$ are obviously required to contribute with allocated loadings of $\{\lambda_i c\}_i$. The problem of optimal allocation is in this context therefore equivalent to determining the optimal weights λ_i for participation in the pool.

We shall say the weights λ_i are optimal when the risk profile of the allocated risks shares $(\lambda_1 Y., \ldots, \lambda_n Y.)$ reflects the original risk profile $(Y_1, \ldots, Y_n)$ as closely as possible - in the sense stated below.

With $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$ being the column vector of risks, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)'$ the corresponding vector of weights, we choose the allocation that minimises the quadratic loss

$$Q(\boldsymbol{\lambda}) = \mathsf{E}[(\boldsymbol{Y} - \boldsymbol{\lambda} Y.)' \boldsymbol{W} (\boldsymbol{Y} - \boldsymbol{\lambda} Y.)] \tag{3.3}$$

where $\boldsymbol{W}$ is a positive definite weight-matrix.

In order to determine the coefficients $\boldsymbol{\lambda}$ such that (3.3) is minimised we differentiate wrt. $\boldsymbol{\lambda}$ and equate to zero yielding the expression

$$0 = 2\mathsf{E}[Y.^2]\boldsymbol{\lambda}'\boldsymbol{W} - 2\mathsf{E}[Y.\boldsymbol{Y}']\boldsymbol{W}$$

or

$$\lambda_i = \frac{\mathsf{E}[Y.Y_i]}{\mathsf{E}[Y.^2]}. \tag{3.4}$$

Note the following points

- $\sum_i \lambda_i = 1$, such that (3.4) represents a genuine risk sharing
- The solution (3.4) does not depend on the weight matrix $\boldsymbol{W}$

The allocated risk according to the risk measure $C(\cdot)$ can then be calculated as the risk measure of the allocated risk, which is $C(\lambda_i Y.) = \lambda_i C(Y.)$ because of (b).

The allocated risk $\lambda_i Y.$, which gives the best approximation to the origimal risk in the sense of (3.3), is given by the coefficients (3.4). Since all the risks Y_i have zero mean in this application, we may also write (3.4) as

$$\lambda_i = \frac{\mathsf{Cov}(Y_i, Y.)}{\mathsf{Var} Y.} \tag{3.5}$$

The optimal allocation principle (3.5) is also known as the Covarince Principle. Readers familiar with portfolio theory will notice that λ_i corresponds to the β coefficient of the CAPM model.

4. Properties of allocation principles

Various criterions can be set up for an allocation principle to be reasonable. The first criterion could be that an allocation should recognize diversification, meaning that the risk allocated to a unit or a group of units can not exceed the corresponding stand-alone risk. This requirement is the immediate analogue to the sub-additivity requirement (c) for risk metrics.

Secondly it should be required that the allocation is consistent, in the sense that the risk allocated to a given unit does not depend on the level at which the allocation is performed. Assume that the company has a hierarchical structure with business units at the first level and sub-units at the second level. The consistency requirement means that the risk allocated to a given unit should be the same irregardless of whether the risk is allocated directly to business units (and subsequently allocated further down the hierarchy) or it is being allocated to sub-units and subsequently aggregated to the business unit level.

Formalising the above gives

Recognition of diversification. For any subset $\mathcal{C} \subseteq \{1, \ldots, n\}$ it holds that

$$\sum_{i \in \mathcal{C}} A(Y_i | Y_1, \ldots, Y_n) \leq C(\sum_{i \in \mathcal{C}} Y_i)$$

Consistency. For any subset $\mathcal{C} \subseteq \{1, \ldots, n\}$ it holds that

$$\sum_{i \in \mathcal{C}} A(Y_i | Y_1, \ldots, Y_n) = A(\sum_{i \in \mathcal{C}} Y_i | Y_1, \ldots, \sum_{i \in \mathcal{C}} Y_i, \ldots, Y_n)$$

The condition "Recognition of diversification" is also considered by Denault (1999), where it is referred to as the "No undercut" condition.

Note that the requirement of "Consistency" in particular implies the "Full allocation" requirement (see Denault, 1999),

$$\sum_{i=1}^{n} A(Y_i | Y_1, \ldots, Y_n) = C(Y.)$$

In order to investigate these properties further, we assume that the risk-metric (norm) $C(\cdot)$ is the usual norm for squared-integrable random variables induced by the inner product

$$\langle X, Y\rangle = \mathsf{E}[XY]$$

Thus,

$$C(Y) = \|Y\| = \langle Y, Y\rangle^{1/2}$$

Since the variables have zero mean, we also have that $\|Y\| = (\mathsf{Var}Y)^{1/2}$ is the standard deviation, such that the risk-metric correponds to the VaR measure in the normal-case. In this setting we have the following:

Theorem 1. *(i) The allocation based on the Covariance Principle is consistent and recognizes diversification. (ii) The relative allocation is not consistent and does not recognize diversification.*

Proof. *(i)* From (3.4) we have that $\lambda_i = \langle Y_i, Y.\rangle / \|Y.\|^2$. With $Y. = \sum_{i\in\mathcal{C}} Y_i$, the condition for recognition of diversification becomes

$$\sum_{i\in\mathcal{C}} \frac{\langle Y_i, Y.\rangle}{\|Y.\|^2} \|Y.\| \leq \|Y.\|$$

or

$$\langle Y., Y.\rangle \leq \|Y.\| \|Y.\|$$

which is simply the Schwarz-inequality.

Consistency of the covariance principle follows readily from the linearity of the inner product.

(ii) For the relative allocation we may write

$$A(Y_i | Y_1, \ldots, Y_n) = \frac{\|Y_i\|}{\sum_i \|Y_i\|} \|Y.\|.$$

The requirement for consistency becomes

$$\frac{\sum_{i\in\mathcal{C}} \|Y_i\|}{\sum_i \|Y_i\|} = \frac{\|Y.\|}{\|Y.\| + \sum_{i\in\{1,\ldots,n\}\setminus\mathcal{C}} \|Y_i\|},$$

and the condition for recognition of diversification becomes

$$\frac{\sum_{i \in \mathcal{C}} \|Y_i\|}{\|Y.\|} \leq \frac{\sum_i \|Y_i\|}{\|Y.\|}.$$

The fact that neither of these conditions are met is easily demonstrated by examples. For instance, consider orthogonal risks X, Y and Z with $\|X\| = \|Y\| = \|Z\| = 1$. If a company consists of four business units with risks X, $-X$, Y and Z, both conditions are violated for the class $\mathcal{C}$ consisting of the three risks X, $-X$ and Y.

□

From the above we see that in practice using the relative allocation for allocating risk in a company might mean that some sub-unit of the company is allocated more EC that it would according to the same risk measure if the sub-unit was not part of the company, but was an independent business.

5. Conclusion

The subject of Capital Allocation and Risk Adjusted Perfomance Measurement has been widely discussed and it has become compulsory for companies in the financial sector to address the issue. The development of the methods applied in practice has been lead by the banking sector and unfortunately it seems that the methods often suffer from a lack of theoretical foundation.

One example of this is that the methods usually do not distinguish between systematic and non-systematicc risk but pools them all together. Another example is the widespread use of relative allocation which, as we have shown, does not satisfy some basic requirements, the most critical probably being that a risk can be punished for contributing to the overall diversification. In practice one would hardly find any management accepting that his risk grows as he enters in to a merger. The approach suggested here, to view the company as a risk sharing arrangement, is in our view a more natural foundation for discussing the allocation issues, which is in keeping with the standard insurance approach to risk, diversification and loading.

As noted the Covariance Principle is similar to the CAPM model familiar to most of the people working with EC and RAPM. This however does not prevent some to advocate for the relative allocation mainly based on a consideration of the Covariance Principle being difficult to understand because actions taken in one part of the company affects another because of the covariation. In our view this argument simply does'nt make sense since the aim must be to optimise the return for the shareholder i.e. the total business.

From a theoretical point of view one can argue against the present approach by noting that the somewhat vague argument of the insurace risk being the key driver behind the loading for frictional costs is unsatisfactory and that the hybrid between a financial approach and an actuarial approach should be replaced by an integrated approach where both systematic and non-systematic risks are priced in the same framework.

From a theoretical point of view an integrated theory would provide a good approach to the problem and we know research is being done in this area (see e.g. Møller (2001) for theory of indifference pricing of composite risks). We also know that the idea of allocating risk loading instead of capital is not new being to our knowledge first presented by Christofides (1999) in the framework of the PH-transform and we know that Wang (2002) in the more general frame of the Wang transform has presented similar ideas.

In practice however we think the approach taken here present a good starting point for measuring the profitability of both the insurance company and different business lines making a good foundation for business decisions. The Covariance Principle provides a straightforward method for allocating the overall risk based on a consistent framework of a risk sharing arrangement and the method is readily applicable.

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