RUIN PROBABILITIES FOR A SHORT TERM DISABILITY MODEL

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Abstract

A simple short-term disability model is developed, where the loss process is comprised of two processes: new claims process and continuing claims process. This results in a surplus function with stationary dependent increments. Three dimensional transition probabilities are needed to model the transitions of this non-Markovian process. An approximating model with stationary independent increments is found. The approximating model leads to bounds for the finite probability of ruin. Numerical comparisons are made.

1 Introduction

A simple discrete time model for short-term disability insurance is consider. Premiums are collected at a constant rate of P per period, where P is assumed to be some positive integer. The premiums paid during the first k periods is $P_k = Pk$. The premiums are to provide wage loss benefits (WLB) to disabled workers while they are unable to work. Let N_k denote the aggregate number of claims as of period k. Claims are assumed to arrive at a Poisson rate λ and hence N_k has a Poisson distribution with mean λk. The notation N_k is used later in section 3. Define the surplus process at time k as $U_k = u_0 + Pk - S_k$, where u_0 is the initial surplus at time 0 (assumed to be a positive integer) and S_k is the aggregate losses paid up until time k. The loss process S_k is quite complicated and requires additional notation, which is developed below.

Claimants receive WLB of 1 per period while disabled. That is if a worker became disabled she would receive 1 unit per period until she was able to return to work. Let X_k denote the number of claimants collecting wage loss benefits in period k. The aggregate loss process is then $S_k = X_0 + X_1 + X_2 + \cdots + X_k$, where X_0 is the initial number of claimants collecting WLB at time 0. The number of claimants in each period depends on two components: the number of claimants continuing to collect from the previous period and the number of new claimants arriving during the period. Let ε_k denote the number of new claims in period k. The total number of claims as of period k is $N_k = \varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_k$.

Suppose that each individual collecting WLB at time $k-1$ has probability α of continuing to collect benefits at time k. Then given the number of claimants at time $k-1$, X_{k-1}, the number who continue to collect at time k, has a binomial distribution with probability of "success" α and number of trials X_{k-1}. This procedure is sometimes called thinning. A time series representation of X_k can be achieved by introducing the thinning operator "$\circ$". Given any non-negative integer X, $\alpha \circ X$ has a binomial distribution with probability of "success" α and number of trials X. The Poisson AR(1) model is written as,

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k.$$

Note that $X_k | X_{k-1}$ is a convolution of a binomial random variable and a

Poisson random variable. The conditional probability function is given by

$$p\left(X_k|X_{k-1}\right) = \sum_{s=0}^{\min(X_k,X_{k-1})} \binom{X_{k-1}}{s} \alpha^s \left(1-\alpha\right)^{X_{k-1}-s} \frac{e^{-\lambda}\lambda^{X_k-s}}{\left(X_k-s\right)!} \tag{1}$$

and the moment generating function and probability generating function are respectively

$$M_{X_k|X_{k-1}}\left(s\right) = \left[1-\alpha+\alpha e^s\right]^{X_{k-1}} e^{\lambda\left(e^s-1\right)}$$

and

$$\gamma_{X_k|X_{k-1}}\left(z\right) = \left[1+\left(z-1\right)\alpha\right]^{X_{k-1}} e^{\lambda(z-1)}.$$

If X_0 has a Poisson distribution with mean $\frac{\lambda}{1-\alpha}$ then the loss process has stationary increments. However the increments are dependent. Further the marginal distribution of $S_k - S_{k-1} = X_k$ is Poisson with mean $\frac{\lambda}{1-\alpha}$. The increments of the loss process can be thought of as a birth & death process or an infinite server queue with Poisson arrivals and geometric service times.

2 Transition probabilities

Equation 1 gives the transition probabilities for $X_k = S_k - S_{k-1} = U_{k-1} + P - U_k$. The state space for U_k expanses with k and is given by $\Omega_{U_k} = \{-1, 0, 1, \ldots, u_0 + Pk - 1, u_0 + Pk\}$. Here the state -1 represents ruin. The transitional probabilities for U_k depend on the previous two states and are related to the transitional probabilities of X_k as follows:

$$P\left(U_k = u|U_{k-1}, U_{k-2}\right) = P(X_k = U_{k-1} + P - u|X_{k-1} = U_{k-2} + P - U_{k-1})$$

Which can be represented as a 3-dimensional transition matrix, where the dimensions are for the current state and the previous two states. The dimensions of the matrix are $\left(U_0 + P\left(k-2\right) + 2\right) \times \left(U_0 + P\left(k-1\right) + 2\right) \times \left(U_0 + Pk + 2\right)$.

3 Approximating model

The model defined in section 1 can be approximated by the following process, which in essence is the same process. Let $S_k^* = \sum_{j=1}^{N_k} X_j^*$, where N_k is the aggregate number of claims and X_j^* is the severity of the jth claim. In this model N_k is assumed to be Poisson with mean λk and the severity distribution is geometric with probability function $P\left(X_j^* = x\right) = (1-\alpha)\,\alpha^{x-1}$, $x = 1, 2, 3, \ldots$. The mean, variance and moment generating function of the severity random variable are $\frac{1}{1-\alpha}$, $\frac{\alpha}{(1-\alpha)^2}$ and $M_{X_j^*}(s) = \frac{(1-\alpha)e^s}{1-\alpha e^s}$ This model has loss increments which are stationary and independent. For reference this model will be referred to as the approximating model and the model in section 1 will be called the dependent model.

The difference between the approximating model and the dependent model is how losses are counted. For the dependent model, the incremental loss is equal to the current number of claims paying (both continuing claims and new claims). This is the amount of losses actually paid during the period. For the approximating model the incremental loss on the number of new claims during the interval and the duration of each of these claims. The full "life" of the claim is counted when the claim occurs.

Since the claims costs are counted sooner in the approximating model $S_k^* \geq S_k$ and $U_k^* \geq U_k$ for all k. Further the probability of ruin for the approximating model forms and upper bound on the probability of ruin in the dependent model.

The adjustment coefficient is the positive solution κ to the equation $1 + (1+\theta)\frac{\kappa}{1-\alpha} = \frac{(1-\alpha)e^\kappa}{1-\alpha e^\kappa}$ or equivalently $1 + (1+\theta)\frac{\kappa}{1-\alpha} = \left[1 + \frac{\alpha}{1-\alpha}(1+\theta)\,\kappa\right] e^\kappa$, where $\theta > 0$ is the load on the actuarially fair premium per period $P = (1+\theta)\frac{\lambda}{1-\alpha}$. Approximating $e^\kappa \simeq 1+\kappa$ leads to the solution $\kappa_1 = \frac{(1-\alpha)\theta}{\alpha(1+\theta)}$. While $e^\kappa \simeq 1+\kappa+\frac{\kappa}{2}$ leads to the solution $\kappa_2 = \frac{\sqrt{[2\alpha(1+\theta)+1-\alpha]^2+8\alpha(1-\alpha)\theta(1+\theta)}-[2\alpha(1+\theta)+1-\alpha]}{2\alpha(1+\theta)}$. Equation (13.2.12) in Actuarial Mathematics (1997) gives the following approximation $\kappa_3 = \frac{2\theta E\left[X_k^*\right]}{E\left[X_k^{*2}\right]} = \frac{2\theta(1-\alpha)}{1+\alpha}$.

4 Numerical Comparisons

Simulation was used to compare the PAR(1) model and the approximating model. The parameters used in the simulation were $\alpha = 0.43$, $\lambda = 1.1$, $P = 2$ and $u_0 = 20$. Based on these parameters the premium load is $\theta = 3.64\%$, the

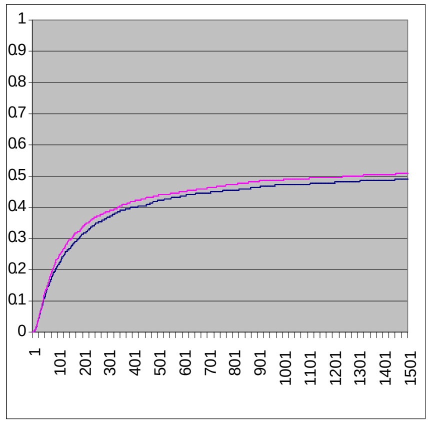

Figure 1: Probability of ruin for the PAR(1) model and the approximating model.

adjustment coefficient is $\kappa = 0.028024$ and the Lundberg upper bound on the probability of ruin is 57.1%. X_0 was selected from the marginal distribution. 2500 repetitions of 1500 periods were simulated. The probability of ruin during the first 1500 periods was found to be 49.0% for the PAR(1) model and 50.8% for the approximating model. Graph 4 show a plot of the ruin probability as a function of the number of periods. Note the approximating model dominates the PAR(1) model and is the higher line on the graph.

Table 4 is a plot of the adjustment coefficient as a function of α, with $\lambda = 1.1$ and $P = 2$. For these parameter value κ_1 is significantly larger than the other two approximations. When $\alpha = 0.1$, κ_2 is about 60% closer to the numerical solution for the adjustment coefficient than κ_3. This improvement in κ_2 over κ_3 increases with α and when $\alpha = .44$, κ_2 is about 90% closer. Alternatively if α is fixed and λ is allowed to vary it is found that the improvement in κ_2 over κ_3 decreases slowly with λ. The main factor in determining the improvement appears to be α and not λ. When α is larger than .5, κ_1 is better than κ_3. However in all the cases examined κ_2 was the best approximation for the adjustment coefficient.

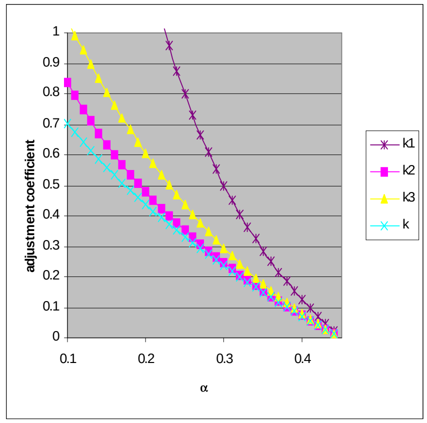

Figure 2: The adjustment coefficient as a function of α.

5 References

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