ANNUITIZATION VERSUS ESTATE CREATION

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1 Abstract

At retirement, people face a choice between annuitization and self management of liquid wealth (also called self-annuitization). The annuitization strategy is to purchase a life annuity by paying a nonrefundable lump sum to an insurance company in exchange for a lifelong income stream to cover living expenses. Under the self-annuitization strategy, the retiree will withdraw from the investment portfolio to cover living expenses. A common belief is that annuitization provides longevity insurance at the cost of liquidity.

In this paper we argue that the desire for liquidity is actually driven by bequest motives and estate value should be the measure on the liquidity of a

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given strategy. The annuity income stream is not necessarily identical to the consumption stream. The difference between these two streams is a balance stream, which in many cases has market value. The estate value of annuitization should be recognized by measuring the value of this balance stream. The annuitization strategy uses the same mutual fund as the self-annuitization strategy to support the variable payout and to grow the balance stream. The estate under self-annuitization is the remaining fund, while the estate under annuitization is the value of the balance stream.

We demonstrate that self-annuitization gives more estate within a critical duration (which is before life expectancy) while annnuitization creates more estate after that. In our second result, we compute the expected ending estate at death under these two strategies and find that, on the average, annuitization creates more estate than self-annuitization under reasonable assumptions. We conclude that, as long as the same type of fund is being used for these two strategies, annuitization and self-annuitization just create different patterns of estates over time. However, annuitization is usually better than self-annuitization for estate creation and liquidity, on average. This is somewhat counter-intuitive to popular belief because payout annuities without a certain period do not have cash surrender values. The creation of the balance stream and the results shown in this paper have important implications on new product and marketing strategies for payout annuities.

2 Introduction

At retirement, people face a choice between annuitization and self management of assets with systematic withdrawals for consumption purposes. The process of annuitization involves purchasing a life annuity by paying a nonrefundable lump sum to an insurance company in exchange for an income stream that cannot be outlived.

Obviously annuitization provides invaluable longevity insurance that cannot be replicated using other investment vehicles. The longevity insurance provided by annuitization guarantees that survivors will never run out of money, no matter how long they live. However, it is an empirical fact that most consumers are reluctant to purchase life annuities.

Ando and Modigliani (1963) life cycle hypothesis(LCH), or Yaari(1965), predicted that individuals would seek to smooth their lifetime consumption by annuitizing wealth. But Modigliani (1986), Friedman and Warshawsky (1990), and Mirrer (1994) have found that only seldom do people choose to annuitize wealth actively. Bernheim(1991), Hurd(1989) simply abandoned the strict form of the life cycle hypothesis and argued that the failure of annuity marketing is due to the fact that individuals have strong bequest motives. Consumers with strong bequest motives are reluctant to annuitize since there is little residual value left after annuitization. Another popular explanation for this phenomenon is that even when individuals have negligible bequest motives, annuities are simply too expensive, as Warshawsky(1988), Friedman and Warshawsky(1990), and Mitchell et al.(1999) have argued. Other explanations have focused on the individual's ability to pool mortality risk in large families, the lack of real (inflation protected) annuities, and nonrational behavioral justifications.

Beyond the academic literature that tries to explain or document the failure of annuity marketing, how can retirees evaluate the risk of self-annuitization, and decide if and when to purchase life annuities? Milevsky and Robinson (2000) modeled the risk of self-annuitization by computing the probability of ruin for a person who chooses self-annuitization. The main idea behind the optimal annuitization of Milevsky(1998), Milevsky(2001) is to compute the probability of "beating" the rate of return from a life annuity. The higher this probability, the longer retirees should wait before annuitizing.

The disadvantages of annuitization assumed in all the work above could be summarized in two points: (1) rates of return from life annuities are too low, so that annuities are expensive; (2) there is a serious loss of liquidity with annuitization. Of course, returns from fixed life annuities are much lower than those of equities, but they are close to the returns from fixed income securities. And the returns of variable life annuities (equity indexed annuities) are close to those from equities. It is unfair to punish all annuities because returns from fixed life annuites are lower than those of equities. When the annuitization strategy and self-annuitization strategy are being compared, similar funds should be used to back them. Moreover, though annuities themselves have no cash value, we can not ignore the fact that they do provide lifelong incomes. And the income stream doesn't have to match the consumption stream exactly. Usually they don't match and the difference is a balance stream, that produces the value of the estate. Annuities should be recognized as part of an individual's investment strategy along with a pure investment portfolio (a combination of stocks, bonds, mutual funds) and their impact on personal wealth should be evaluated together with pure investment and consumption.

In contrast to the academic literature that tries to explain the disadvantages of annuitization and show people how to avoid annuitization, the object of this paper is to (1) demonstrate that the two strategies, annuitization and self-annuitization, create different patterns of estate, neither of them necessarily superior to the other, and (2) on the average, annuitization creates more estate than self-annuitization. The main ideas behind this paper give life annuities and pure investment portfolios the same return characteristics and pool the impact of annuities, pure investment portfolios and consumptions on ending estate.

3 Relative Estate Patterns Over Life Spectrum

In the following, we used these standard actuarial notations:

(x): an individual at age x

$T(x)$: the future life time of (x)

ω: the limiting age

${}_{(t-1)|}q_x$: the probability of dying in the $t-th$ year for (x)

e_x: the life expectancy of (x)

a_x: the actuarial present value of a life annuity for (x)

$a_{\overline{t}|}$: the actuarial present value of an annuity certain with duration t

A retiree who has initial assets A will consume a certain amount of money, C_t, in the $t-th$ time period. We call $\{C_t\}$ the *consumption process*. The self-annuitization strategy is to put all assets into a pure investment portfolio and withdraw C_t at the end of the $t-th$ period. We called the sequence of cash values of the pure investment portfolio at the end of every period after withdrawal the *estate process under self-annuitization strategy*, $\{M_t\}$.

Alternatively, the annuitization strategy will use all initial assets A to buy a life annuity and open a side fund account. At the end of period t, the individual has income from this life annuity, P_t. If P_t is more than C_t, he will reinvest the surplus $P_t - C_t$ into the side fund account; otherwise, he will have to cash out the amount $C_t - P_t$ from the side fund account to cover living expenses. There is no cash value for a life annuity and we call the sequence of cash values in the side fund account the *estate process under annuitization strategy*, $\{A_t\}$.

Suppose a certain type of fund is being used by the pure investment portfolio under the self-annuitization strategy, and the life annuity and side fund account under annuitization strategy. Denote the rate of return on this fund during the $t-th$ period by R_t. Then the estate process under self-annuitization strategy is

$$\begin{aligned} M_0 &= A, \\ M_1 &= M_0(1+R_1) - C_1, \\ M_2 &= M_1(1+R_2) - C_2, \end{aligned}$$

$$\cdots$$

$$M_t = M_{t-1}(1+R_t) - C_t.$$

For the annuitization strategy, suppose the assumed interest rate (AIR) of the life annuity is i, then the initial payment P_0 satisfies $A = P_0 a_x$. The actual payments from this life annuity are

$$\begin{aligned} P_1 &= P_0 \frac{1+R_1}{1+i}, \\ P_2 &= P_1 \frac{1+R_2}{1+i}, \\ &\cdots \\ P_t &= P_{t-1} \frac{1+R_t}{1+i}. \end{aligned}$$

The estate process under annuitization strategy is

$$\begin{aligned} A_0 &= A - P_0 \cdot a_x = 0, \\ A_1 &= A_0(1+R_1) + P_1 - C_1, \\ &\cdots \\ A_t &= A_{t-1}(1+R_t) + P_t - C_t. \end{aligned}$$

Theorem 1 *There exists a duration n such that $a_x = a_{\overline{n}|}$ and $n \leq e_x$. Furthermore, for this duration n,*

$$\begin{aligned} M_t &> A_t \quad \textit{if} \quad t < n; \\ M_t &= A_t \quad \textit{if} \quad t = n; \\ M_t &< A_t \quad \textit{if} \quad t > n. \end{aligned}$$

Proof. We notice that the actuarial present value of a life annuity is no greater than that of an annuity-certain with duration e_x, i.e., $a_x \leq a_{\overline{e_x}|}$. The reason is simple.

$$\begin{aligned} f(t) &= a_{\overline{t}|} = (1-v^t)/i, \\ f''(t) &= -(\ln v)^2 v^t / i < 0. \end{aligned}$$

By Jensen's Innequality, we have

$$a_x = E[a_{\overline{T(x)}|}] \leq a_{\overline{E[T(x)]}|} = a_{\overline{e_x}|}.$$

It is obvious that the present value of an annuity certain $a_{\overline{t}|}$ as a continuous function of duration t is strictly increasing and $a_{\overline{0}|} = 0$. Recognizing $a_x \leq a_{\overline{e_x}|}$, we know there must exist a duration n, such that $a_x = a_{\overline{n}|}$ and $n \leq e_x$.

Next, we considered the relationship between estate processes under annuitization and self-annuitization strategies.

$$\begin{aligned} M_t &= M_{t-1}(1+R_t) - C_t \\ &= M_0 \prod_{j=1}^{t}(1+R_j) - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t \\ &= A \prod_{j=1}^{t}(1+R_j) - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t. \\ A_0 &= 0 \\ &= A - P_0 \cdot a_x = A - P_0 \cdot a_{\overline{n}|}. \end{aligned}$$

$$\begin{aligned} A_t &= A_{t-1}(1+R_t) + P_t - C_t \\ &= \sum_{j=1}^{t-1}\left[P_j \prod_{i=j+1}^{t}(1+R_i)\right] - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] + P_t - C_t \\ &= P_0 \prod_{j=1}^{t}(1+R_j) \sum_{j=1}^{t}(1+i)^{-j} - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t \\ &= P_0 \cdot a_{\overline{t}|} \prod_{j=1}^{t}(1+R_j) - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t \end{aligned}$$

$$
\begin{aligned}
&= (P_0 \cdot a_{\overline{n}|} + P_0 \cdot a_{\overline{t}|} - P_0 \cdot a_{\overline{n}|}) \prod_{j=1}^{t}(1+R_j) - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t \\
&= A\prod_{j=1}^{t}(1+R_j) - \sum_{j=1}^{t-1}\left[C_j \prod_{i=j+1}^{t}(1+R_i)\right] - C_t + (a_{\overline{t}|} - a_{\overline{n}|})P_0 \prod_{j=1}^{t}(1+R_j) \\
&= M_t + (a_{\overline{t}|} - a_{\overline{n}|})P_0 \prod_{j=1}^{t}(1+R_j).
\end{aligned}
$$

Therefore, the difference of estates for these two strategies at the end of period t is

$$A_t - M_t = (a_{\overline{t}|} - a_{\overline{n}|})P_0 \prod_{j=1}^{t}(1+R_j).$$

$P_0 \prod_{j=1}^{t}(1+R_j)$ is always positive, so the sign of this difference depends on the sign of $a_{\overline{t}|} - a_{\overline{n}|}$. Again, $a_{\overline{t}|}$, as a function of duration t, is strictly increasing.

$$
\begin{aligned}
a_{\overline{t}|} &< a_{\overline{n}|} \quad \text{if} \quad t < n; \\
a_{\overline{t}|} &= a_{\overline{n}|} \quad \text{if} \quad t = n; \\
a_{\overline{t}|} &> a_{\overline{n}|} \quad \text{if} \quad t > n.
\end{aligned}
$$

The proof is complete.

The duration n indicates an important point in this changing pattern and we call it the *critical duration* from now on. If the retiree dies early, prior to the critical duration, the self-annuitization strategy has a larger ending estate than the annuitization strategy. The ending estate under the annuitization strategy starts low, continues to increase, equals the estate under the self-annuitization strategy at the critical duration, and then increases exponentially beyond that. The annuitization strategy beats the self-annuitization strategy in terms of the amount of estate created even before life expectancy.

What is interesting about this result is that it is independent of the assumed interest rate (AIR) i, the actual rates of return $\{R_t\}$, and even the consumption process $\{C_t\}$. In another words, an individual is assured that if he annuitizes his assets now, he will have more estate beyond life expectancy than if he invests his fortune in a pure investment portfolio as long as he selects a life annuity backed by the same portfolio. He can spend in any way he wishes; he doesn't have to predict the portfolio performance; he doesn't need to know the purchase rate for his annuity. If he needs more estate in the early years, he may choose self-annuitization; otherwise, he may be better off choosing annuitization. Since there is about a 50-50 chance of dying before or after life expectancy, the choice is solely up to personal preference.

4 Expected Ending Estate at Death

The self-annuitization strategy creates more estate early while the annuitization strategy creates more estate later. Neither of them is necessarily superior over the other one on a year by year basis. We next investigate ways to compare the two strategies.

The two ending estates A_T and M_T are random because of the uncertainties of the future life time T and the return process $\{R_t\}$, and we write them as $A(T;\{R_j\})$ and $M(T;\{R_j\})$. The expected ending estate at death is the expectation over both the future life time and the return process. The two expected ending estates are denoted by $E[A(T;\{R_j\})]$ and $E[M(T;\{R_j\})]$, respectively.

$$
\begin{aligned}
E[A(T;\{R_j\})] &= \sum_{t=1}^{\omega-x} E_{\{R_j\}}[A(t;\{R_j\})]_{(t-1)|}q_x, \\
E[M(T;\{R_j\})] &= \sum_{t=1}^{\omega-x} E_{\{R_j\}}[A(t;\{R_j\})]_{(t-1)|}q_x.
\end{aligned}
$$

First, we consider the case of a conservative investor. He is facing two options: invest in a certificate of deposit(CD) account or buy a fixed life annuity which assumes the same interest rate. That is to say, the guaranteed interest rate of the CD is the same as the AIR from this annuity, $R_t = i$ for any t. We find in this case annuitization is better than self-annuitization in terms of expected ending estate at death.

Theorem 2 *If $R_t = i > 0$ for any t, then $E[A(T;\{R_j\})] \geq E[M(T;\{R_j\})]$.*

Proof. Let $f(T;\{R_j\}) = (a_{\overline{T}|} - a_{\overline{n}|})P_0 \prod_{j=1}^{T}(1 + R_j)$. This is the difference between the ending estates at death under the two strategies. Let

$$
\begin{aligned}
f(t) &= f(T = t; \{R_j = i\}) \\
&= (a_{\overline{t}|} - a_{\overline{n}|})P_0(1+i)^t \\
&= (\frac{1-v^t}{i} - \frac{1-v^n}{i})P_0(1+i)^t \\
&= \frac{P_0}{i}((1+i)^{t-n} - 1), \\
f''(t) &= \frac{P_0}{i}\ln^2(1+i)v^{n-t} \\
&> 0. \\
E[A(T;\{R_j\}) - M(T;\{R_j\})] &= E[f(T;\{R_j\})] \\
&= E[E[f(T;\{R_j\}) \mid T]]
\end{aligned}
$$

$$\begin{aligned} \text{(use the assumption)} &= E[(a_{\overline{T}|} - a_{\overline{n}|})P_0(1+i)^T] \\ &= E_T[f(T)]. \end{aligned}$$

By Jensen's Innequality, we have

$$E_T[f(T)] \geq f(E(T)) = \frac{P_0}{i}((1+i)^{e_x-n} - 1) \geq 0.$$

This result tells us that when a retiree wants to invest in a CD type of portfolio and finds a fixed life annuity which assumes the same rate of return as that of the CD, he'd do better to annuitize initial wealth to seek higher expected estate value, instead of putting all the money in the CD directly.

Most people want to invest in an equity type of portfolio for its upside potential. The rate of return of the equity fund goes up and down over time but on the average it is higher than that of the CD. Usually a conservative AIR is being used for a life annuity, either fixed or variable, and it is close to the rate of return of a CD. If a retiree is facing a choice between an equity type portfolio and a variable life annuity backed by the same portfolio, we see in the next two results that annuitization is still better than self-annuitization in many cases. We extend the result above to the case in which the periodic returns have the same expected value and they are mutually independent.

We assume that $E[R_j] = R > i$ for any j and $Cov(R_k, R_j) = 0$ for any $k \neq j$, then

$$\begin{aligned} E[A(T;\{R_j\}) - M(T;\{R_j\})] &= E[f(T;\{R_j\})] \\ &= E_T[E_{\{R_j\}}[f(T;\{R_j\}) \mid T]] \\ \text{(use assumptions)} &= E_T[(a_{\overline{T}|} - a_{\overline{n}|})P_0(1+R)^T] \end{aligned}$$

$$= E_T[f(T)].$$

The next result is true for any mortality assumptions.

Theorem 3 *If (1)* $E[R_j] = R$ *for any* j *and* $Cov(R_k, R_j) = 0$ *for any* $k \neq j$,

and (2) $(1+i)^{1/(1+v^{n/2})} - 1 = L < R < U = (1+i)^{1/(1-v^{n/2})} - 1$,

then $E[A(T;\{R_j\})] \geq E[M(T;\{R_j\})]$.

Proof.

$$(1+i)^{1/(1+v^{n/2})} - 1 < R < (1+i)^{1/(1-v^{n/2})} - 1 \Longleftrightarrow v^n - (1 - \frac{\ln(1+i)}{\ln(1+R)})^2 > 0.$$

And we observe that $L < i < U$.

$$\begin{aligned}
\text{Let} \quad f(t) &= (a_{\overline{t}|} - a_{\overline{n}|})P_0(1+R)^t \\
&= (\frac{1-v^t}{i} - \frac{1-v^n}{i})P_0(1+R)^t \\
&= \frac{P_0}{i}[v^n(1+R)^t - (v(1+R))^t] \\
\text{then} \quad f''(t) &= \frac{P_0}{i}[v^n \ln^2(1+R)(1+R)^t - \ln^2(v(1+R))(v(1+R))^t] \\
&= \frac{P_0(1+R)^t}{i}[v^n \ln^2(1+R) - \ln^2(v(1+R))v^t]
\end{aligned}$$

Next, we prove $f''(t) > 0$ if $t \geq n$ (which implies $v^n \geq v^t$).

When $R \geq i > 0$,

$$1+R > v(1+R) \geq 1 \Longrightarrow \ln(1+R) > \ln(v(1+R)) \geq 0 \Longrightarrow \ln^2(1+R) > \ln^2(v(1+R)).$$

When $R < i$, we use the lower bound condition.

$$\begin{aligned}
& R > (1+i)^{1/(1+v^{n/2})} - 1 \\
\Longrightarrow \quad & \ln(1+R) > \frac{1}{1+v^{n/2}} \ln(1+i) > \frac{1}{2} \ln(1+i)
\end{aligned}$$

$$\Longrightarrow \ln(1+R)^2 > \ln(1+i)$$

$$\Longrightarrow 1+R > \frac{1+i}{1+R}$$

$$\Longrightarrow \ln^2(1+R) > \ln^2(v(1+R)).$$

Considering $v^n \geq v^t$, we have

$$v^n \ln^2(1+R) > \ln^2(v(1+R))v^t.$$

We have proved that $f''(t) > 0$ if $t \geq n$. When $t < n$, using the second condition we have

$$\begin{aligned}
f''(t) &= \frac{P_0(1+R)^t}{i}[v^n \ln^2(1+R) - \ln^2(v(1+R))v^t] \\
&= \frac{P_0(1+R)^t v^t}{i}[v^{n-t} \ln^2(1+R) - \ln^2(v(1+R))] \\
&= \frac{P_0(1+R)^t v^t \ln^2(1+R)}{i}[v^{n-t} - (\frac{\ln(v(1+R))}{\ln(1+R)})^2] \\
&> \frac{P_0(1+R)^t v^t \ln^2(1+R)}{i}[v^n - (1 - \frac{\ln(1+i)}{\ln(1+R)})^2] \\
&> 0
\end{aligned}$$

Again, by Jensen's Innequality, we have

$$E_T[f(T)] \geq f(E(T)) = \frac{P_0(1+R)^{e_x}}{i}(v^n - v^{e_x}) \geq 0.$$

The result above gives us a sufficient condition under which the annuitization strategy outperforms the self-annuitization strategy. We emphasize that the upper bound U and the lower bound L on R are used to secure the concavity of the difference function $f(t)$. When the average rate of return is over the upper bound or below the lower bound, self-annuitization is not necessarily better but the concavity condition of $f(t)$ is often violated. Though these bounds make the result less general, they work well since they allow a wide range for the expected value of the rate of return. For instance, suppose $n = 20$,

$$L = 1.7\% \quad \text{and} \quad U = 12.2\% \quad \text{if} \quad i = 3\%;$$

$$L = 2.4\% \quad \text{and} \quad U = 12.9\% \quad \text{if} \quad i = 4\%;$$

$$L = 3.1\% \quad \text{and} \quad U = 13.5\% \quad \text{if} \quad i = 5\%;$$

$$L = 3.8\% \quad \text{and} \quad U = 14.1\% \quad \text{if} \quad i = 6\%.$$

We further analyze the necessity of the bounds for this result. It seems that a lower bound is necessary to achieve this result. For example, we assume mortality follows De Moivre's Law and the maximum future life time is fifty years, which implies that the life expectancy is twenty five years. Also assume that the initial wealth is exactly the actuarial present value of the variable life annuity with yearly amount of payment \$ 1000 and AIR 4% and the yearly expense is \$800. If $R = 1.7\%$, $E_T[f(T)] = \$ - 37$. The self-annuitization strategy is better than the annuitization strategy.

We conjecture that the upper bound may not be necessary. As we will see in the next result, higher expected return always favors the annuitization strategy if the symmetry of the unconditional mortality distribution is assumed. A symmetric unconditional mortality distribution means ${}_{(t-1)|}q_x = {}_{(\omega-x-t)|}\ q_x$ for any $1 \leq t \leq \omega - x$. Usually the unconditional mortality rate, ${}_{t|}q_x$, keeps increasing until it reaches a peak around the life expectancy and then decreases with time, so the symmetric mortality assumption is reasonable.

Theorem 4 *If (1)* $\quad {}_{(t-1)|}q_x = {}_{(\omega-x-t)|}\, q_x$ *for any* $1 \le t \le \omega - x$,

(2) $\quad E[R_j] = R$ *for any* j *and* $Cov(R_k, R_j) = 0$ *for any* $k \neq j$,

and (3) $\quad R \ge L_2 = \sqrt{1+i} - 1$,

then $\quad E[A(T; \{R_j\})] \ge E[M(T; \{R_j\})]$.

Proof. Without loss of generality, suppose the future life time spectrum is an odd interger, $2m-1$, and death occurs at year end. Since the mortality distribution is symmetric, we have $e_x = m$ and ${}_{(t-1)|}q_x = {}_{(2m-1-t)|}\, q_x$ for any $1 \le t \le m-1$.

Let $g(t) = (a_{\overline{t}|} - a_{\overline{m}|})P_0(1+R)^t$.

$$
\begin{aligned}
E_T[f(T)] &= E[(a_{\overline{T}|} - a_{\overline{n}|})P_0(1+R)^T] \\
&\ge E[(a_{\overline{T}|} - a_{\overline{m}|})P_0(1+R)^T] \\
&= E_T[g(T)] \\
&= \sum_{t=1}^{2m-1} g(t)\,{}_{(t-1)|}q_x \\
&= \sum_{t=1}^{m-1} g(t)\,{}_{(t-1)|}q_x + g(m)\,{}_{(m-1)|}q_x + \sum_{t=m+1}^{2m-1} g(t)\,{}_{(t-1)|}q_x \\
&= \sum_{t=1}^{m-1} g(t)\,{}_{(t-1)|}q_x + \sum_{j=1}^{m-1} g(2m-j)\,{}_{(2m-1-j)|}q_x + g(m)\,{}_{(m-1)|}q_x \\
&= \sum_{t=1}^{m-1} [g(t) + g(2m-t)]\,{}_{(t-1)|}q_x + g(m)\,{}_{(m-1)|}q_x.
\end{aligned}
$$

$g(m) = (a_{\overline{m}|} - a_{\overline{m}|})P_0(1+R)^m = 0$. To prove $E_T[f(T)] \ge 0$, it is sufficient to prove $g(t) + g(2m-t) \ge 0$ for any $1 \le t \le m-1$.

$$
\begin{aligned}
& g(t) + g(2m-t) \\
= \;& (a_{\overline{t}|} - a_{\overline{m}|})P_0(1+R)^t + (a_{\overline{2m-t}|} - a_{\overline{m}|})P_0(1+R)^{2m-t}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{P_0}{i}[v^m(1+R)^t - v^t(1+R)^t + v^m(1+R)^{2m-t} - v^{2m-t}(1+R)^{2m-t}] \\
&= \frac{P_0 v^t(1+R)^t}{i}[v^{m-t} - 1 + v^{m-t}(1+R)^{2(m-t)} - v^{2(m-t)}(1+R)^{2(m-t)}] \\
&= \frac{P_0 v^t(1+R)^t}{i}[1 - v^{m-t}][(v(1+R)^2)^{m-t} - 1].
\end{aligned}
$$

We use the third condition here,

$$
R \geq \sqrt{1+i} - 1 \Longleftrightarrow (1+R)^2 \geq 1+i \Longleftrightarrow v(1+R)^2 \geq 1.
$$

Since $1 - v^{m-t} \geq 0$ and $(v(1+R)^2)^{m-t} - 1 \geq 0$, we have $g(t) + g(2m-t) \geq 0$. The proof is complete.

The result above demonstrates that if the mortality distribution is symmetric, the annuitization strategy outperforms the self-annuitization strategy in a very broad range since there is no upper bound on R. We need a very small positive lower bound, which is independent of the life expectancy and the critical duration, to guarantee the result at the expense of giving up the freedom of choosing the mortality assumption. This result supports our conjecture that an upper bound may not be necessary.

In all the work above, we assume the mortality status of the individual coincides with the mortality table used in pricing the life annuity. But the fact is when we are computing the expected ending estate at death, we should use the mortality distribution of the individual, which is usually different from the one used in pricing the life annuity. Insurance companies believe that healthy people are more likely to buy life annuities so that the mortality tables used in pricing recognize this anti-selection factor. Will this anti-selection strategy change the result we have? Probably not. Given the purchase rate of the life annuity an insurance company offers and the mortality distribution of the individual, there exists a certain valuation interst rate for that person, which is usually different from the AIR. If we replace the AIR by this valuation interest rate in all the formulas above, we keep all the results free from the impact of the anti-selection strategy. Though this valuation interest rate can't be calculated, it can be estimated. For example, an individual is looking for a life annuity. A insurance company offers a purchase rate 12 by using the AIR 4% and its own mortality table. Purchase rate 12 implies a valuation interest rate 3% given the health status of this person. Then we can apply this 3% as the i to all the results above.

5 Conclusion

The annuitization strategy and the self-annuitization strategy create different patterns of ending estates: the self-annuitization strategy creates more estate early while the annuitization strategy creates more estate later. To choose one of them is really a problem of personal perference. But if more average ending estate is desired, a retiree is better off choosing the annuitization strategy: investing all initial assets in a life annuity backed by the type of fund he prefers and reinvesting the disposable income in that fund, versus investing all assets in the fund directly. This is even before he considers the different tax treatment of annuities versus mutual funds or the longevity risk protection which annuitization offers. The only concern to the retiree is the spread insurance companies charge on the returns of variable annuities.

References

[1] ANDO, A. AND F. MODIGLIANI. 1963. "The Life Cycle Hypothesis of Saving." *American Econimic Review* 53(1):55-74

[2] BERNHEIM, B.D. 1991. "How Strong Are Bequest Motives: Evidence Based on Estimates of the Demand for Life Insurance and Annuities." *Journal of Political Economy* 99(5):899-927

[3] BOWERS, N., H. GERBER, J. HICKMAN, D. JONES, AND C. NESBITT. 1997. *Actuarial Mathematics.* SChaumburg, IL: Society of Actuaries.

[4] FRIEDMAN,B.M. AND M.J. WARSHAWSKY. 1990."The Cost of Annuities: Implications for Saving Behavior and Bequests." *Quarterly Journal of Economics* 105: 135-54.

[5] HURD, M. D. 1989. "Mortality Risk and Bequest." *Econometrica* 54(4):779-813.

[6] MILEVSKY, M. A. 1998. "Optimal Asset Allocation Towards the End of the Life Cycle: To Annuitize or Not to Annuitize?" *Journal of Risk and Insurance* 65(3):401-36.

[7] MILEVSKY, M. A. AND C. ROBINSON 2000. "Self-Annuitization and Ruin in Retirement" *North American Actuarial Journal* 4(10):112-23.

[8] MILEVSKY, M. A. 2001. "Optimal Annuitization Policies: Analysis of the Options" *North American Actuarial Journal* 5(1):57-69.

[9] MIRER, T. W. 1994."The Dissaving of Annuity Wealth and Marketable Wealth In Retirement." *Review of Income and Wealth* 40(1):87-97.

[10] MITCIIELL, O. S., J. M. POTERBA, M. J. WARSHAWSKY, AND J. R. BROWN. 1999. "New Evidence On the Money of Individual Annuities." *American Economic Review* 89(5):1299-1318.

[11] MODIGLIANI, P. 1986. "Life Cycle, Individual Thrift and the Wealth of Nations." *American Economic Review* 76(3):297-313.

[12] WARSHAWSKY, M. 1998. "Private Annuity Markets in the United States." *Journal of Risk and Insurance* 55(3):518-28.

[13] YAARI, M. E. 1965. "Uncertain Lifetime, Life Insurance and the Theory of the Consumer." *Review of Economic Studies* (April):137-50