

Stress and Resiliency Testing: Mandelbrotian Grey Swan Scenarios*

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Abstract

In “The Black Swan, The Impact of the Highly Improbable,” Taleb (2010) makes a distinction between extreme scenarios that can be modeled (Mandelbrotian Grey Swans) and those that cannot (Black Swans). A Grey Swan model would consider the power law fractal nature of the markets that Mandelbrot first described in the 1960s. In this paper, we discuss the generation of Mandelbrotian Grey Swan scenarios by using dependent multivariate fractional Brownian motion (DMFBM), as implemented from the methodology in “Basic properties of the Multivariate Fractional Brownian Motion” by Amblard et al. (2010).

We discuss how real world Grey Swan scenarios are excellent choices for stress and resiliency testing. In addition, we provide an example of a set of Grey Swan scenarios, which correspond to the RBC C-3 Phase II Wealth Factors of 2005 Bennet et al. (2006).

Additional advanced material has been added as an addendum.

Key words: Hurst Exponent, Fractional Brownian Motion, Black Swan, Grey Swan, Stress Testing, Wealth Factors, Multifractal Model of Asset Returns (MMAR).

1 Introduction

Since 2007, we have watched the instability of the derivatives and the associated insurance on the U.S. subprime mortgage market lead to the interna-

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tional financial meltdown. As economies have moved to greater globalization, they have become so tightly interdependent they are entangled. Mandelbrot and Hudson (2004) prophetically said:

“In a networked world, mayhem in one market spreads instantaneously to all others—and we only have the vaguest of notions how this happens, or how to regulate it.”

The management of risk during this entanglement is vastly more difficult, since the majority of the pre-2007 risk models are not adequate to model the interdependence of the market, let alone frequent and highly volatile events. The economic scenarios used are too tame, where the interdependence between the separate series was measured by static correlations and did not consider that the market correlations moved toward one as mayhem spreads. This is all obvious when we look backwards from our present vantage point. However, prior to the meltdown, risk managers had created models that best encapsulated their knowledge of a relatively stable market. Suddenly, the world as we knew changed forever, as we encountered in *The Black Swan*, Taleb (2010).

Though it is impossible to anticipate these Black Swan events, we still need to create economic scenarios that can be used in risk management. Taleb (2010) does distinguish between extreme scenarios that can be modeled (Mandelbrotian Grey Swans) and those that cannot (Black Swans). A Grey Swan scenario model considers the power law fractal nature of the markets that Mandelbrot first described in the 1960s. Mandelbrot’s models have evolved from simple Fractional Brownian Motion (FBM) models discussed in Mandelbrot and Van Ness (1968); Mandelbrot (1983) to more realistic market models as discussed in Mandelbrot et al. (1997); Mandelbrot and Hudson (2004).¹

1. These later models are made up of compound processes, where each compound process is the composition of an univariate FBM and a multifractal stochastic trading time process. This trading time process contracts and expands time, so that in a contracted time period, frequent market trading occurs with extreme volatility events rapidly occurring. However, in an expanded trading time period, the market trading becomes slow and sluggish with periods of low volatility. The univariate FBM introduces the fractal, long-term dependence to the model. In a collection of market indices, each index would have its own FBM model, and the stochastic trading time process is common to all indices. This interdependence of the indices comes only from the trading time process. In the past dependent multivariate fractional Brownian motion (DMFBM) models have been difficult to simulate, and this restriction is one reason for the emphasis on univariate FBM models.

With the crisis, the use of market models within ERM has forever changed. Consideration for extreme market interdependence with stress and resiliency testing have become paramount.

In this paper, we provide a means of generating Mandelbrotian Grey Swan scenarios through DMFBM by using the methodology in Amblard et al. (2010). Though these DMFBM models do not consider a stochastic trading time process, it does create a stepping stone to take existing dependent Brownian motion and normal copula market models into the realm of Mandelbrotian Grey Swan scenarios.

In Section 2, we discuss stress and resiliency testing, and in Section 2.1, how to extract severe scenarios from extreme scenario sets.

In Section 3, the use of Hurst exponents in the modeling of fractional Brownian motion and the methodology of Amblard et al. (2010) is outlined along with the introduction of the DMFBM R function.

Section 4 contains the data sources and descriptive statistics of the funds used to parameterize the model.

A discussion of the model construction is in Section 5, followed by a comparison of fund growth using the model scenarios against the RBC C3 Phase II Wealth Factors Bennet et al. (2006) scenarios in Section 6. Section 6.1 examines the impact of two means of extracting severe scenarios from an extreme scenario set.

Section 7 closes the paper with conclusions and future research.

The addendum includes results of additional research that has occurred since the original publication of this paper.

2 Financial Stress and Resiliency Testing

Much of the use of scenarios in ERM start from the implementation of regulatory and rating agency requirements around external reserve adequacy and risk-based capital. These scenarios could be deterministic, reflecting only initial market condition, used for stress testing. Other required tests could be around testing the sensitivity of a product or line of business to a small change in an assumption, such as mortality or lapse. As the use of market guarantees have increased as components of various products, full stochastic evaluations are now required in the determination of reserves and risk-based capital. The scenarios being used for these models may be prescribed or some leeway may be granted to allow for the use of internal company scenarios.

There are several benefits in the use of internal scenarios. A few of these are:

1. Strategic planning – Shell Oil was able to use strategic scenarios to great profit to determine how to position itself when oil supplies tightened before the 1973 Oil Crisis.
2. “What ifs?” – What happens to the enterprise’s bottom line, if there is a sudden drop in the market, or if market volatility dramatically increases?
3. Educational –
 - (a) Risk Appetite – Disclosed results from the use of extreme or severe scenarios can be used as an educational tool, to see how management reacts to extreme or severe events. The use of both extreme and severe sets of scenarios can aid in determining the best and extreme case corporate risk appetite.
 - (b) Interdependence – The use of severe or extreme dependent scenarios reveals the second level effects arising from the entangled networked market.

In the next section, we examine some techniques to reduce a set of extreme stochastic scenarios into a set of severe scenarios.

2.1 Extreme vs. Severe Scenarios

Extreme scenarios cannot be used in basic product pricing or in financial reporting, due to the fact that some scenarios are so extreme that they indicate that any governmental underpinning of the economy no longer exists. Though this is very possible, since there are very few governments that have existed longer than 200 years, societal stability is critical to allow proper business function to continue. Severe scenario sets should be designed so that societal “killer” scenarios are not possible. Also, other extreme scenarios may be industrial “killers,” where specific industries cease to exist. For example, U.S.-based scenarios that indicate that the NYSE fails would be excluded. Ultimately, there is no one indicator of when a scenario set is severe or extreme, since the consumers of the scenario sets have varying risk

appetites. Corporate culture is the final determining factor as to what is severe and extreme.

For the scenario designer, below are several methods on how to extract severe scenarios sets from extreme scenario sets.

1. Conduct cluster analysis on the extreme scenario set.
 - (a) Examine the membership size within each cluster. Clusters with small membership size will identify extreme scenarios.²
 - (b) The inspection for reasonableness on each cluster's representative scenario also aids in the determination of which clusters should be eliminated.
 - (c) This inspection will also point out how severe the interdependence is between separate indices. Candidates of removal may be scenarios where the behavior of one index dominates all of the other indices.
2. Remove all scenarios that indicate that the market of a specific index has completely failed. For instance, failure may be indicated when an index value drops below a specific target level, say 5 percent. Separate scenario collections may be created by varying the target level.
3. Conduct a simple test, such as a non-life contingent return of premium European option, on each separate scenario. For example, invest \$1 in each index at time 0, and examine the undiscounted total fund value at some future point in time. Based on the distribution of the resulting net amount at risk (NAR), discard scenarios where the NAR falls below a specific level. Similarly, remove extreme upside scenarios by eliminating scenarios where the total fund value exceeds a specific threshold.
4. Use existing corporate predictive or replicating portfolios models to quickly examine the effectiveness of a new scenario set. This can indicate if there is an over preponderance of corporate "killer" scenarios within the severe set.

2. To obtain a collection of stress test scenarios just use each cluster's representative scenario. If the number of stress scenarios exceeds 200 the collection may be used as scenario training sets for predictive modeling Craighead (2008) or replicating portfolio analysis.

In the next section, we will look at the key fractal measurement of market self-similarity and a public domain function that can be used to generate interdependent extreme scenarios.

3 Description of Fractal and DMFBM Methodology

The Hurst Exponent (H) is a fractal statistic that is collected on a time series. It indicates long-term dependency, self-similarity through a power law, roughness of a series, and the color of noise. An FBM process $B_H(t)$ is self-similar through a power rule, in how it rescales time t . For instance, for $a > 0$, $B_H(at)$ is equivalent in finite dimensional distributions to $a^H B_H(t)$.

Normally, $0 < H < 1$, but occasionally (especially in financial data) it can take values greater than 1. In these situations, the simple transformation, $H' = H - 1$, resolves this Anonymous (2004).

FBM in all forms produce continuous non-differential functions for $0 < H < 1$. For $H = 1/2$, the corresponding FBM has no memory of the past, and the FBM is just Brownian Motion and produces white noise. If $H < 1/2$, the corresponding FBM has a long-term memory effect where if there is an uptick, the next tick tends to be downward and vice versa. As $H \rightarrow 0$, this tendency increases. Because the innovations are constantly changing directions, the series will be very rough. For $H > 1/2$, an uptick will tend to be followed by another uptick (and vice versa). As $H \rightarrow 1$, this tendency increases. Since the ticks tend to be in the same direction, the resulting series will not be as rough as with lower values of H . For $H \geq .5$, the color of the noise darkens as $H \rightarrow 1$. See Mandelbrot (1983) for a further discussions regarding the noise color.

Examples of $H = 1/5$, $H = 1/2$ and $H = 4/5$ are in Figure 1. For each H , the first graph is a time series of the cumulated sum of the innovations and the second is a time series graph of the actual innovations. Notice how the graphs display the increase of smoothness as $H \rightarrow 1$.

Additional discussion around the determination of the Hurst exponent for the model is in Section 5.

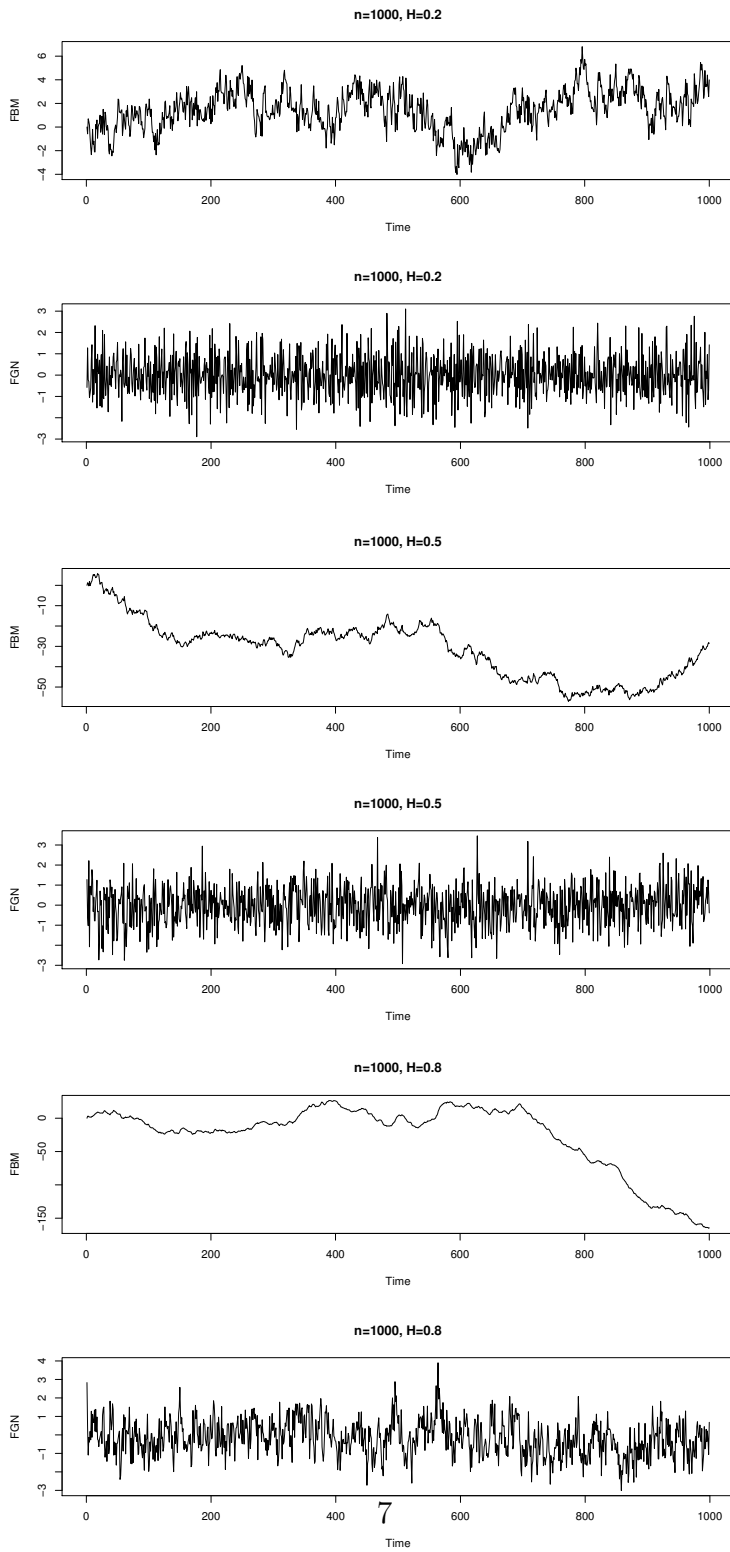


Figure 1: $H = 1/5$, $H = 1/2$, $H = 4/5$

3.1 The DMFBM function

The methodology of Amblard et al. (2010) implements a tensor approach to allow for correlation through time and across indices by the use of block matrices. They also use a technique by Wood and Chan (1994) that converts a multivariate normal simulation into a phase space problem by the use of the fast Fourier transform (FFT). Once transformed, simulations occur in phase space and the final results in the time domain are obtained by reusing the FFT.

For a full understanding of the DMFBM process, this part of the paper should be read in tandem with Amblard et al..

Let N_A denote the number of funds modeled in the DMFBM. For the model example in Section 5, $N_A = 6$.

The public domain DMFBM function is available at Craighead (2010). Its input is described in Table 1.

Parameter Name	Description
HurstList	A list of Hurst exponents, which are determined for each asset on the cumulated sum of the total returns.
covar	The covariance matrix of the price return portfolio.
correl	The correlation matrix of the price return portfolio.
projperiod (T)	The projection period associated with the scenarios. This defaults to 120 months.
numscen (n)	The number of scenarios generated. The default value is 1.
delta	This variable allows the change of modality. For example, if the price return portfolio has daily frequency, using $\text{delta} = 21$, the scenarios will be approximately monthly (there are approximately 21 working days a month).
GenKernel	Generation kernel. The default value is NULL. If this is supplied in subsequent runs, and if all of the parameters (other than n) are identical, then the DMFBM function will not regenerate the kernel and will proceed directly to scenario generation.

Table 1: DMFBM Calling Parameters

The output of DMFBM is outlined in Table 2.

Parameter Name	Description
cusumX (S_i)	The cumulative sum of the innovations, which corresponds to hydrological applications. The scenario array dimension is $n \times N_A \times (T + 1)$. $S_i = \sum_{j=0}^i r_j$ where $r_0 = 0$.
cuprodX (P_i)	The cumulative product of (1+innovations), which corresponds to return price indices. Its dimension is $n \times N_A \times (T + 1)$. $P_i = \prod_{j=0}^i (1 + r_j)$ where $r_0 = 0$.
DelX (r_i)	The individual innovations from the projection. Its dimension is $n \times N_A \times (T + 1)$. These innovations correspond to the total return series.
GenKernel	The Generation Kernel used in the projection process. This is a stored array of the circulant matrix discussed in Amblard et al. (2010). This array can be reused as input to DMFBM to generate additional scenario sets and reduce processing time.

Table 2: DMFBM Output Parameters

The DMFBM R function is internally documented by the use of internal R functions that implement the six steps in (Amblard et al., 2010, pages 14-15).

The list of internal functions are in Table 3.

Note: The indices i and j indicate separate funds.

Function Name	Description
nu(i,j)	This function should have been called eta. It corresponds to the $\eta_{i,j}$ causal anti-symmetrical term component on (Amblard et al., 2010, page 8). This implements the causal case.
nuhat(i,j)	This function models $\tilde{\eta}_{i,j}$, which is used when some $H_i + H_j = 1$. Again see (Amblard et al., 2010, page 8). Since the causal assumption creates undefined correlation, this function implements the well-balanced case.
w(i,j,h)	This function is a component function to the gamma(i,j,h,delta) function. This corresponds to $w_{i,j}(h)$ in (Amblard et al., 2010, Equation 5).
gamma(i,j,h,delta)	This is the cross-correlation of the increments of size δ of the components i and j . This corresponds to $\gamma_{i,j}(h, \delta)$ in (Amblard et al., 2010, Equation 6).
CapG(h,delta=1)	This is the $p \times p$ matrix $G(h) := (\gamma_{j,k}(h))_{j,k=1,\dots,p}$.

Function Name	Description
CapC(j,m,delta=1)	This is the block circulant matrix $C(j)$ in (Amblard et al., 2010, Equation 20).
CapB(k,m,delta=1)	This is $B_{u,v}(k)$ function in (Amblard et al., 2010, step 1).
CapBHat(j,m,delta=1)	This corresponds to the $\tilde{B}(j)$ function obtained from (Amblard et al., 2010, steps 2-3).
CapBHatMat	This is a matrix storing the results from CapBHat. This is the generation kernel GenKernel.
CapZ(j,m)	This corresponds to the multivariate normal simulation $Z(j)$ in (Amblard et al., 2010, step 4).
CapWMat	This is an internal matrix that corresponds to the $W(j)$ in (Amblard et al., 2010, step 4).
CapXMat	This is an internal matrix that corresponds to $\Delta X_u(k)$ in (Amblard et al., 2010, step 5).

Table 3: DMFBM Output Parameters

In the design and use of the DMFBM function two technical assumptions are made. These are:

1. The anti-symmetrical term η is assumed to be causal and not well-balanced (Amblard et al., 2010, page 8). Also, see Coeurjolly et al. (2010) for additional discussions around the anti-symmetrical parameter.
2. Negative eigenvalues were set to zero as the generation kernel is produced. This was done in a consistent manner as mentioned in (Amblard et al., 2010, Remark 11, page 15).

In the next section, we examine the sources and characteristics of the data used to parameterize the model.

4 Data

Based on a personal communication from Strommen (2010), the original source of the data for the parameterization of the 2005 RBC C3 Phase II Wealth Factors was from Ibbotson Associates (2006). The six funds that are modeled and their sources are in Table 4.

The data is on the total return performance for each fund; the data spans from January 1926 to December 2005. The associated basic statistics for the funds are in Table 5. Note that the estimated Hurst Exponent H for each

fund is also contained in this table. A discussion on how these are estimated is in the next section.

The static correlation matrix of the six funds' total returns as of December 2005 is in Table 6.

RBC C3 Fund Name	Ibbotson Series Name	Series ID
Small	U.S. Small Stk TR	IBOR003368
US	S&P 500 TR	IBOR003450
AGGR	Domestic Hi-Yld Corp TR	IBOR006433
LTCorp	U.S. LT Corp TR	IBOR001753
ITGvt	U.S. IT Gvt TR	IBOR001145
Money	U.S. 30 Day TBill	IBOR003667

Table 4: Ibbotson Funds and Sources

Fund	Min	q₁	\bar{x}	\tilde{x}	q₃	Max	s	H
Small	-0.3674	-0.0256	0.0134	0.0143	0.0500	0.7346	0.0849	0.6401
US	-0.2973	-0.0169	0.0098	0.0130	0.0397	0.4256	0.0556	0.5253
AGGR	-0.1705	-0.0068	0.0059	0.0068	0.0180	0.2294	0.0302	0.6374
LTCorp	-0.0890	-0.0030	0.0050	0.0040	0.0127	0.1376	0.0200	0.5672
ITGvt	-0.0641	-0.0007	0.0044	0.0027	0.0095	0.1198	0.0127	0.5783
Money	-0.0006	0.0009	0.0030	0.0027	0.0045	0.0135	0.0025	0.3029

Table 5: Fund Basic Statistics with Hurst Exponent

Fund	Small	US	AGGR	LTCorp	ITGvt	Money
Small	1.0000	0.8268	0.6991	0.1618	0.04378	-0.0435
US	0.8268	1.0000	0.6450	0.2053	0.1004	-0.0189
AGGR	0.6991	0.6450	1.0000	0.4192	0.2833	0.0111
LTCorp	0.1618	0.2053	0.4192	1.0000	0.7795	0.0915
ITGvt	0.0438	0.1004	0.2833	0.7795	1.0000	0.2124
Money	-0.0435	-0.0189	0.0111	0.0915	0.2124	1.0000

Table 6: Total Return Correlation Matrix

See Ibbotson Associates (2006) for further information on the descriptions and graphical displays of the funds.

5 Model Construction

To construct the model, several assumptions are made. These are:

1. For simplicity, the DMFBM RBC model will be a correlated noise model having no deterministic time component unlike the RBC C3 Phase II scenarios. This implies that the model scenarios do not mean revert. With no mean reversion, volatility will become infinite as the time horizon approaches infinity.
2. The entire total return history (from 1926 to 2005) of the six funds Ibbotson Associates (2006) will be used in the parameterization. The RBC C3 Phase II Wealth factors were parameterized from data from December 1955 to December 2005 (Bennet et al., 2006, page 7). Since the total returns during the Great Depression are included, the resultant scenarios have additional extreme behavior.
3. There will not be a term structure model for interest rates in the model, since we are only interested in modeling the equity components..
4. Inflation will not be modeled.
5. In the model the order of fund models are Small, US, AGGR, LTCorp, ITGvt and Money Market where the ordering is based on decreasing volatility. The order of generation is critical because if a low volatility fund is used first, the higher volatility funds will become very tame and produce counterintuitive results. There are two reasons why this occurs. The first is that all of the simulated prices are scaled to start at \$1 at time 0. Second, because of the scaling, the multivariate dependent normal simulations process used in the simulation step of DMFBM, the first fund has all of the variability, where the second fund's variability is reduced by correlation with the first fund. This cascade effect reduces each subsequent fund's variability. The implication of this assumption is that the model results in the Section 6 are mostly influenced by the Small fund's behavior.

Creating a DMFBM model of the six funds consists of first determining the variance and correlation statistics on the funds, as summarized in the prior section. The second step is to estimate each fund's Hurst exponent. Finally, the DMFBM R function Craighead (2010) is used in a two-step process, where first the generation kernel is created, and then the scenarios are produced. All subsequent modeling is done within the R environment R Development Core Team (2010).

The Hurst exponents for the funds are estimated by using the R function `dvFBM` from the `dvFBM` package, which was developed and maintained by Coeurjolly (2009). This function is an implementation of various estimation techniques for H in the presence of outliers by Achard and Coeurjolly (2010). Since estimations of H are made on attained levels of a time series and not on the changes in the values of the time series, each separate fund's total return time series $\{r_i\}$ is transformed into a cumulative sum time series $\{S_i\}$ by the formula $S_i = \sum_{j=1}^i r_j$. This time series is then processed within the `dvFBM` function using the "B1-ST" method.

Note: In the estimation of the Hurst exponent for the Money fund, H is greater than 1.0. The Hurst exponent that is used in the model construction is obtained by subtracting 1.0 from the estimated value.

First process the DMFBM function with the list of Hurst exponents, covariances correlations and $T = 360$, which produces the generation kernel. Second, the effectiveness of the generation kernel is tested by generating small scenario sets with small n . Finally, the final set of $n = 10,000$ scenarios is produced. Table 7 provides the basic statistics on the prices series P_i for each fund at 12, 24, 60, 120, 240 and 360 months across all 10,000 scenarios. Observe how the standard deviation increases through time for each fund. Note, also, how the minimum and maximum values of the funds expand through time as well, especially for the Small fund at 360 months. The median value stays near 1 for all of the funds at all times. This is due to the fact that there is no upward biased mean reversion in the model.

Figures 2 and 3 display scenarios 1 and 2 of the fund prices through the entire 30-year projection horizon.

Notice in Figure 2 that the Small and the U.S.funds climb then fall in the last 100 months, but the AGGR and the LTCorp funds have a stable trading range in that same 100-month time frame. The ITGvt fund first falls and then grows during that period. Since the Hurst exponent for the Money fund is less than 0.5, observe that the fund prices have greater volatility than the other funds.

Regarding Scenario 2 in Figure 3, all of the funds except Money decline over the 30 years. The LTCorp and ITGvt hold steady for the first 175 months and then sharply drop, while the Money fund sharply increases.

To examine how well the fractal nature is simulated by DMFBM, the `dvFBM` function estimates each fund's Hurst exponent for each scenario S_i . The basic statistics on each fund's H across the 10,000 scenarios are in Table 8. Note how the average H for each fund is accurate to two places when

compared to the original Hurst exponents in Table 5. In separate experiments with increased numbers of scenarios, the accuracy of the average Hurst exponent to the actual Hurst exponent improves as is expected. The variability, indicated by s and IQR , is very stable for each fund, but this does reveal that DMFBM creates a spectrum of Hurst exponents, that are both persistent ($H > 0.5$) and anti-persistent ($H < 0.5$), except for the Money fund which always is anti-persistent.

Examining the stability of the cross-correlation modeling in the model, the Pearson correlation of the funds from each scenario is produced from the r_i total returns. Each correlation in the upper matrix portion of the correlation matrix is then subtracted from the corresponding static correlation in Table 6 and then squared. These squared differences are then summed over all of the correlations and displayed in Table 9. Notice how average squared difference is 0.031 and the standard deviation s and IQR are stable at .02 and .023. The results of this table demonstrate that the cross-correlations are well modeled over the 10,000 scenarios.

In the next section, we use a simple return of premium feature on the fund values and observe how the results compares to the corresponding values from using the 10,000 RBC C3 Phase II Wealth Factor scenarios.

Fund	Months	Min	q₁	\tilde{x}	\bar{x}	q₃	Max	s	IQR
Small	12	0.605	0.922	0.999	1.005	1.081	1.577	0.117	0.158
Small	24	0.445	0.875	0.994	1.012	1.133	2.000	0.192	0.258
Small	60	0.275	0.775	0.985	1.049	1.250	3.776	0.389	0.475
Small	120	0.121	0.652	0.969	1.150	1.441	8.501	0.733	0.789
Small	240	0.025	0.481	0.922	1.476	1.800	42.811	1.813	1.318
Small	360	0.005	0.362	0.888	2.102	2.215	144.797	4.365	1.853
US	12	0.800	0.962	0.999	1.001	1.039	1.206	0.057	0.077
US	24	0.686	0.943	0.998	1.001	1.056	1.363	0.085	0.114
US	60	0.551	0.900	0.995	1.004	1.094	1.899	0.145	0.195
US	120	0.459	0.857	0.987	1.009	1.137	2.332	0.217	0.280
US	240	0.277	0.779	0.970	1.020	1.206	3.184	0.339	0.427
US	360	0.147	0.721	0.956	1.041	1.267	4.706	0.454	0.546
AGGR	12	0.833	0.972	1.000	1.001	1.028	1.160	0.042	0.056
AGGR	24	0.765	0.955	1.000	1.002	1.046	1.294	0.068	0.091
AGGR	60	0.622	0.918	1.000	1.008	1.092	1.601	0.128	0.174
AGGR	120	0.470	0.871	1.000	1.019	1.145	1.976	0.207	0.274
AGGR	240	0.275	0.790	0.994	1.047	1.249	4.378	0.355	0.458
AGGR	360	0.157	0.731	0.989	1.091	1.340	7.688	0.516	0.609

Fund	Months	Min	q₁	$\tilde{\bar{x}}$	\bar{x}	q₃	Max	s	IQR
LTCorp	12	0.921	0.985	1.000	1.000	1.016	1.086	0.023	0.031
LTCorp	24	0.866	0.977	1.000	1.001	1.024	1.151	0.035	0.047
LTCorp	60	0.766	0.959	1.000	1.002	1.042	1.300	0.062	0.083
LTCorp	120	0.678	0.937	1.000	1.003	1.065	1.393	0.095	0.127
LTCorp	240	0.561	0.902	0.998	1.008	1.102	1.825	0.150	0.200
LTCorp	360	0.473	0.871	0.998	1.014	1.137	2.117	0.201	0.266
ITGvt	12	0.940	0.990	1.000	1.000	1.010	1.050	0.015	0.020
ITGvt	24	0.915	0.985	1.000	1.000	1.016	1.089	0.023	0.031
ITGvt	60	0.856	0.972	1.000	1.001	1.028	1.205	0.041	0.055
ITGvt	120	0.786	0.958	1.000	1.001	1.043	1.259	0.064	0.085
ITGvt	240	0.684	0.933	0.999	1.003	1.069	1.448	0.101	0.135
ITGvt	360	0.605	0.911	0.999	1.006	1.092	1.658	0.135	0.181
Money	12	0.994	0.999	1.000	1.000	1.001	1.005	0.002	0.002
Money	24	0.993	0.999	1.000	1.000	1.001	1.007	0.002	0.003
Money	60	0.990	0.998	1.000	1.000	1.002	1.010	0.003	0.003
Money	120	0.988	0.998	1.000	1.000	1.002	1.016	0.003	0.004
Money	240	0.985	0.997	1.000	1.000	1.003	1.015	0.004	0.005
Money	360	0.983	0.997	1.000	1.000	1.003	1.018	0.005	0.006

Table 7: Fund Values Statistics through Time

Variable	Min	q₁	$\tilde{\bar{x}}$	\bar{x}	q₃	Max	s	IQR
Small	0.452	0.603	0.638	0.638	0.673	0.826	0.052	0.070
U.S.	0.332	0.488	0.523	0.523	0.557	0.703	0.051	0.069
AGGR	0.441	0.601	0.636	0.636	0.671	0.837	0.052	0.070
LTCorp	0.371	0.530	0.565	0.565	0.599	0.743	0.052	0.070
ITGvt	0.386	0.541	0.575	0.576	0.611	0.764	0.052	0.070
Money	0.138	0.270	0.300	0.301	0.331	0.480	0.045	0.060

Table 8: Simulated Funds Hurst Exponent Basic Statistics

Variable	Min	q₁	$\tilde{\bar{x}}$	\bar{x}	q₃	Max	s	IQR
SqDiff	0.002	0.017	0.025	0.031	0.040	0.202	0.020	0.023

Table 9: Sum of Squared Differences of Sample and Model Correlation

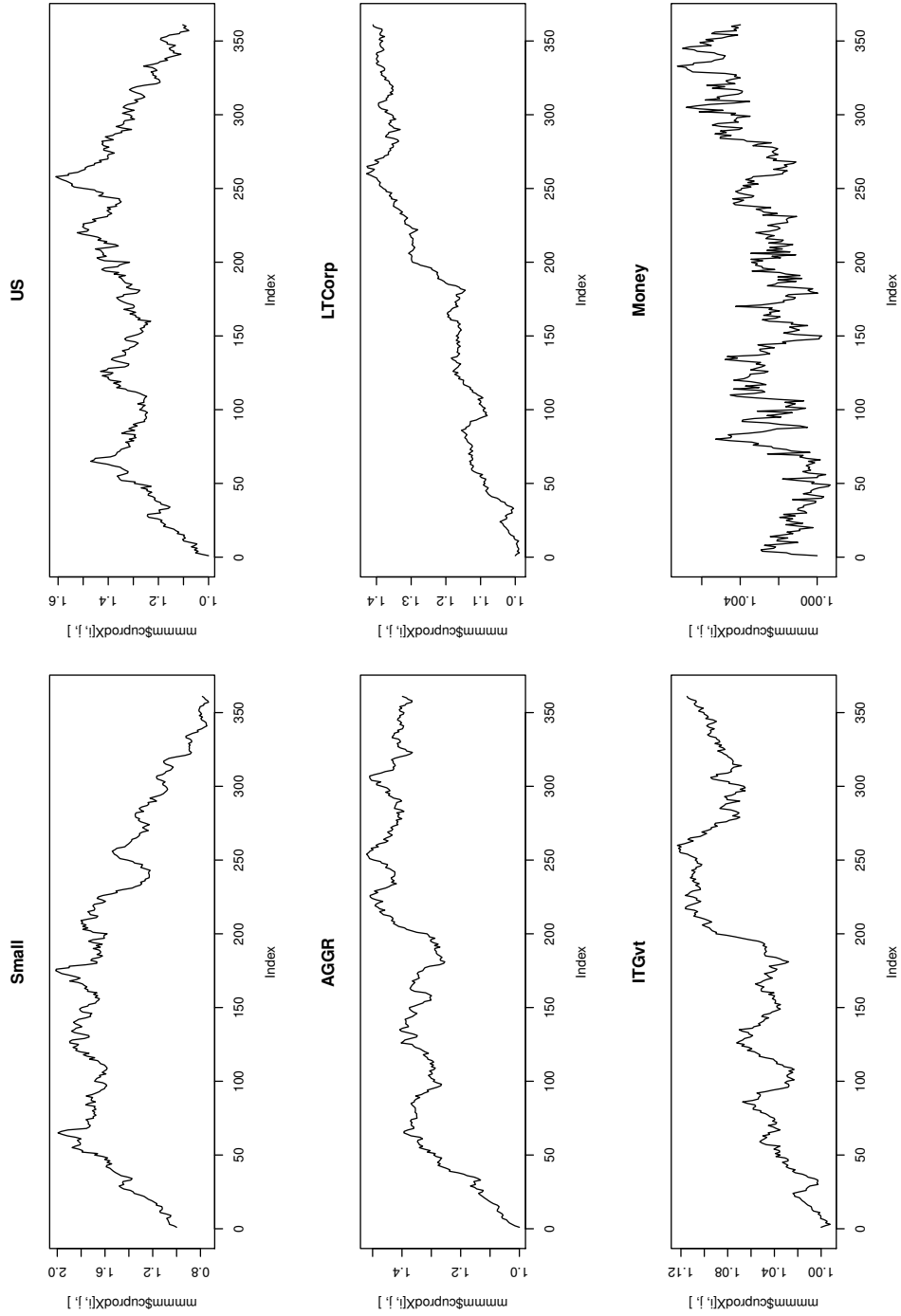


Figure 2: Scenario 1

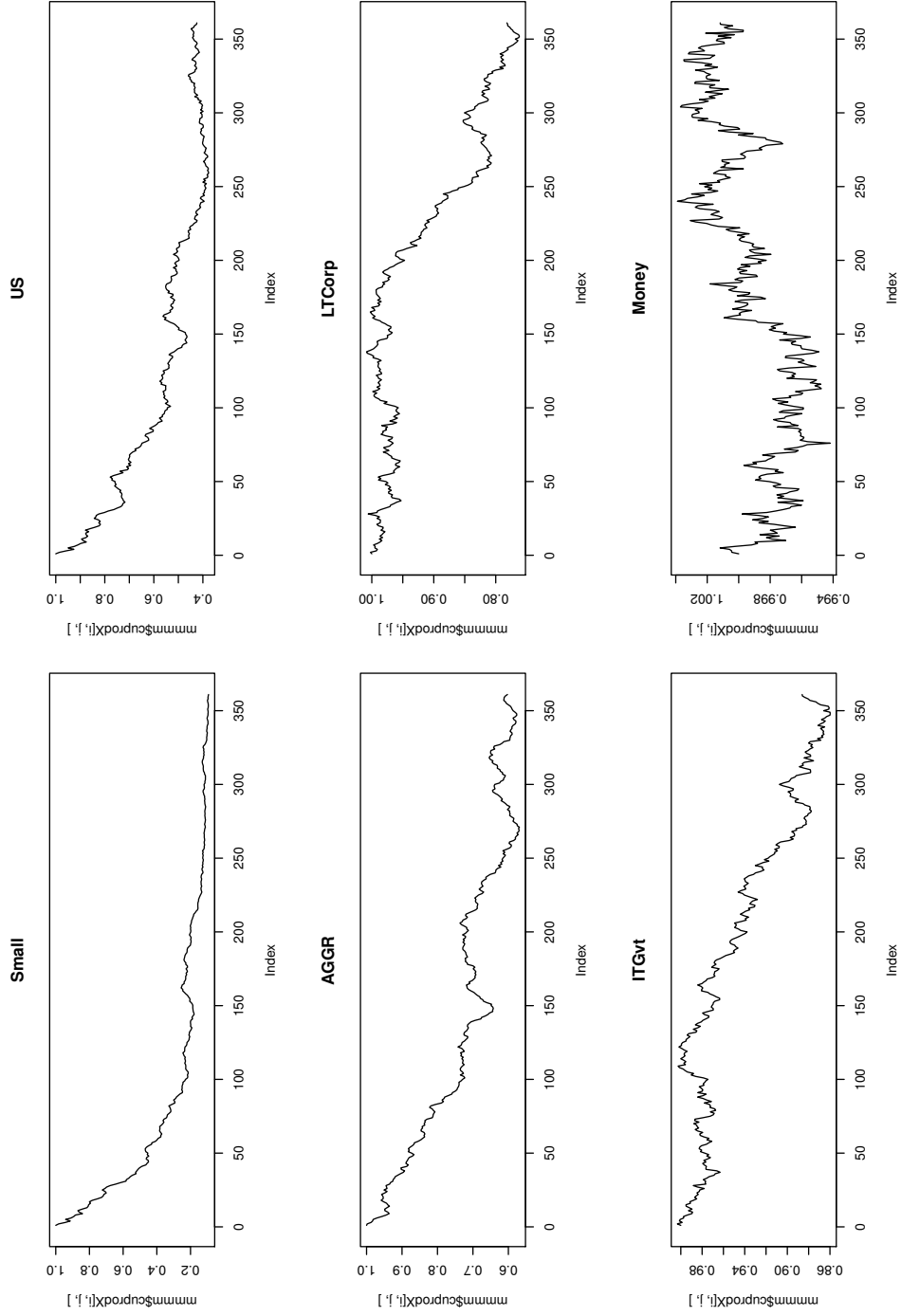


Figure 3: Scenario 2

6 Comparison between DMFBM and RBC C3 Phase II Scenarios

To make a simple comparison of the RBC C3 Phase II and DMFBM models, assume that \$1 is invested in each fund at time 0, with no additional deposits. Also, assume that there are no deaths or lapses. Below we examine the values of the aggregate fund portfolio arising from the DMFBM and RBC C3 Phase III Wealth Factors models. To accomplish this, first the P_i prices series for each scenario must be produced on each of the RBC Wealth Factors fund's total return. For both models, take each scenario, and sum each fund's P_{360} prices to produce the accumulated fund portfolio at time 360. This accumulated sum is not discounted.

Figure 4 is a graph of the probability density of the RBC accumulated fund values. The RBC statistics are in Table 10. Notice how the minimum value is \$8.34.

Figure 5 is the corresponding plot of the probability density of the DMFBM accumulated fund portfolio. The first row of Table 11 contains the statistics of the accumulated fund values. Note the minimum portfolio value of \$2.85.

If there is a return of premium guarantee on the accumulated fund, the net amount at risk (NAR) would be $\max\{6.00 - FundVal, 0.00\}$. For each RBC scenario, $NAR = 0.0$.

Figure 6 displays the probability density associated with the DMFBM NAR and its statistics are in the second row of Table 11. Note how the return of premium is in the money at $T = 360$ for 5,060 scenarios.

We can see from this simple example, the impact of how no mean reversion and extreme scenarios reveal a risk that is underpriced with the use of the RBC Wealth Factors.

Next we look at two examples of moving from extreme scenarios to severe scenarios as discussed in Section 2.1.

6.1 Extreme vs. Severe Scenario Analysis

The first approach is to conduct cluster analysis on the entire DMFBM scenario set. This is done in R using the `clara` function within the `cluster` package Maechler et al. (2005). The analysis consisted of using the Manhattan absolute difference metric to measure the closeness between the separate scenarios. A target of 100 separate clusters is made. The smallest resulting

cluster has only two scenarios. Table 11 displays the statistics on the accumulated fund portfolio value and the associated NAR by removing all scenarios associated with clusters less than or equal to specific sizes. Of course, size 0 has no scenarios removed, where size 50 has removed all scenarios belonging to clusters of size 50 and lower.

Observe the impact on the NAR statistics is almost unchanged, with the average NAR staying at 1.10, though the average and the standard deviation of the fund value does drop. It appears that this technique is only effective in removing scenarios with large fund values. The cluster analysis was conducted on the price, but should have been done instead on the total returns. If this was done, both extreme scenarios from both high- and low-prices would have been removed.

The second approach is to remove scenarios whose fund's values drop below a specific threshold at any time in the time horizon. This appears to be more effective regarding the return of premium option, where the average NAR values drops in Table 12 as scenarios are removed. Note how many scenarios are removed as the threshold percentage increases, where at 50 percent 4,431 scenarios have been removed. Observe that the average fund value increases, where the average NAR drops.

In the next section, we discuss our conclusions and future research and development ideas.

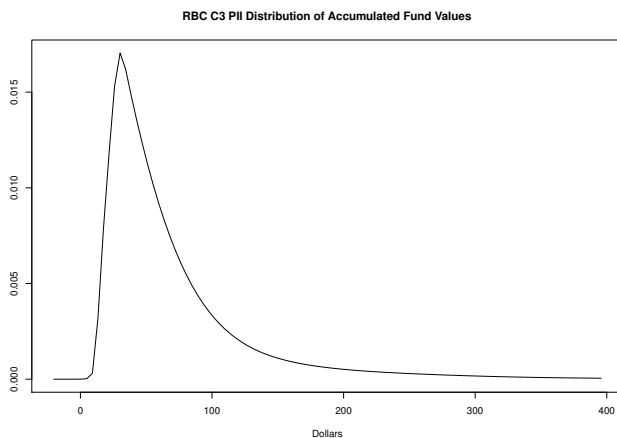


Figure 4: RBC Accumulated Fund Value Distribution

Variable	Min	q_1	\tilde{x}	\bar{x}	q_3	Max	s	IQR
FundValue	8.341	33.281	51.396	73.022	83.710	2733.694	81.065	50.429

Table 10: RBC C3 Phase II Accumulated Fund Values

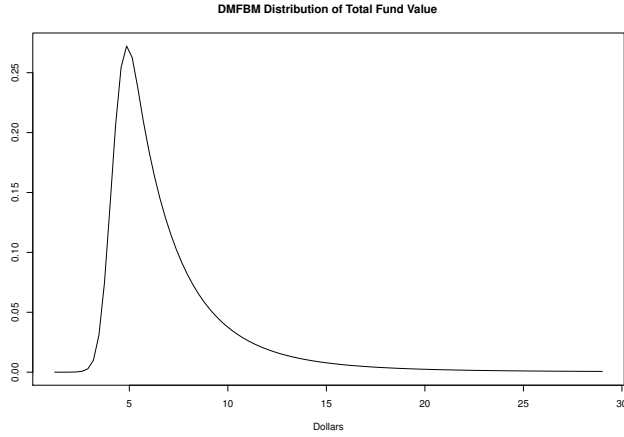


Figure 5: DMFBM Accumulated Fund Value Distribution

Size	Variable	n	Min	q_1	\tilde{x}	\bar{x}	q_3	Max	s	IQR
0	FundVal	10,000	2.853	4.922	5.974	7.264	7.861	155.760	5.062	2.939
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
2	FundVal	9998	2.853	4.922	5.973	7.263	7.860	155.760	5.062	2.938
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
3	FundVal	9989	2.853	4.922	5.971	7.248	7.853	155.760	5.034	2.931
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
4	FundVal	9985	2.853	4.921	5.971	7.242	7.848	155.760	5.025	2.926
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
5	FundVal	9970	2.853	4.919	5.968	7.218	7.833	155.760	4.973	2.914
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
10	FundVal	9845	2.853	4.908	5.940	7.066	7.716	155.760	4.667	2.808
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
15	FundVal	9648	2.853	4.892	5.890	6.807	7.563	56.392	3.459	2.671
	NAR	5,060	0.000	0.567	1.067	1.103	1.601	3.147	0.656	1.033
20	FundVal	9466	2.853	4.877	5.847	6.595	7.447	33.417	2.690	2.570
	NAR	5,059	0.000	0.568	1.067	1.103	1.601	3.147	0.656	1.033

Size	Variable	n	Min	q ₁	\tilde{x}	\bar{x}	q ₃	Max	s	IQR
30	FundVal	9073	2.853	4.841	5.758	6.302	7.172	19.667	2.145	2.330
	NAR	5054	0.000	0.570	1.068	1.104	1.601	3.147	0.656	1.032
40	FundVal	8940	2.853	4.831	5.725	6.212	7.089	18.947	1.976	2.258
	NAR	5053	0.000	0.570	1.068	1.104	1.602	3.147	0.656	1.032
50	FundVal	8899	2.853	4.824	5.717	6.180	7.062	18.947	1.918	2.238
	NAR	5053	0.000	0.570	1.068	1.104	1.602	3.147	0.656	1.032

Table 11: Cluster Removal Results on DMFBM Fund and NAR Statistics

Threshold	Variable	n	Min	q ₁	\tilde{x}	\bar{x}	q ₃	Max	s	IQR
1%	FundVal	9998	2.908	4.922	5.974	7.265	7.862	155.760	5.062	2.940
	NAR	5058	0.000	0.567	1.066	1.102	1.599	3.092	0.655	1.032
5%	FundVal	9833	2.971	4.975	6.015	7.322	7.908	155.760	5.084	2.932
	NAR	4893	0.000	0.547	1.029	1.067	1.549	3.029	0.631	1.002
10%	FundVal	9422	3.187	5.092	6.123	7.462	8.044	155.760	5.148	2.952
	NAR	4482	0.000	0.507	0.951	0.992	1.431	2.813	0.592	0.924
20%	FundVal	8480	3.187	5.339	6.402	7.790	8.403	155.760	5.324	3.064
	NAR	3547	0.000	0.423	0.816	0.858	1.234	2.813	0.537	0.810
30%	FundVal	7441	3.712	5.631	6.742	8.195	8.845	155.760	5.559	3.215
	NAR	2562	0.000	0.336	0.678	0.728	1.061	2.288	0.476	0.725
40%	FundVal	6534	4.079	5.912	7.065	8.597	9.303	155.760	5.812	3.391
	NAR	1785	0.000	0.282	0.581	0.636	0.928	1.921	0.432	0.647
50%	FundVal	5569	4.205	6.249	7.493	9.108	9.863	155.760	6.139	3.614
	NAR	1082	0.000	0.227	0.467	0.524	0.773	1.795	0.368	0.546

Table 12: Threshold Results on DMFBM Fund and NAR Statistics

7 Conclusions and Future Research

The use of DMFBM, as a source of creating scenario sets with interdependence between funds and a long-term dependence replicating the fractal nature of the market, shows great promise. Though DMFBM does produce scenarios with infinite variance, the use of these scenarios in risk analysis reveals actual risks that scenarios generated with mean reversion do not.

In addition, we observed the effectiveness of controlled scenario removal by either cluster analysis or fund thresholds. These methods allow the risk manager a means to dial their results from extreme to severe.

Though DMFBM has greater complexity than simpler dependent Brow-

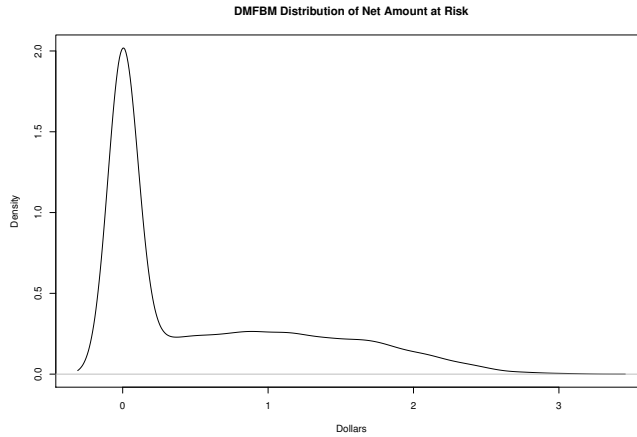


Figure 6: DMFBM Net Amount at Risk Distribution

nian models, it does create a bridge between these simpler models into the are of Mandelbrotian Grey Swan scenarios.

Below is a list of future improvements to the DMFBM model:

1. Revise the DMFBM model to allow for deterministic components such as drift and mean reversion. This would entail using the DelX output in tandem with a deterministic component. The addendum does examine the results of this using an R model of the AAA generator.
2. As understanding increases regarding the estimation of the anti-symmetric component η , the causal assumption for DMFBM may be replaced by an estimated parameter or a functional family in the DMFMB R function.
3. Create additional functions within the R language to implement the multifractal stochastic trading time approach Mandelbrot et al. (1997); Mandelbrot and Hudson (2004) with the DMFBM market model. Currently, this is used with univariate FBM to create market models with finite variance. By combining the stochastic trading time and DMFBM, greater market realism is attained. This methodology is implemented in the addendum.
4. Enrich the DMFBM process to model interest rate term structures in a fashion similar to Jamdee (2005).

5. Create a R package containing all of the related functions and methods associated with DMFBM to simplify user access to these modeling tools.

8 Addendum

Upon further analysis of the various papers by Mandelbrot, Fisher and Calvet, this addendum discusses the implementation of the Multifractal Model of Asset Returns (MMAR) Mandelbrot et al. (1997); Calvet and Fisher (2008) to DMFBM. Though the DMFBM model is a step toward scenario generation of Mandelbrotian Grey Swan scenarios, the addition of MMAR completes that process.

An MMAR model modifies an FBM model by allowing the FBM model behavior to have greater realism around volatility. FBM models without MMAR (or without other advanced models such as Markov Switching Multifractal Asset Return (MSM) models Calvet and Fisher (2004); Lux (2008)) exhibit unrealistic volatility structure over the entire time horizon. However in the market, asset returns reflect various trading ranges. There are times when the market is slow and the volatility is insignificant. At the other extreme, when the market is fast, the volatility changes frequently and it becomes difficult for market observers to keep up with what is happening. Also, during an extremely fast market, the market resets, correlations tighten and prices are discontinuous.

An MMAR model is a composition of an FBM $B_H(t)$ (or a DMFBM $B_{H_1, \dots, H_n}(t)$) with a cumulative distribution function $\theta(t)$ of a multifractal measure. The MMAR resulting process is $X(t) = B_H(\theta(t))$. The function $\theta(t)$ supplies a random trading time to the MMAR that models both slow and fast markets and the corresponding volatility.

The $\theta(t)$ function is also known as a cascade model. The next five paragraphs will describe the construction of some cascade models, by the use of the outer product of two vectors.

The outer product of two density functions (dfs) is similar to a convolution of the dfs.³ For instance, let $F = \{0.1, 0.9\}$ and $G = \{0.2, 0.8\}$ be two

3. Convolution of two probability density functions produce the probability density function of the sum of the individual random variables. For example, the probabilities associated with the sum of two dice is obtained from the convolution of probabilities associated with the throwing of one die by itself.

examples. The convolution of $F * G = \{0.2 * 0.1, 0.2 * 0.9 + 0.8 * 0.1, 0.8 * 0.9\} = \{0.02, 0.26, 0.72\}$. An outer product does not do any intermediate sums, but instead adds another element to the new df. For instance, the outer product would be $\{0.2 * 0.1, 0.2 * .09, 0.8 * 0.1, 0.8 * 0.9\} = \{.02, .18, .08, .72\}$.

A cascade model of base b and order k has k outer products with density functions of length b . For a deterministic binomial distribution example, let the df be fixed at $F = \{0.4, 0.6\}$. So, $b = 2$. A nil outer product ($k = 0$) would produce just the boring uniform df $\{1\}$. When $k = 1$, this is the outer product of $\{1\}$ and F , which is just F . When $k = 2$, the outer product of F with itself is $F^2 = \{0.16, 0.24, 0.24, .36\}$. When $k = 3$, the outer product would be $F^3 = \{0.064, .096, 0.096, 0.096, 0.144, 0.144, 0.144, 0.216\}$. The actual cascade model is the cumulative distribution function (cdf) of the final F^k outer product. This cdf is fractal in behavior. For instance, the graph of the cdf over the entire region $1 \dots b^k$ vs. the graph of the first $1 \dots b^{k-1}$, the graphs will appear to be identical, which demonstrates scale invariance. Figure 7 demonstrates this scale invariance.

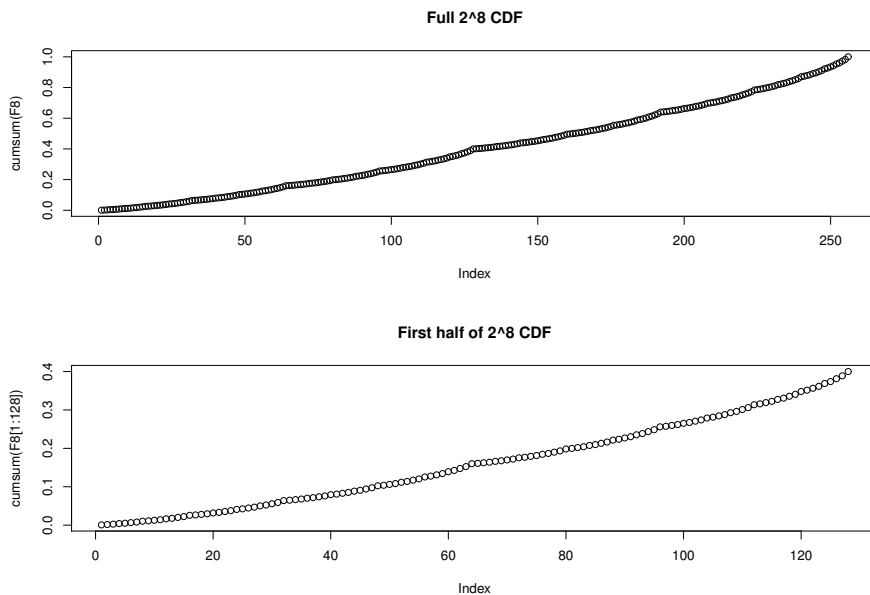


Figure 7: Binomial Distribution Cascade

The binomial distribution example above is deterministic, however, if the

df F randomly changes at each outer loop calculation, the resulting cascade is much more complex in nature. This type of cascade model, though still fractal in nature, does not have scale invariance, since the df changes at each level.

All of the subsequent MMAR examples use either a univariate 9^4 lognormal cascade model or a 9^4 multivariate correlated lognormal cascade model. This is because at each level, nine new samples are drawn from the lognormal distribution with $\mu = 0$, and $\sigma^2 = 1$. The nine samples are normalized to ensure that a density function is being used. Next, the outer product of the prior level's results and the new normalized samples produce the new level's df. Figure 8 is a graph of one random 9^4 cdf. This process has been implemented in the Logcascade R function and is available at Craighead (2011c). This R function is a good template to use in the creation of other random univariate cascade models.

The multivariate correlated lognormal cascade allows for correlation effects to be added between each of the states. For instance assume that you have three assets (or states) that are modeled with DMFBM $B_{H_1, H_2, H_3}(t)$. If the trading volatility of each corresponds to that of the other two states, you could just use the same lognormal cascade model $\theta(t)$ for the MMAR calculation of each asset. However, if each asset has a different trading volatility structure, you may want to use a separate lognormal cascade model for each asset. If b^k is the same for each asset, then you could use correlated lognormal draws in the construction of each of the separate lognormal cascades. This type of model could be denoted as $B_{H_1, H_2, H_3}(\theta_1(t), \theta_2(t), \theta_3(t))$. The corresponding R function, Correllogcascade, is available at Craighead (2011a).

Once, the cascade cdf has been constructed, the composition of the FBM and the cascade produces the MMAR $X(t)$. To do this, first the scale multiplier S is determined, which is b^k divided by the projection period p (360 months for our examples in Section 8.1). Now, for each time t in $1..p$ transform it into $t' = \theta(tS) * p$. Let $w_1 = \theta(tS)$ and $w_2 = 1 - w_1$. Now, $X(t) = w_1 * B_H(\lfloor t' \rfloor) + w_2 * B_H(\lceil t' \rceil)$. This $X(t)$ will have long memory correlations arising from H of $B_H(t)$ and $X(t)$ exhibits both slow and fast markets. This composition is implemented in the two R functions SingleMMAR and CorrelMultiMMAR.⁴ These R functions use the corresponding Logcascade and Correllogcascade functions from above. The two MMAR functions are

4. These functions are based on a R conversion and modification of the MATLAB mmarm.m function written by Wengert (2010).

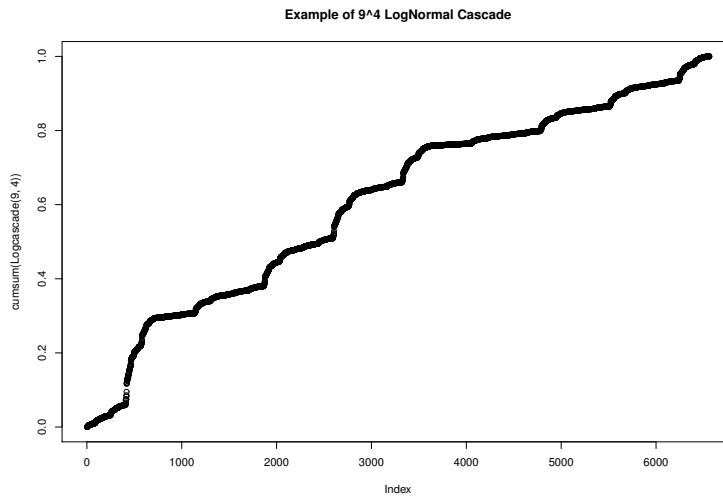


Figure 8: Lognormal Distribution Cascade

available at Craighead (2011d,b).

In the next section, the American Academy of Actuaries Scenario Generator model is used to compare the use of Brownian Motion, DMFBM, and MMAR models.

8.1 AAA

Using a R function that replicates the 2010 Interest Rate and Wealth Factors workbook of the Academy of Actuaries generator Strommen et al. (2010), each type of noise model was used to produce 10,000 scenarios. These noise models are:

1. Brownian Motion - Generated by DMFBM with all Hurst exponents set to 0.5. The correlation matrix used corresponds to that of the academy's generator.
2. DMFBM - This uses the academy's correlation matrix and the Hurst exponents from Table 5.
3. Brownian Motion - Single-MMAR applied to the Brownian Motion model from item 1.

4. DMFBM - Single-MMAR applied to the DMFBM model from item 2.
5. Brownian Motion - Multi-MMAR applied to the Brownian Motion model from item 1.
6. DMFBM - Multi-MMAR applied to the DMFBM model from item 2.

To ensure that each noise model has the same random deviates, the same random seed is used when the DMFBM function generates both the Brownian Motion and the DMFBM models. Notice in Figure 9 how the Scenario 1 Small fund returns have similar patterns for both the Brownian Model and the DMFBM model. But, also note that the Brownian model fund returns are noisier than that of the DMFBM. This is evidence that the extra restrictions being placed on the DMFBM model by the use of Hurst exponents tends to reduce the overall noise level.

Again for consistency between models, a common random number seed is set before creating the four Single- and Multi-MMAR models. Observe in Figure 10 the impact of using the Single Lognormal Cascade on the Scenario 1 Small fund return series. Notice that the market is very slow the majority of the time, but there are sudden bursts of activity when the market is fast. It is interesting that in the DMFBM model there is a severe downward spike early in the series, but the corresponding spike in the Brownian model is not prominent. Notice that in the middle of the series, this behavior switches.

Though the Multi-MMAR model has the capability to use correlations between the separate lognormal cascade models, the covariance matrix is set as the identity matrix. Observe in Figure 11 the use of the Correlated Lognormal Cascade. The fast periods correspond between the two separate models, but the Brownian Model actually has greater volatility. The full impact of the use of the Multi-MMAR approach can only be observed when comparing two separate funds against the same funds when using Single-MMAR models. This is because the Single-MMAR assumes that one $\theta(t)$ applies to all funds, where Multi-MMAR has separate $\theta_i(t)$ for each fund. However, the additional random draws required by the Multi-MMAR does not allow a clear one-to-one comparison with Single-MMAR scenarios.

8.2 Portfolio Comparison

As in Section 6, comparison portfolios are created by investing \$1 in each of the same six funds. A portfolio is built for each model constructed in

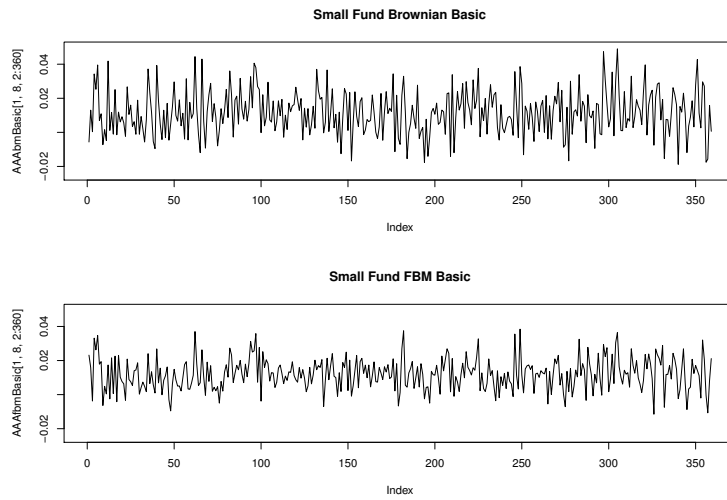


Figure 9: Simple Small Fund Scenario 1 Returns

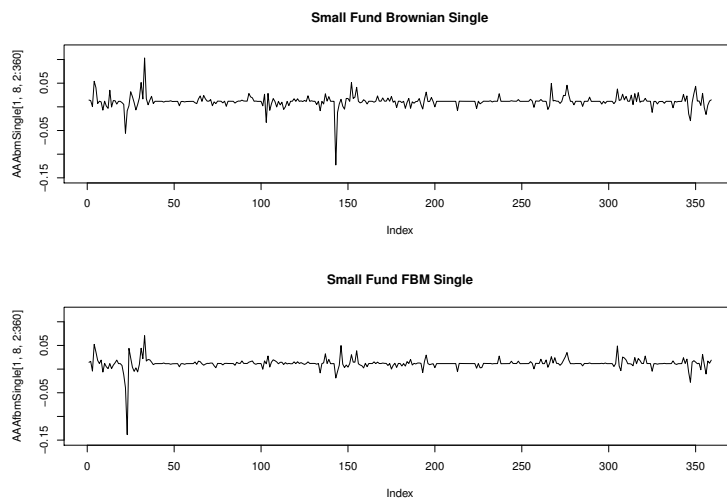


Figure 10: Single-MMAR Small Fund Scenario 1 Returns

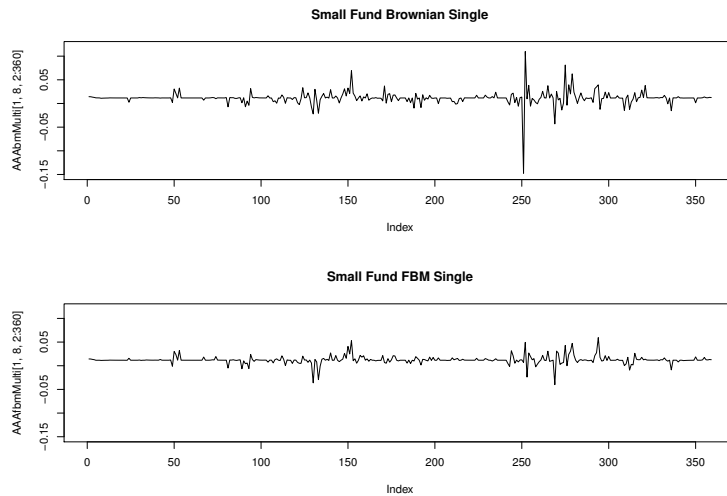


Figure 11: Multi-MMAR Small Fund Scenario 1 Returns

Section 8.1. Again the models are listed with the corresponding portfolio names here:

1. Brownian Motion Basic - BMBas
2. DMFBM Basic - FBMBas
3. Brownian Motion Single MMAR - BMSgl
4. DMFBM Single MMAR - FBMSgl
5. Brownian Motion Multi MMAR - BMMlt
6. DMFBM Multi MMAR - FBMMlt

Basic statistics on the portfolios are in Table 13. Figure 12 displays the box plots associated with each portfolio. The removal of the high outlier in the BMMlt portfolio results in Figure 13. Notice how that the interquartile range are very similar between all of the portfolios. Observations from both the statistics and Figure 13 reveal that the interquartile range of FBMBas and FBMSgl portfolios are slightly wider than the corresponding BMBas and BMSgl portfolios. What is interesting is that the BMSgl and BMMlt results

have outliers lower than the FBMSgl and FBMMlt portfolios. The last four portfolios demonstrate that the return of premium option from Section 6 do go in-the-money where BMBas and FBMBas do not, which is consistent with the results in Section 6.

Variable	Min	q ₁	\bar{x}	\tilde{x}	q ₃	Max	IQR	s
BMBas	11.101	16.060	18.487	17.870	20.194	42.293	4.134	3.510
FBMBas	9.772	15.216	17.820	17.110	19.509	49.239	4.293	3.856
BMSgl	1.477	16.171	18.242	17.776	19.699	52.868	3.529	3.238
FBMSgl	1.772	15.402	17.634	17.095	19.094	49.668	3.691	3.613
BMMlt	1.615	16.278	18.319	17.851	19.736	130.651	3.457	3.280
FBMMlt	3.879	16.048	18.293	17.708	19.845	51.015	3.797	3.514

Table 13: Basic Statistics of the 30-year values of the Six Portfolio Series

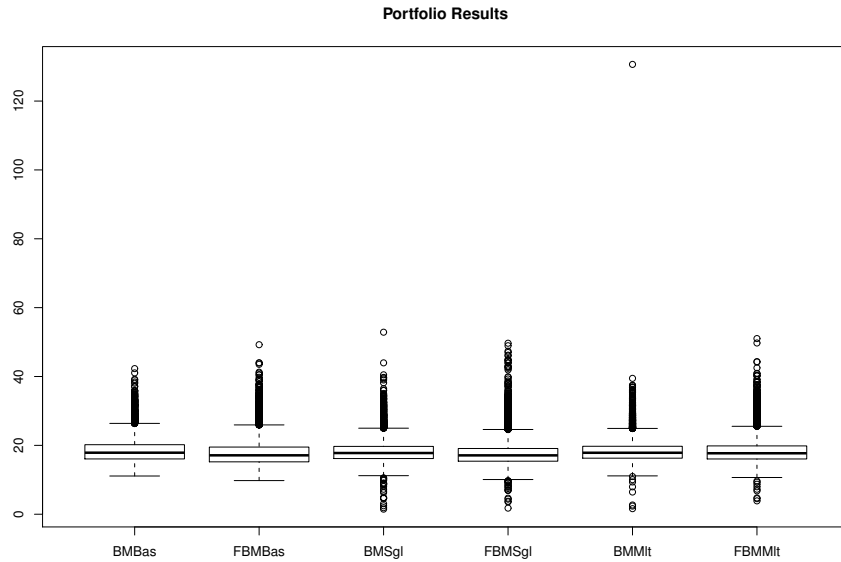


Figure 12: Portfolios Box Plots

8.3 Comments and Future Work

The use of MMAR in the modeling of market models completes the implementation of Mandelbrotian Grey Swan scenario generation. A important observation is that when using just Brownian Motion models combined with MMAR that portfolios with low fund values are obtained. So, the addition of MMAR reveals that a return of premium option is not free.

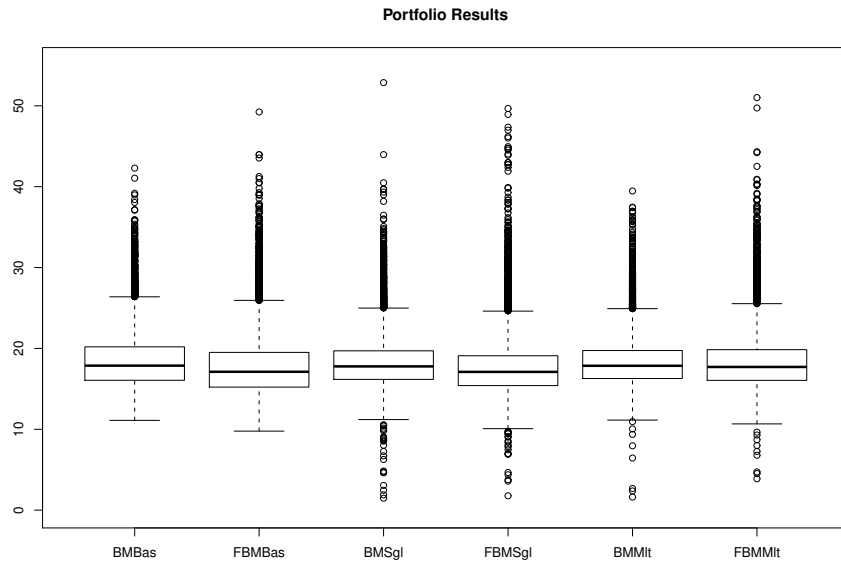


Figure 13: Portfolios BoxPlots Outlier Removed

This addendum has not discussed the proper parameterization of the various MMAR models, nor has it discussed the determination of the proper b^k in the lognormal cascade models. This addendum used 9^4 arbitrarily for the base and level of the cascade models and all lognormal draws have been based on a mean of 0 and a variance of 1. Primarily the models in this addendum were created as a proof of concept and there has been no attempt to ensure that the MMAR parameters correspond to actual market conditions.

The parameterization of MMAR models is discussed in Mandelbrot (1997) and in Calvet and Fisher (2008), and in many other publicly available papers. There is a need to create R functions to aid in the determination of the parameters.

Additional areas for development are:

1. Create an R function that models correlated Multi-MMAR, that allow the correlations to go to 1 at times of extreme volatility.
2. Develop other cascade R functions using various statistical distributions.

3. Develop copula cascade R functions.
4. Create an R function that models Markov-Switching Multi-fractal asset returns.
5. Create R functions to aid in the parameterization and calibration of MMAR or MSM models.

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