

# Longevity Risk and Regular Discount Sequence

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## **Abstract**

Life expectancy for males and females in many countries has been increased significantly since the turn of the 20<sup>th</sup> century. The elderly are expected to have the largest mortality improvements in the future, and the study of elderly mortality has received a lot of attention recently. However, because there were not enough elderly data before 1990, it is still unknown whether there is a reliable (stochastic) mortality law that can solve the problem of longevity risk.

In this study, we adapt the idea of regular discount sequence in the bandit problem, and use it to interpret life expectancy, as well as to develop a model for survival probabilities. We found that many frequently used mortality models, such as the Gompertz law and the Coale-Kisker model, and famous mortality assumption (uniform distribution of death, constant force and hyperbolic assumption) all satisfy the requirement of regular discount sequence. It seems that the discount sequence provides another possibility for the mortality models.

We conducted an empirical analysis for the data from the Human Mortality Database, including Japan, the United States, England and Wales, Sweden, France and Taiwan, and found that the discount sequences of life expectancy and mortality ratio do satisfy the regularity condition. In addition, we use the Brownian motion stochastic differential equation to model the discount sequence to predict the future mortality rates and life expectancy. The simulation study shows some promising results. We expect the proposed approach can provide a possibility for predicting mortality rates for the elderly.

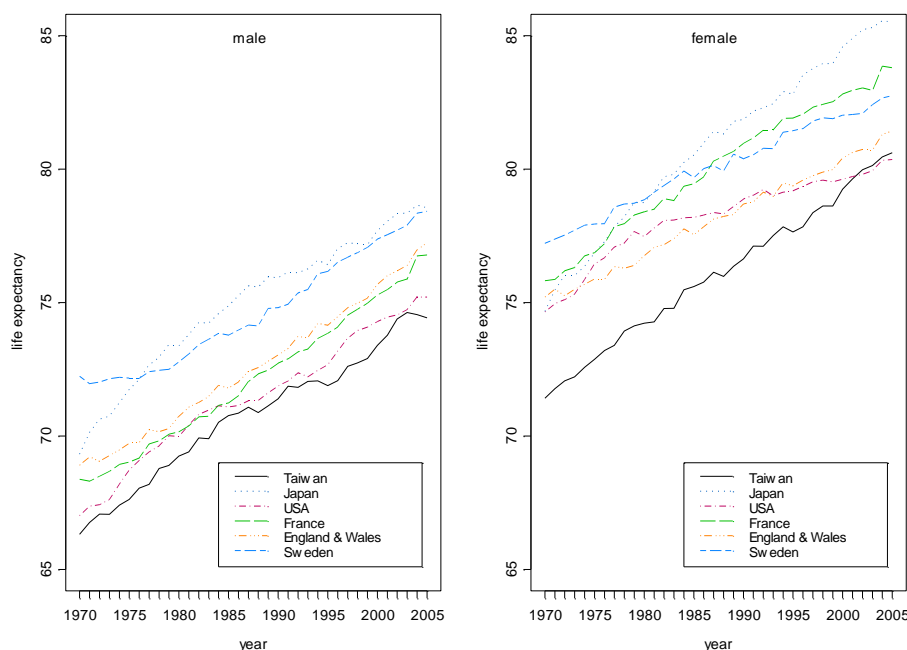
In insurance product pricing, Taiwan insurance companies at present use the Taiwan Standard Ordinary Experience Mortality Table of the static approach in calculating the traditional insurance expenses. This causes overestimating the insurance expense for life insurance products and underestimating the risk of annuity products. We apply our proposed model, with Monte Carlo simulation, to build the mortality confidence interval to evaluate if the current practice adequacy in insurance companies. In addition, we also explore if there is an overestimate or underestimate premium in different insurance products or possible natural hedging strategy.

**Key words:** Bandit Problem, Brownian Motion, Longevity Risk, Monte Carlo Simulation, Mortality Improvement

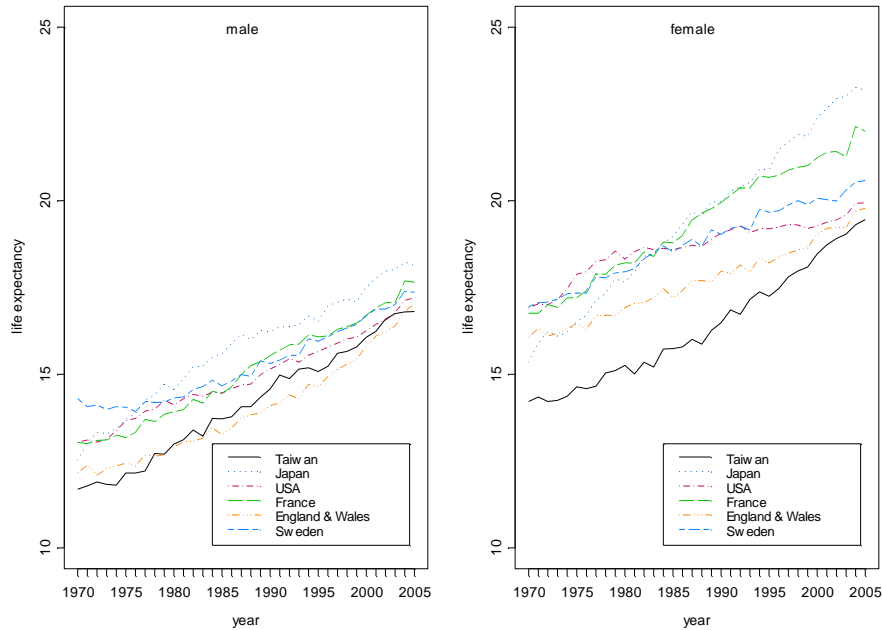
## I. Introduction

Due to economical and medical progresses, life expectancy in Taiwan has reached a historic high and now is about at the same level as other developed countries. Figures 1 and 2 show the life expectancies at ages 0 and 65 for Japan, the United States, England and Wales, Sweden, France and Taiwan, from 1970 to 2005. We can see that Taiwan's life expectancies are approaching those of the developed countries and Taiwan's female life expectancy has the largest growth. (Human Mortality Database [HMD] from University of California-Berkeley) As a result of increasing life expectancy, the proportion of people 65 and older in Taiwan increases significantly. It reached 7 percent for the first time in 1993 and surpassed 10.5 percent by the end of 2009. According to the official population projection in Taiwan (Council for Economic Planning and Development [CEPD] "Population Projections for Taiwan Areas: 2010-2060"), the proportion will be more than 20 percent in 2025 and 30 percent in 2039. The population aging in Taiwan is even faster than in Japan. It is impossible to ignore the influence of population aging in Taiwan.

**Figure 1**  
**Life expectancy in six countries at the age 0, 1970-2005**



**Figure 2**  
**Life expectancy in six countries at the age of 65, 1970-2005**



Low fertility is the other cause of an aging population in Taiwan. According to the Ministry of the Interior in Taiwan, the total fertility rate was about 1.1 between 2006 and 2009. This is about the lowest in the world, comparing to 1.34 in Japan. Traditionally, care of the elderly in Taiwan relies on family support. The low fertility rate makes it impossible for family support to be the only option for the elderly. Thus, social welfare and social insurance have become more and more important in Taiwan. However, to guarantee the financial solvency of these social systems, we need to handle the mortality forecast with care. For this reason, we want to build mortality models that can provide solutions for dealing with longevity risk in Taiwan.

Among the mortality models, the De Moivre's survival model in 1729, Gompertz's model in 1825, Makeham's model in 1860, Weibull's model in 1939, and Heligman and Pollard's model in 1980, can be defined as nondynamic deterministic models. These models consider gender and age as variables but do not include the mortality improvement over time. As an alternative, stochastic mortality models have been proposed, which include a time-varying factor. The well-known Lee-Carter model (Lee and Carter 1992) is an example of a stochastic model that has time and age variables and where the trend of mortality is an approximated linear function of time. The model receives a lot of attention because it is easy to use and has good accuracy.

Several modifications to the Lee-Carter model have been proposed. For example, Renshaw and Haberman (2006) incorporated a generation (or age group) factor, in addition to the time and age variables. Another modification is via the variable reduction, such as principal component analysis (PCA) (Yang et al. 2010; Chen and Cox 2009) and functional PCA (van der Linde 2008) because the concept of the Lee-Carter model is to consider one principal component. In addition to using more than one principal component, a time-varying parameter in Lee-Carter's model, geometric Brownian motion and compound Poisson process have been used to predict future mortality rates. There are other modifications. Cairns et al. (2006a; 2006b), for example, considered that the mortality model is the same as a yield model that can be described by average tendency, slope and curvature and the parameters from both models are affected by time. They proposed a short-term mortality model, a long-term mortality model, a market mortality model and a two-factor stochastic mortality model.

All of the above modifications were made to provide better methods for projecting mortality. However, every country faces a different situation; for example, we lack sufficient and good quality data in Taiwan. Thus, the Gompertz law is often assumed for the elderly mortality rates but it produces very conservative mortality rates, especially for the oldest-old. It is also not easy to apply the popular Lee-Carter model in Taiwan due to the data availability. For example, the highest age group of mortality rates recorded is 95 to 99 and this can only be traced back to 1990. No official data for people 100 and older are available, and the ultimate age for life tables in Taiwan is usually 100. To solve the problem of modeling mortality for the elderly in Taiwan, we propose a new approach in this study. We will adapt the idea of the discount sequence in the bandit problem and use it to develop a mortality model.

This paper is organized as follows: In Section II, we introduce the discount sequences and their connection with mortality rates. We use the data of Japan, the United States, England and Wales, Sweden, France and Taiwan from the HMD to verify the behavior of empirical discount sequence in Section III. The variables considered include the ratios of the life expectancy, death number and number of survivors. In Section IV, we apply Brownian motion to the discount sequences and to the forecasted mortality via Monte Carlo simulation. Section V contains conclusions and recommendations.

## II. Discount Sequences and Mortality

The discount sequence is from the bandit problem (choosing a rule to maximize the expected payoff) and the idea is similar to that of life expectancy, which will be shown later. Berry and Fristedt (1985) proved that if the discount sequences possess regularity, the optimal decision can be made on the one-armed bandit problem. The discount sequences possess regularity when the sequences have an increasing failure rate. This paper will use these sequences to describe the character of the survival function. First, we introduce the discount sequences and then apply these sequences to the mortality model and assumption.

### 1. Introduction of discount sequences

In the bandit problem, Berry and Fristedt (1979) defined that a discount sequence  $(\alpha_1, \alpha_2, \dots)$  is regular, i.e., for each  $n$ ,  $\gamma_n = \sum_{i=n}^{\infty} \alpha_i$  satisfies the following function:

$$\gamma_n \cdot \gamma_{n+2} \leq (\gamma_{n+1})^2. \quad (1)$$

To establish the relationship of regular discount sequence and mortality, we regard  $N$  as the survival time  $T$  and assume  $\gamma_n = P(N \geq n)$  as the survival function  $S(x) = P(T \geq x)$ . Then function (1) equals to the following function (2).

$$l_n \cdot l_{n+2} \leq (l_{n+1})^2 \quad \text{or} \quad \frac{l_n \cdot l_{n+2}}{(l_{n+1})^2} \leq 1, \quad (2)$$

where  $l_n$  is the number of survivors at age  $n$  in the life table. If we take a radix  $l_0$ , usually 100,000, to calculate the number of surviving  $l_n$ . The equation (2) is equivalent to  $\frac{p_{n+1}}{p_n} \leq 1$  or  $q_{n+1} \geq q_n$ , and it means that the survival time satisfies the regularity is equivalent to the change in mortality trend when age is nondecreasing.

Since the discrete life expectancy at age  $x$  satisfies  $e_x = \sum_{k=1}^{\infty} k p_x$ ,  $S(x)$  equals to  $\mathcal{V}_x$  and is nondecreasing. Under the stationary population assumption and  $\mathcal{V}_n \geq \mathcal{V}_{n+1}$ , i.e., condition of regularity on function (1), we can use life expectancy as replacement. Then function (1) equals to function (3).

$$e_n \cdot e_{n+2} \leq (e_{n+1})^2. \quad (3)$$

Furthermore, we can check the following functions to see if they are tenable.

$$e_n \cdot e_{n+2} \leq (e_{n+1})^2 \quad \text{or} \quad \frac{e_n \cdot e_{n+2}}{(e_{n+1})^2} \leq 1 \quad (4)$$

$$d_n \cdot d_{n+2} \leq (d_{n+1})^2 \quad \text{or} \quad \frac{d_n \cdot d_{n+2}}{(d_{n+1})^2} \leq 1 \quad (5)$$

$$q_n \cdot q_{n+2} \leq (q_{n+1})^2 \quad \text{or} \quad \frac{q_n \cdot q_{n+2}}{(q_{n+1})^2} \leq 1 \quad (6)$$

Note that, although life expectancy, number of death, the surviving number and mortality rate can be used to verify the regularity condition, we need to construct life tables. Empirically, if the population is stationary, we suggest using the mortality rate in (6). Since there is not much elderly population, we expect the results computed via (6) would have greater volatility.

## 2. The verification of the mortality models and mortality assumptions

### (1) Gompertz model

The Gompertz law is the famous elder population mortality model:

$$\mu_x = BC^x, \quad B > 0, C > 1, \quad (7)$$

where  $x$  is the age and  $\mu_x$  is the force of mortality.

Using the survival probability combined with the Gompertz law, we can get

$${}_t p_x \equiv P(T > x+t | T > x) = \exp\left(-\int_0^t \mu_{x+s} ds\right) = \exp\left(-\frac{BC^x}{\log(C)}(C^t - 1)\right).$$

Therefore, the Gompertz Law is equivalent to the following function (8).

$$\frac{\log(P_{x+1})}{\log(P_x)} = C \quad (8)$$

If we use the central death rate  $m_x$  to replace  $\mu_x$ ,

then  $k_{x+1} \equiv \log(m_{x+1}/m_x) = \log(C)$  is a constant. Since  $\mu_x = BC^x$  with  $C > 1$ , we

can get  $\gamma_n = {}_n p_0 = \exp\left(-\frac{B}{\log(C)}(C^n - 1)\right)$  and

$$\frac{\gamma_n \cdot \gamma_{n+2}}{(\gamma_{n+1})^2} = \exp\left(-\frac{BC^n}{\log(C)}(C^2 - 2C + 1)\right) = \exp\left(-\frac{BC^n}{\log(C)}(C-1)^2\right) < 1. \quad \text{That is, if the}$$

mortality defers to Gompertz law, then the function (1) is established.

(2) Coale-Kisker (CK) model

CK model is an extended model of the Gompertz model. It assumes

$$m_x = m_{65} \cdot \exp\left(-\sum_{y=66}^x k_y\right), \quad x = 66, 67, \dots, \quad (9)$$

where  $k_{x+1}$  may not be a constant. Brown (1997) introduced a model similar to the CK model to construct the U.S. 1989-91 life table. For the population

older than 94, they use the mortality ratio  $\frac{q_{x+1}}{q_x} = 1.05$  (male) or 1.06 (female)

to explain the higher age mortality. That means  $q_n \cdot q_{n+2} = (q_{n+1})^2$  and thus satisfies the requirement of function (6).

(3) The uniform distribution of death (UDD) assumption

If mortality under the UDD assumption, i.e., any  $t, m, 0 \leq t \leq m$ ,  $l_n$  satisfies

$$l_{n+t} = \frac{m-t}{m} \cdot l_n + \frac{t}{m} \cdot l_{n+m}, \text{ then function (2) is equivalent to } \frac{l_{n+2}/l_{n+1}}{l_{n+1}/l_n} \leq 1, \text{ or}$$

$P_{n+1} \leq P_n$ . That is, the UDD assumption satisfied function (2).

(4) Constant force (CF) assumption

We postulate the force of mortality is a constant, i.e., every age  $x$ ,  $\mu_x = \mu$ .

$$\text{Then, } {}_n P_x = e^{-n\mu} = \frac{l_{x+n}}{l_x} \text{ and } \frac{l_n \cdot l_{n+2}}{(l_{n+1})^2} = \frac{e^{-(2n+2)\mu}}{e^{-(2n+2)\mu}} = 1. \text{ That is, the CF}$$

postulation satisfied function (2).

(5) Hyperbolic assumption

If mortality satisfy hyperbolic assumption, i.e., any  $t, m, 0 \leq t \leq m$ ,  $l_n$  satisfies

$$\frac{m}{l_{n+t}} = \frac{m-t}{l_n} + \frac{t}{l_{n+m}}, \text{ or } l_n \cdot l_{n+2} = l_{n+2} \cdot \frac{l_n + l_{n+2}}{2}, \text{ then, similar to the UDD}$$

assumption, we can get  $l_{n+1} \geq \frac{l_n + l_{n+2}}{2}$  through calculation. Therefore, the

hyperbolic assumption also satisfies  $l_n \cdot l_{n+2} = l_{n+1} \cdot \frac{l_n + l_{n+2}}{2} \leq (l_{n+1})^2$ .

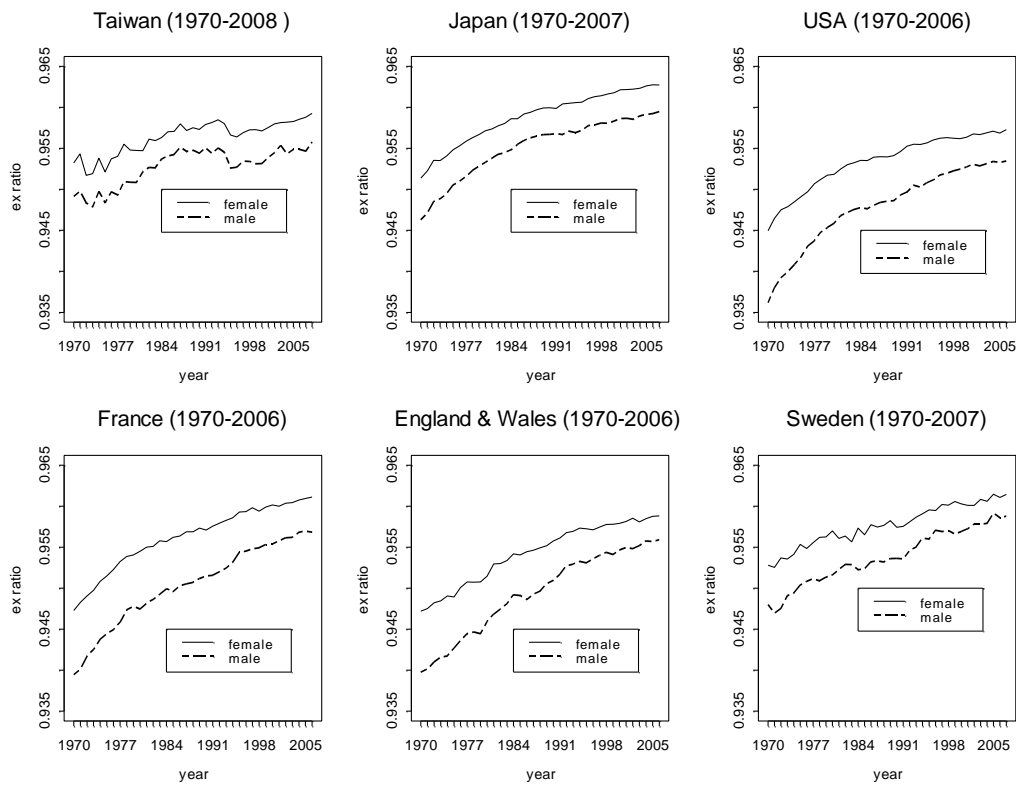


We've checked different mortality assumptions, including Gompertz and other models, and they all satisfied the regularity  $\frac{l_n \cdot l_{n+2}}{(l_{n+1})^2} \leq 1$  or  $p_{n+1} \leq p_n$ . Therefore, this character can be treated as a limit to the mortality improvement at all ages. We then use the experience data and continue to test the proposed method in the next section.

## Empirical Data Analysis and Discovery

We will use the empirical data to evaluate the regularity condition in this section. We use the life table data of Japan, the United States, England and Wales, Sweden, France and Taiwan from the Human Mortality Database (HMD) of the University of California-Berkeley. The variables considered for checking the regularity are males' and females' life expectancy, mortality rate, death number and number of survivors. We shall check the ratios of these variables for adjacent ages, by first transforming empirical data into the discount sequence ratio. Also, to have updated results, we will only use the data since 1970.

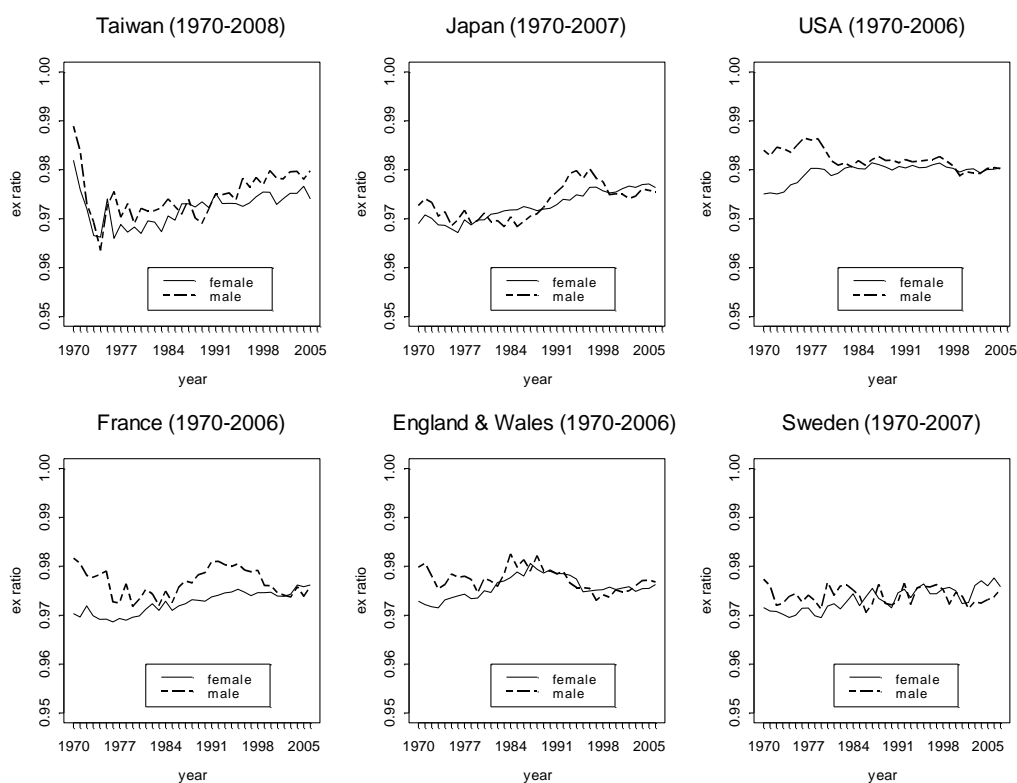
**Figure 3**  
**Life expectancy discount sequence ratios at the ages of 0, 1 to 4, and 5 to 9**



We will only show the results of five-year age groups in this paper because their trends are more obvious. We first show the results of the life expectancy discount sequence. The results of life expectancy for the age groups (0, 1 to 4, 5 to 9) and (55 to 59, 60 to 64, 65 to 69) are shown in figures 3 and 4 as a demonstration. The discount sequence ratios for the age groups 0, 1 to 4, and 5 to 9 are all less than 1 and approaching 1 (Figure 3). The ratios of the female are always larger than those of the male. It seems that the regularity condition is satisfied in all six countries. Also, the results of Taiwan and Sweden have the largest fluctuations, probably due to smaller populations. The fluctuations in Taiwan are especially obvious and it might be because Taiwan has the largest increase in life expectancy since 1970.

**Figure 4**

**Life expectancy discount sequence ratio at the ages of 55 to 59, 60 to 64, and 65 to 69**

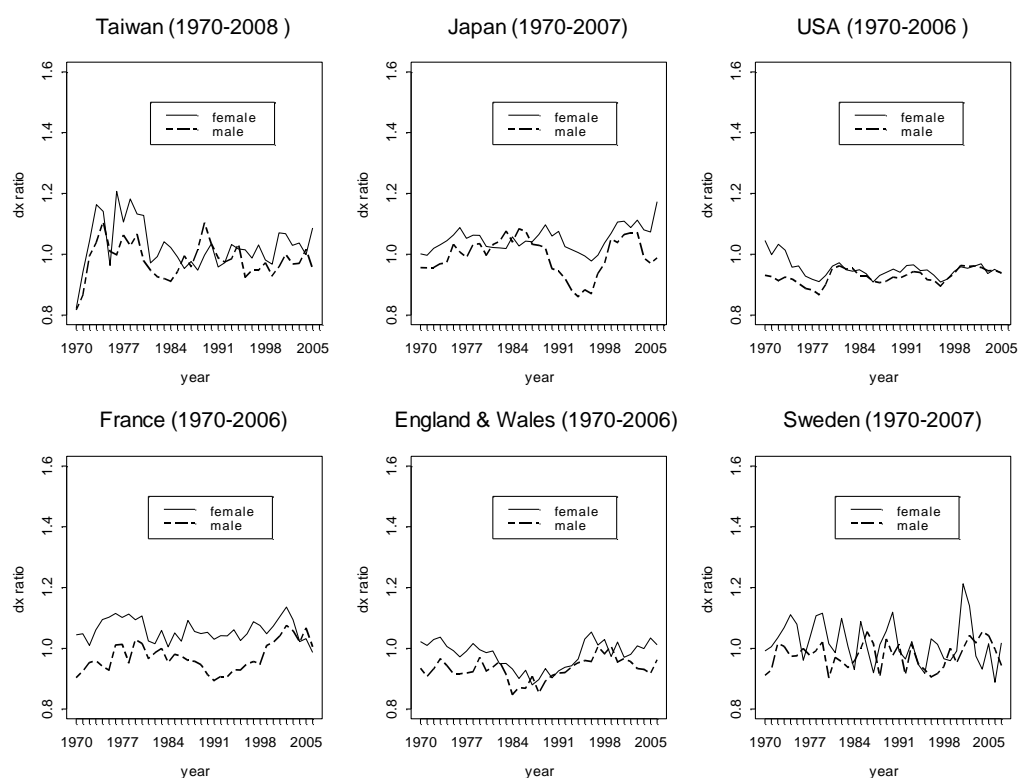


The ratios of life expectancy become more stable as the ages increase, and they are more volatile for ages 80 and older due to fewer populations. As we can see from comparing Figure 3 to Figure 4, the ratios are all smaller than 1 but the increasing trend of the ratios at ages 0, 1 to 4, and 5 to 9 are much more obvious. Also, the ratios of life expectancy at ages 55 to 59, 60 to 64, and 65 to 69 (Figure 4) lie around the value 0.97 and 0.98. The ratios of the United States are almost constant, and those of Taiwan and Japan have the most obvious increasing trend.

Next, we look at the ratios of death number and mortality rate. The ratios of death number have larger oscillations, comparing to those of life expectancy. The ratios at ages 55 to 59, 60 to 64, and 65 to 69 can be used to evaluate the differences. Comparing figures 4 and 5, it is obvious that ratios of death number are less stable and they show much larger fluctuations, even in the cases of Japan and the United States. The ratios of Taiwan and Sweden again have the largest variations, due to a smaller population. Also, the ratios are not always smaller than 1, although on average the ratios are very close to 1.

**Figure 5**

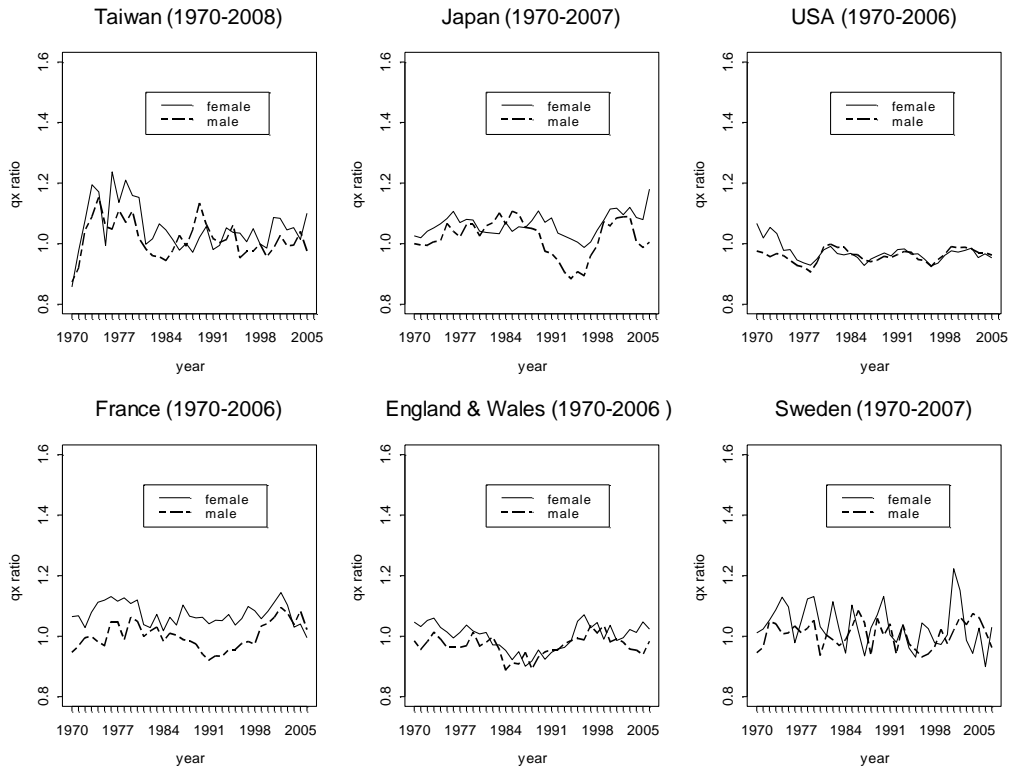
**Death number discount sequence ratios at the ages of 55 to 59, 60 to 64, and 65 to 69**



Just like the ratios of death number, the ratios of mortality rates have a similar behavior and they show larger fluctuations. Again, the ratios are not always smaller than 1. As the observed ratios of death number and mortality have variations, the results from figures 4 and 5 do not violate the regularity condition of discount sequence. But it seems that using the ratios of life expectancy is easier to obtain the regularity condition.

**Figure 6**

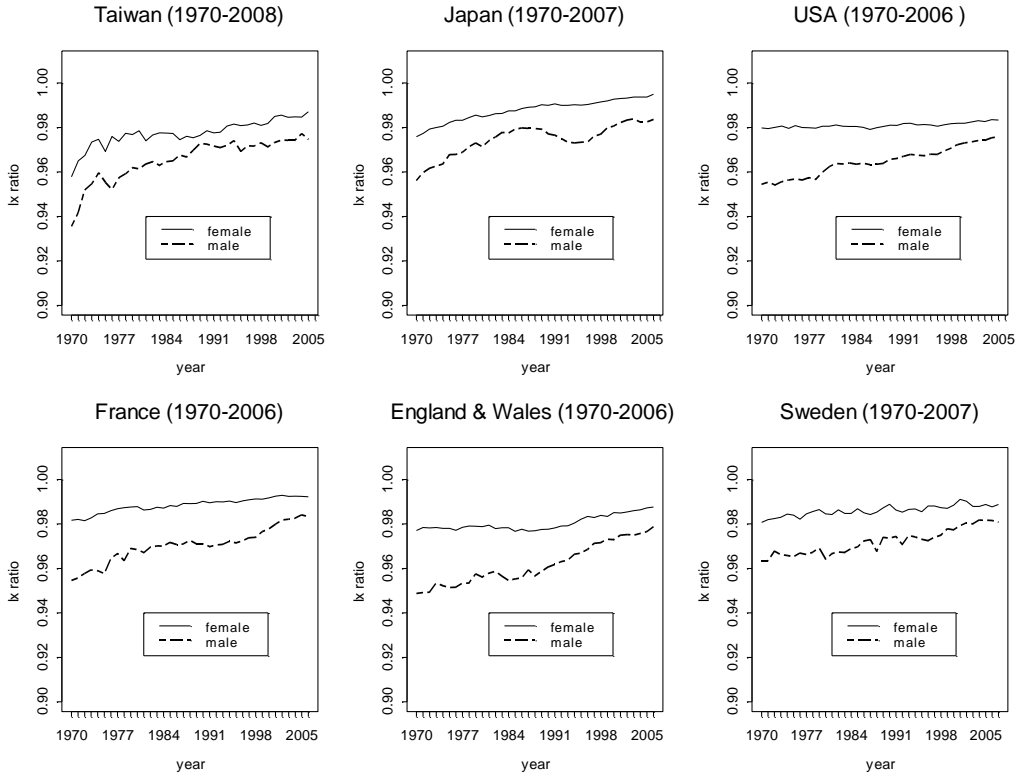
**Mortality rate discount sequence ratios at the ages of 55 to 59, 60 to 64, and 65 to 69**



The ratios of number of survivors show similar patterns as those of life expectancy, and we use the ratios at ages 55 to 59, 60 to 64, and 65 to 69 as a check (Figure 7). The ratios of survivor numbers are always smaller than 1, supporting the regularity condition, and show an increasing trend. The countries with fewer populations, Taiwan and Sweden, have larger fluctuations (or less smooth curves). The ratios of the female are generally larger than those of the male. Further, from the ratios of survivor numbers, we can observe another result: The differences of ratio values between males and females become smaller, especially for the United States, France, and England and Wales.

**Figure 7**

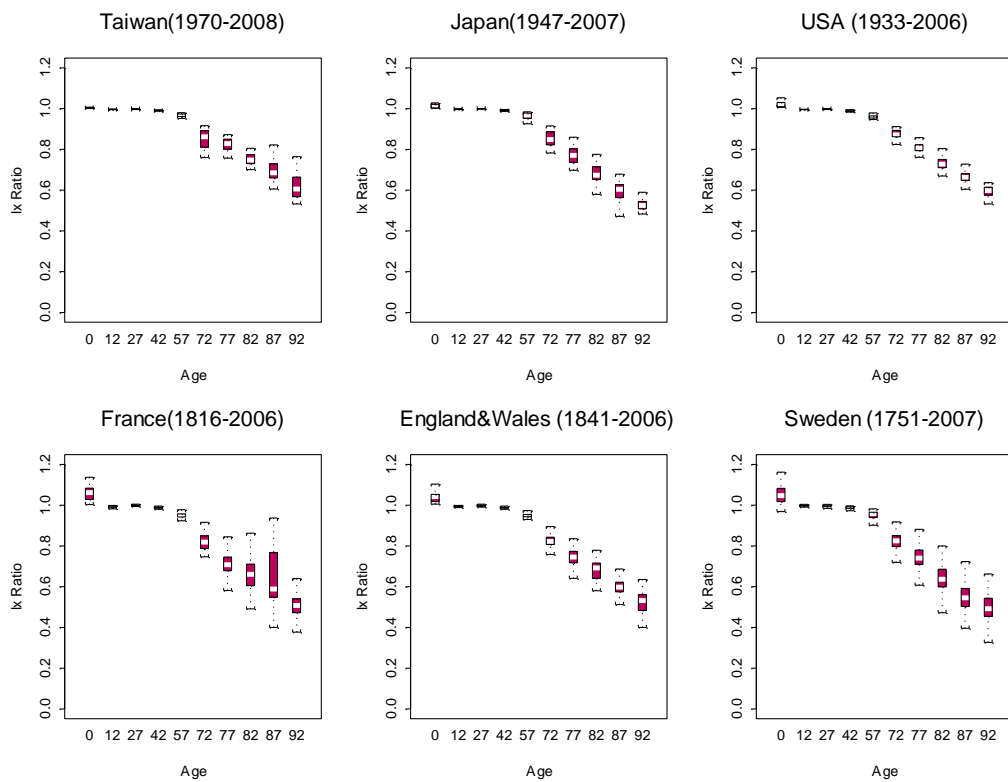
**Survivor number discount sequence ratio at the ages of 55 to 59, 60 to 64, and 65 to 69**



In the following discussions, we will continue exploring the properties of ratios of survivor numbers since they show more information. As the results of both sexes are similar, we will only show those of the male. The boxplots of the ratios of male survivor numbers are shown in Figure 8. All six countries show a common trend, which is that the centers of boxes (or the medians) decrease exponentially as the age increases. This phenomenon tells us the breadth of the mortality improvement at the elder groups and thus we can calculate the mortality or life expectancy at different ages. For example, suppose  $\frac{l_{x+2}/l_{x+1}}{l_{x+1}/l_x} = 0.5 \leq 1$ . This suggests that  $\frac{l_{x+2}}{l_{x+1}} = 0.5 \frac{l_{x+1}}{l_x}$  is  $q_{x+1} = 0.5 + 0.5q_x$ . When we know the mortality rate of age  $x$ , the mortality rate age  $x+1$  can be calculated accordingly.

Another observation from Figure 8 is that the median of boxplot at the aged. We can see that the lower bound of the ratios for the United States and Taiwan are about 0.6, while it is about 0.5 for the other four countries. The life expectancies of the United States and Taiwan are smaller, and this might suggest that there is a connection with the lower bound and life expectancy. We should continue exploring their relationship.

**Figure 8**  
**Boxplot of survivor number discount sequence ratios (males)**



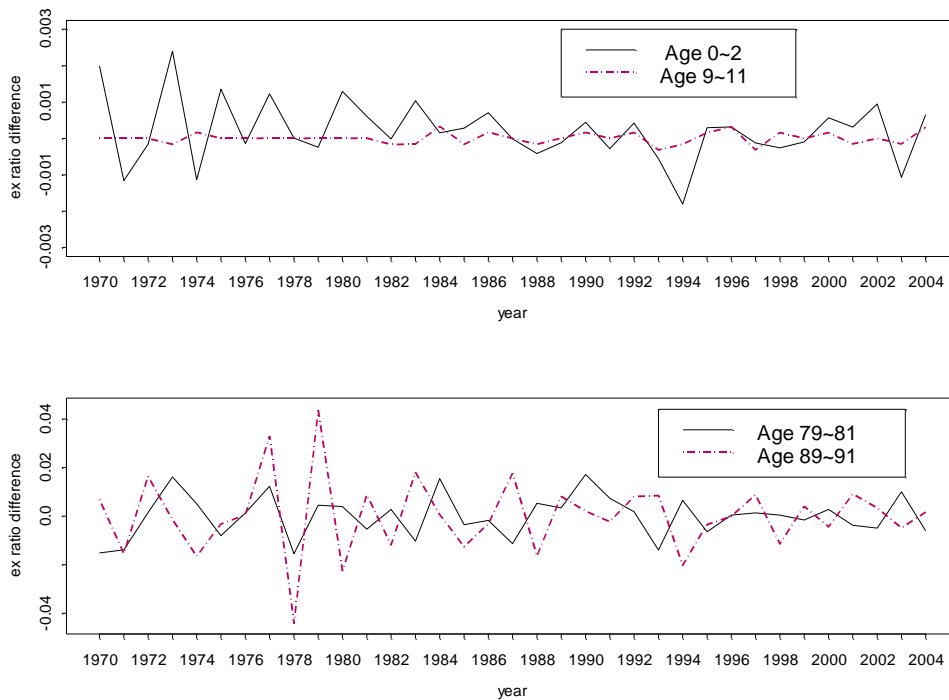
From the preceding empirical data analysis, we found that the discount sequence ratios, especially of the life expectancy and the survivor numbers, do support the conjecture of regularity condition. In other words, the regular discount sequence provides alternative for modeling mortality rates. We can use the regularity to estimate the mortality rates for the elderly, when there are not sufficient data, and to project the future mortality. In the next section, we will propose a method to model the discount sequence.

### III. Modeling the Discount Sequence

In the previous section, we found that the empirical data and many mortality models can be well described by the regular discount sequence. We shall use the regularity condition to model mortality rates in this section. In specific, we use Brownian motion stochastic differential equations (BMSDE) to capture the change of ratios (for life expectancy and the survivor numbers). We will only use the data from Taiwan in this section.

The reason for using the BMSDE is that the change of ratios behaves like stochastic process. Suppose  $r_x(t)$  is the life expectancy discount sequence ratio (or death number, mortality, number of survivors) at the age of  $x$  and also suppose  $\Delta r_x(t) = r_x(t+1) - r_x(t)$  is the difference of ratios. The differences of ratios at ages 0 to 2, 9 to 11, 79 to 81, and 84 to 86 (single age group) behave like stochastic processes (Figure 9, Taiwan males).

**Figure 9**  
**Taiwan males' life expectancy  $\Delta r_x(t)$ , 1970 to 2005**



Since the fluctuation of  $\Delta r_x(t)$  seems random at all ages, a natural choice of modeling is the Wiener process, or  $\Delta r_x(t) = a\Delta t + b\Delta z(t)$ . In this study, we choose the BMSDE to describe the change of  $r_x(t)$ , using the following equation:

$$dr_x(t) = k(\theta - r_x(t)) \cdot dt + \sigma \cdot dz, \quad dz \sim N(0, dt) \quad (10)$$

The parameters in equation (10) are:

$k$ : the strength, strong or weak, for  $r_x(t)$  returning to the average level  $\theta$ , i.e. adjusted speed.

$\theta$ : the long-term average level of  $r_x(t)$ . When  $r_x(t) > \theta$ , the mean value of the changes of the sequence instantaneous ratio is negative. Therefore,  $r_x(t)$  fluctuates toward the decreasing direction evenly. Conversely, if  $r_x(t) < \theta$ , the mean value of the changes of the sequence instantaneous ratio is positive. Therefore,  $r_x(t)$  fluctuates toward the increasing direction evenly.

$\sigma$ : diffusion coefficient.

To make this continuous model be calculated easily, add the following assumptions (discretization):

$$\Delta r_x(t) = k(\theta - r_x(t)) + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2).$$

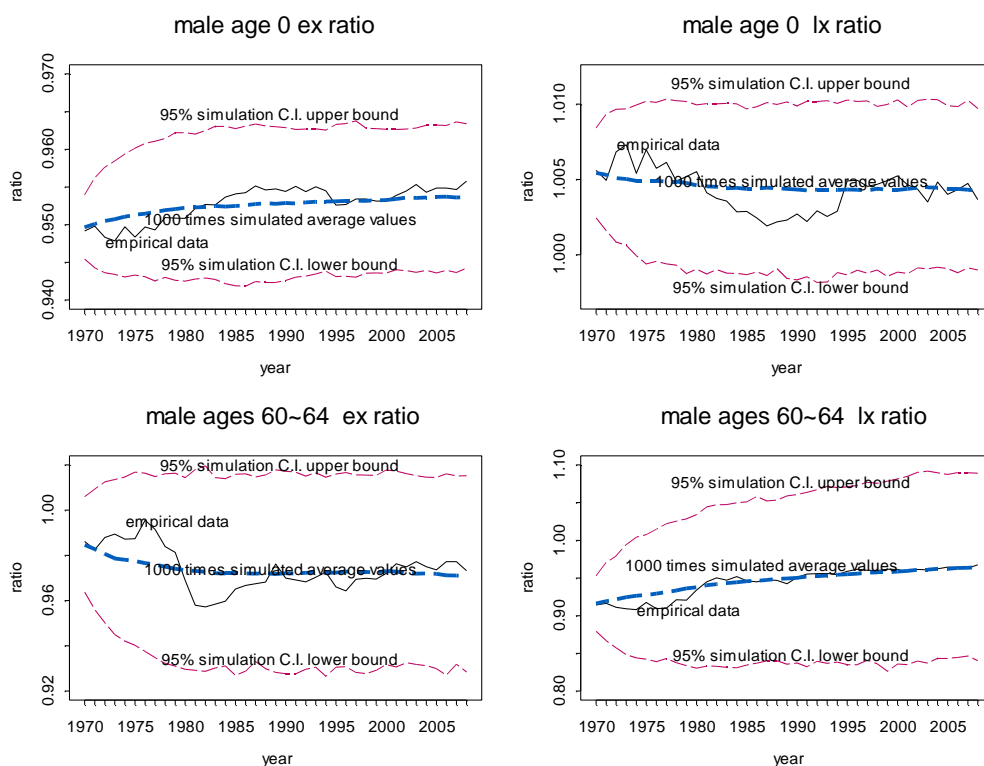
Under this assumption, we have  $r_x(t+1) = k\theta + (1-k)r_x(t) + \varepsilon$ . The estimates of the parameters  $k$  and  $\theta$  can be derived via the least squares method. Then, we can obtain the ratio for the next stage by  $\hat{r}_x(t+1) = \hat{k}\hat{\theta} + (1-\hat{k})\hat{r}_x(t)$ .

We will use the life expectancy and the number of survivors for the Taiwan males (five-year age group) as an example. We first estimate the parameters in the BMSDE, and then conduct computer simulation and build confidence intervals for the ratios. Figure 10 shows the observed ratios of life expectancy and the number of survivors at ages 0 and 60 to 64, and their 95 percent confidence interval computed via the Monte Carlo simulation (1,000 runs). The observed ratios all lie in the 95 percent confidence intervals, and it seems that the BMSDE is one of the possibilities for modeling the ratios. Also, in Figure 10, the widths of confidence intervals at age 0 are smaller, comparing to those of ages 60 to 64. The confidence intervals are much wider for the higher ages due to fewer populations.



**Figure 10**

**Using BMSDE to model life expectancy and survivor numbers (Taiwan males)**



In addition to using simulation to verify the model fit of BMSDE, we will continue to evaluate the BMSDE via cross-validation. We separate the 1970-2008 Taiwan male data into two periods: training period (1970-2003) and testing period (2004-08). We first use the training data to acquire model parameters and then find the predicted values of parameters for the testing period. Then, the mortality rates for the testing period can be derived from the estimated parameters. We will compare the performance of estimation and prediction from the BMSDE with that from the Lee-Carter model.

In applying the BMSDE model, we first use the data and the least squares method to estimate the parameters  $\hat{\alpha}$  and  $\hat{\beta}$ , and then convert  $\hat{k}$  and  $\hat{\theta}$  as the following equation:

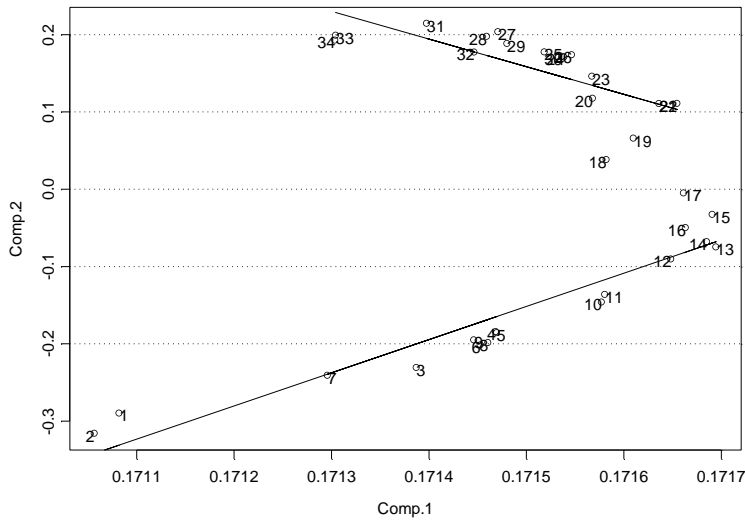
$$\hat{r}_x(t+1) = \hat{k}\hat{\theta} + (1-\hat{k})\hat{r}_x(t) = \hat{\alpha} + \hat{\beta} \cdot \hat{r}_x(t).$$

Next, we use the equation (10) and the Monte Carlo simulation, to find the estimate of  $r_x(t)$  for the training period. The estimates of ratios,  $\hat{r}_x(1970), \dots, \hat{r}_x(2003)$ , can be the

averages or the medians of  $r_x(t)$  computed from 1,000 simulation runs. In this study, we used the averages of 1,000 runs.

The predicted values of  $r_x(t)$  can be obtained from the time series method or other prediction methods. In this study, we use the principal component analysis (PCA). First, we apply the PCA on the original mortality rates and decide the number of time periods. As the changes of mortality rates are not necessarily constant of time, the estimation accuracy can be improved if we use data with similar property (i.e., using homogeneous data). Then, the estimates of  $\hat{k}$  and  $\hat{\theta}$  are derived for each period. The parameter estimates of the testing period will be obtained for the most recent period.

**Figure 11**  
**Loadings of the first vs. second principal components**



In the following, we shall demonstrate the analysis process. First, we plot the loadings of the first principal component (PC) and the second principal component in Figure 11. We found that the first two PCs are positively correlated for data points 1 to 15, and negatively correlated for data points 16 to 34. Therefore, we shall find the estimates  $\hat{k}$  and  $\hat{\theta}$ , separately for time periods 1 to 15 and 16 to 34. The future  $\hat{k}$  and  $\hat{\theta}$  can be obtained from the second time period 16 to 34. In this study, the data are separated into one, two, three and four periods. We will compare the accuracy of mortality estimation for the training and testing periods, with the Lee-Carter model.

The comparison is based on the mean absolute percentage error (MAPE), or

$$MAPE = \frac{1}{n} \sum_{i=1}^n \frac{|\varepsilon_i|}{Y_i} \times 100\% ,$$

where  $Y_i$  and  $\varepsilon_i$  are the observed value and residual of observation  $i$ ,  $i=1,2,\dots,n$ . We shall only compare the mortality rates of ages 66 to 85 because the Taiwan data are less reliable for higher ages.

Because the original Lee-Carter model (Lee and Carter 1992) is to model the central death rates, we need to convert them to the mortality rates. Under the uniform distribution of

death, the mortality rate  $q_x = \frac{m_x}{1 + \frac{1}{2}m_x}$ , where  $m_x$  is the central death rate at age  $x$ . The

period parameter  $\kappa_t$  in the Lee-Carter model, i.e.,  $\log(m_{x,t}) = \alpha_x + \beta_x \cdot \kappa_t$ , is estimated via the autoregressive integrated moving average (ARIMA) model (1,1,0). For the BMSDE method, the mortality rates are derived from the ratios. Because  $\frac{l_{x+2}/l_{x+1}}{l_{x+1}/l_x} = r_x$ , we can get

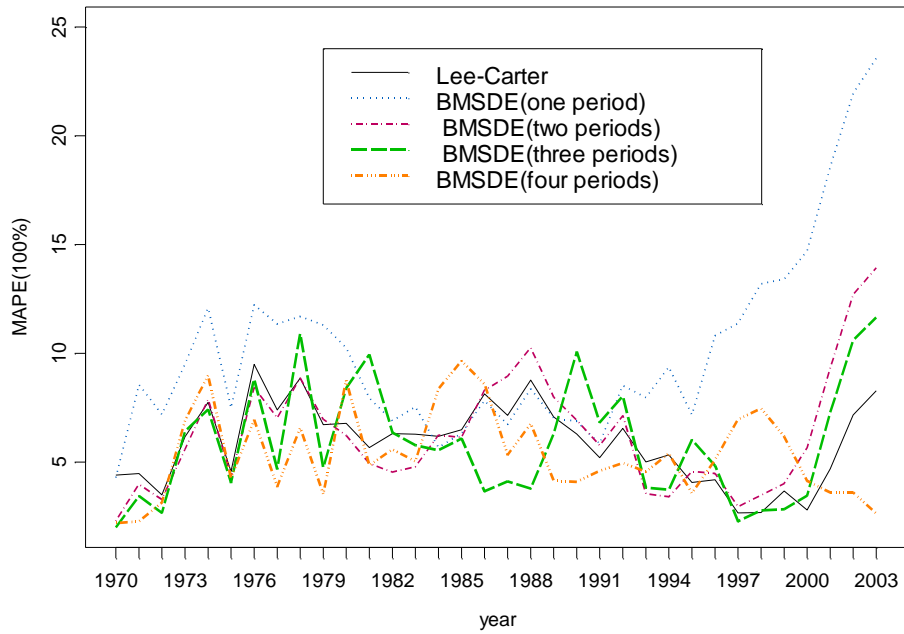
$\frac{l_{x+2}}{l_{x+1}} = r_x \frac{l_{x+1}}{l_x}$ , that is,  $q_{x+1} = 1 - r_x(1 - q_x)$ . Because the goal is the mortality rates for ages 66 to 85, we first obtain the mortality rates at age 65 and then calculate the mortalities from age 66 to 85 by iteration.

We first show the MAPE of the training period (1970-2003). From Table 1, the Lee-Carter model has smaller error, comparing to the BMSDE method with one or two time periods. But the BMSDE method has smaller error if there are three and four periods. Figure 12 shows the yearly fitted errors of all methods. The one-period BMSDE is not recommended because the errors are much larger since year 1996. Empirically, the BMSDE with two, three and four periods can be used. Interesting, similar to one-period BMSDE method, the errors of the Lee-Carter model can be separated into two parts: smaller at the beginning and the end but larger at the middle. Figure 13 shows the errors with respect to ages, and again the BMSDE with one period is not recommended because the errors vary with ages. However, the errors of the Lee-Carter model can also be divided into two parts, suggesting that the Lee-Carter might have room for improvement.

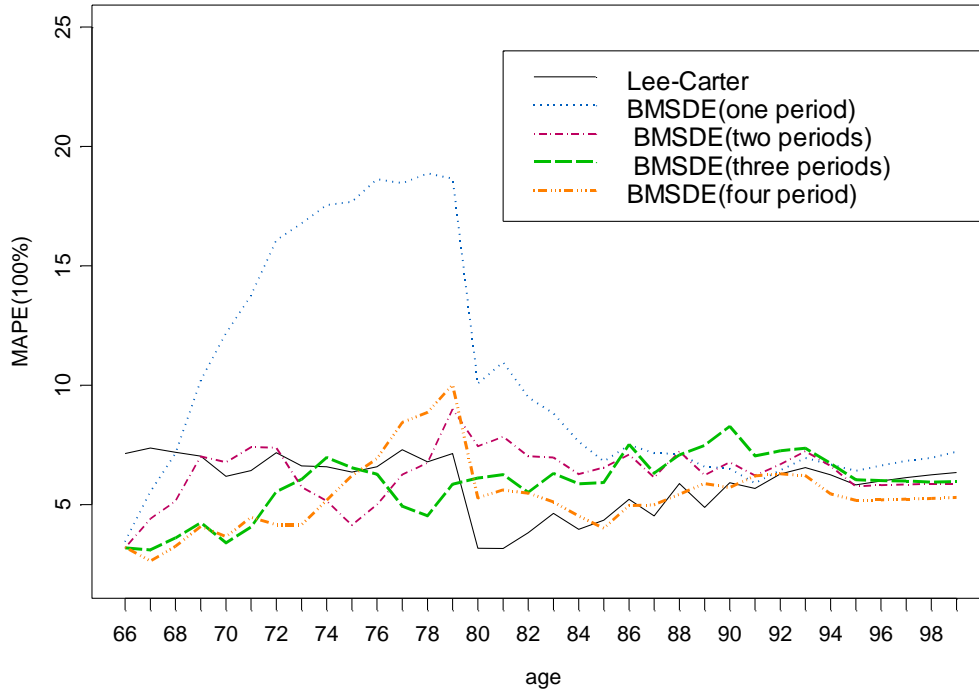
**TABLE 1**  
**MAPE of mortality for the Lee-Carter and BMSDE (training)**

Lee-Carter	BMSDE (one period)	BMSDE (two periods)	BMSDE (three periods)	BMSDE (four periods)
5.95%	12.44%	6.28%	5.22%	5.27%

**Figure 12**  
**Errors comparison with respect to time in training period (1970-2003)**



**Figure 13**  
**Errors comparison with respect to ages in training period (1970-2003)**

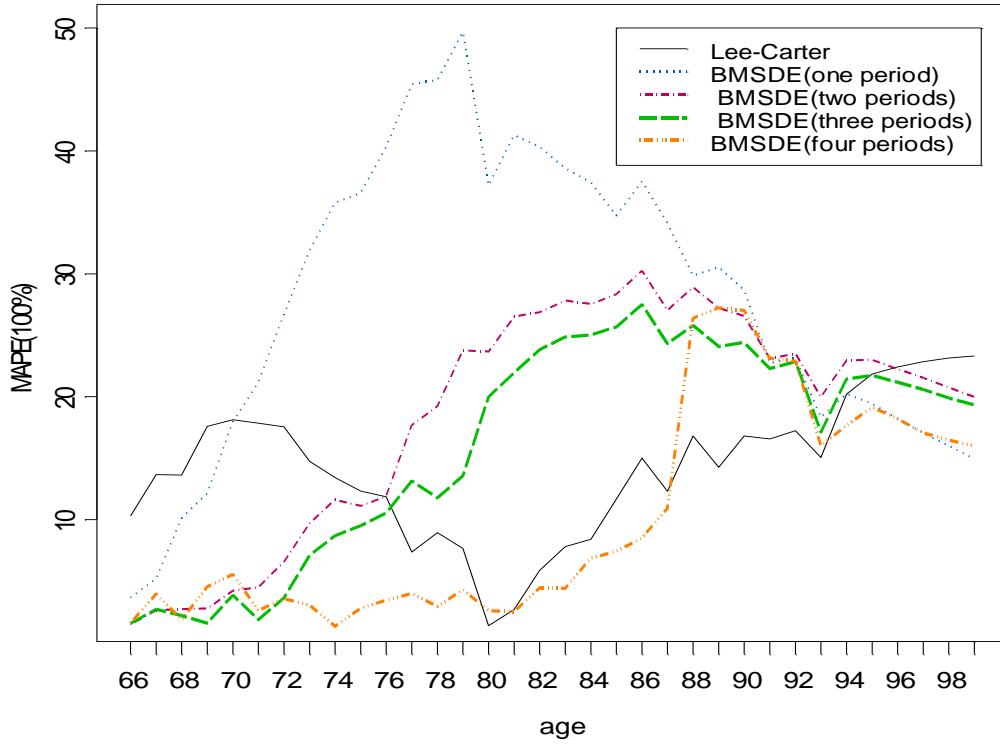


**TABLE2**  
**MAPE of mortality for the Lee-Carter and BMSDE (testing)**

Lee-Carter	BMSDE (one period)	BMSDE (two periods)	BMSDE (three periods)	BMSDE (four periods)
11.14%	30.59%	14.53%	11.64%	3.68%

The MAPE of the testing period can be seen from Table 2 and Figure 14. The MAPE of BMSDE method with one or two periods is also larger than that of the Lee-Carter model. The BMSDE method is competitive if three or four periods are used, similar to the case in the training period. The errors with respect to ages in Figure 14 show similar results, and the BMSDE method with one period has inhomogeneous errors with respect to ages. But the BMSDE with two or three periods are not satisfactory because the errors increase with ages. The errors of the Lee-Carter model can be separated into a combination of two straight lines. This is very similar to the results in Yang et al. (2010), where they found the parameter  $\kappa_t$  is also a combination of two straight lines in many countries.

**Figure 14**  
**Errors comparison with respect to ages in testing period (2004-08)**



#### **IV. Conclusions and Recommendations**

Since the end of World War II, population aging has become a common phenomenon in most countries. Population aging itself is not a problem but it incurs the burdens of, for example, medical expenses, workforce shortage and demand for senior care. Traditionally, the family plays an important role in taking care of the elderly in Asian countries. However, low fertility and longer lifespans increase the burden, and it is almost impossible for the family to deal with the problem alone. Thus, planning the retirement life by oneself is an inevitable task and the social system becomes very important. For example, Taiwan has had its own national health insurance since 1995 and national pension since 2009. These social insurance and welfare systems can provide better protection for people after retirement. Still, we need to handle the mortality projection with care, in order to avoid financial insolvency.

However, because of the rapid increase in prolonging of life, there are not enough elderly data and this increases the difficulty in modeling mortality. There are popular and well-known mortality models, such as Gompertz law and Lee-Carter model, but there is still room for improvement. As an alternative, in this study, we introduce the regular discount sequences from the bandit problem to describe the survival function. Many common mortality models, such as Gompertz law, Coale-Kisker model, uniform distribution of death, constant force and hyperbolic assumption, meet the regularity condition. We also use the data from six countries to verify the regular sequences and found that they all satisfy the regularity condition. Therefore, the discount sequence can provide a possible choice for modeling mortality. In addition, we also have the following three findings:

The first observation is that the discount sequence ratios of the life expectancy, number of survivors, death number and mortality rates met the regularity condition in empirical analysis. That is, the ratio for all ages will gradually approach 1, which suggests that 1 can be used as a limiting value for mortality projection. Also, we can use the regularity of the discount sequence ratio to compare different mortality models and use it as a selection criterion.

The second is we found that the discount sequence ratio of the number of survivors in each age group for all six countries will gradually decrease to 1 with time. This is especially obvious for the elderly. Therefore, we can take the median of the discount sequence ratio of the number of survivors at the same age, and use it to forecast the mortality rates via the least squares method.

The third observation is that we can use the discount sequence to interpolate missing values. The discount sequence ratio can grasp the changes among age levels and can be used to estimate the missing values. Of course, the discount sequence can be used for extrapolation, i.e., forecast, by plugging the mortality rates and life expectancy, and it can be incorporated into methods such as time series analysis.

Note that the current approach only considers the age and time effects and does not include the age group effect. We can modify the current discount sequence model, such as adding the age group effect in the original Lee-Carter model (Renshaw and Haberman 2006). In specific, we can use the spatial statistic method to check if there is an age group effect on the mortality improvement. Moreover, because the discount sequence ratio will gradually converge, we can use it to calculate the mortality index for the longevity risk securitization or use it to construct the life table. To refine the parameter estimates, such as modifying the Brownian motion mortality stochastic differential equation (BMSDE) used in this study or combining the discount sequence model with other mortality models (e.g. the Lee-Carter model), is also on the list of our future study.



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