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COMPLETE ANNUITIES

EUGENE A. RASOR* AND T. N. E. GREVILLE

INTRODUCTION

In general, a complete annuity of one per annum may be defined as a curtate annuity of one per annum with an additional final payment, made a fraction of a year after the last payment of one, representing the accrued portion of the succeeding payment not yet due. If the payments under a curtate immediate annuity of one per annum are regarded as interest at the rate i on a principal sum of $1/i$, and if the annuity is to run for a period of $n + t$ years (where n is integral and t fractional), then it is reasonable to take the accrued portion of the succeeding payment as the accrued interest on the principal $1/i$ for t years. This is $(1/i)$ $[(1+i)^t - 1]$ if compound interest is assumed, or $(1/i)it = t$ if simple interest is assumed. In other words, under the assumption of simple interest for a fractional part of a year, the amount of the partial payment is exactly proportional to the time elapsed. This corresponds to the definition of the complete annuity given by Spurgeon.

However, the assumption of compound interest leads to some very interesting relationships, which it is the purpose of this note to explore. It turns out, for example, that the relation

$$\bar{A}_x = 1 - i\mathring{a}_x , \qquad (1)$$

which Spurgeon derives by general reasoning and considers to be an approximate formula, is exact if compound interest is assumed. It may be pointed out that the additional partial payment is slightly smaller under the compound interest assumption.

Accordingly, it will be assumed in what follows that the additional partial payment is $[(1+i)^t - 1]/i$ when the annuity is 1 per year. The corresponding payment for an annuity with payments of $1/m$ payable m times per year is

$$\frac{(1+i)^t - 1}{i^{(m)}} , \qquad (2)$$

based on a principal of $1/i^{(m)}$.

* Eugene A. Rasor, not a member of the Society, is Actuarial Mathematician in the Social Security Administration.

ANNUITIES CERTAIN

The value of a complete annuity certain would be

$$\mathring{a}^{(m)}_{\overline{n+t}|} = a^{(m)}_{\overline{n}|} + v^{n+t}\,\frac{(1+i)^{t}-1}{i^{(m)}} = \frac{1-v^{n+t}}{i^{(m)}}, \tag{3}$$

where n is a multiple of $1/m$ and $t < 1/m$. It will be noted that this is exactly the result which would be obtained by the usual formula for the present value of an annuity certain if regarded as valid for a fractional term.

If δ is the force of interest corresponding to i,

$$\bar{a}_{\overline{n+t}|} = \frac{1-v^{n+t}}{\delta}.$$

Therefore,

$$\mathring{a}^{(m)}_{\overline{n+t}|} = \frac{\delta}{i^{(m)}}\,\bar{a}_{\overline{n+t}|}.$$

Similarly,

$$\mathring{s}^{(m)}_{\overline{n+t}|} = \frac{\delta}{i^{(m)}}\,\bar{s}_{\overline{n+t}|}.$$

In particular, for $n = 0$,

$$\mathring{s}^{(m)}_{\overline{t}|} = \frac{\delta}{i^{(m)}}\,\bar{s}_{\overline{t}|},$$

which represents the amount of the partial payment to be made at the end of a fractional period of t years.

It follows that formula (3) also gives the present value of an annuity certain a fraction t of a year before the first payment is due, provided this first payment is a fractional payment given by the expression (2). Therefore, we may conclude that

$$\mathring{a}^{(m)}_{\overline{n}|} = \frac{\delta}{i^{(m)}}\,\bar{a}_{\overline{n}|}, \tag{4}$$

whether n is a multiple of $1/m$ or not, and where the time interval between the nominal commencement of the annuity and the first payment is anything between 0 and $1/m$ years, provided the payment made for any fractional period, either at the beginning or the end, is

$$\frac{(1+i)^{t}-1}{i^{(m)}} = \frac{\delta}{i^{(m)}}\,\bar{s}_{\overline{t}|},$$

where t is the length of the fractional period.

LIFE ANNUITIES

The present value of a continuous temporary life annuity at age x of one per annum for a term of n years may be written in the form

$$\bar{a}_{x:\overline{n}|}=\int_0^n {}_tp_x\mu_{x+t}\bar{a}_{\overline{t}|}\,dt+{}_np_x\bar{a}_{\overline{n}|}. \tag{5}$$

Since $\bar{a}_{\overline{t}|}=(1-v^t)/\delta$, this gives

$$\bar{a}_{x:\overline{n}|}=\frac{1}{\delta}\left({}_nq_x-\bar{A}^{1}_{x:\overline{n}|}+{}_np_x-{}_nE_x\right)=\frac{1-\bar{A}_{x:\overline{n}|}}{\delta},$$

a well known result. If we consider, instead of a continuous annuity, an annuity consisting of payments of $1/m$ payable m times a year with an additional payment immediately on death of the amount indicated by the expression (2), the expression corresponding to the equation (5) is

$$\mathring{a}^{(m)}_{x:\overline{n}|}=\int_0^n {}_tp_x\mu_{x+t}\mathring{a}^{(m)}_{\overline{t}|}\,dt+{}_np_xa^{(m)}_{\overline{n}|}.$$

In view of the equation (4), this gives, at once,

$$\mathring{a}^{(m)}_{x:\overline{n}|}=\frac{\delta}{i^{(m)}}\,\bar{a}_{x:\overline{n}|}, \tag{6}$$

or

$$\mathring{a}^{(m)}_{x:\overline{n}|}=\frac{1-\bar{A}_{x:\overline{n}|}}{i^{(m)}}. \tag{7}$$

Equation (7) immediately gives the relation

$$\bar{A}_{x:\overline{n}|}=1-i^{(m)}\mathring{a}^{(m)}_{x:\overline{n}|}. \tag{8}$$

Similar relations are obtained for whole life annuities, merely omitting the "angle n" in each case. Thus, the equation (1) given by Spurgeon is a special case of equation (8).

There is a practical situation, arising in connection with the valuation of estates, in which it is desirable to be able to regard equation (8) as an exact relationship. Suppose it is provided that the income from an estate is to go to an heir A during his lifetime and that principal is to pass upon A's death to another heir B (or to B's estate if B is not then alive), and that it is desired to evaluate for tax purposes the interests of A and B in the estate. Taking the principal of the estate as unity, we may take A's interest as $i^{(m)}\mathring{a}^{(m)}_x$ and B's interest as $\bar{A}_x$, where x is A's age at the time the income to him commences. As the sum of the two interests should equal the total value of the estate, we should have $1=i^{(m)}\mathring{a}^{(m)}_x+\bar{A}_x$, in conformity with equation (8).

If the assumption of compound interest for fractional periods is adopted and, in consequence, equations (6) and (7) are accepted as being exact relations, they do not, however, immediately provide a means of computing $\mathring{a}_x^{(m)}$, since neither $\bar{A}_x$ nor $\bar{a}_x$ is directly obtainable from the usual mortality table without introducing some approximation or assumption. Spurgeon shows (1938 ed., p. 158) that, on the assumption of uniform distribution of deaths over each year of age,

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x ,$$

and, thus,

$$\bar{A}_x = \frac{i}{\delta} A_x \tag{9}$$

$$= A_x \left(1 + \frac{i}{2} - \frac{i^2}{12} + \ldots\right). \tag{10}$$

Substituting the right member of equation (9) in formula (7) gives

$$\mathring{a}_x^{(m)} = \frac{1}{i^{(m)}} \left(1 - \frac{i}{\delta} A_x\right).$$

When $m = 1$, this reduces to

$$\mathring{a}_x = \frac{1}{i} - \frac{A_x}{\delta}$$

$$= \frac{d}{\delta} a_x + \left(\frac{1}{i} - \frac{v}{\delta}\right)$$

$$= a_x + A_x \left(\frac{1}{d} - \frac{1}{\delta}\right). \tag{11}$$

New estate and gift tax regulations recently issued by the Bureau of Internal Revenue contain tables based on the formula

$$\mathring{a}_x = a_x + \tfrac{1}{2} A_x , \tag{12}$$

which is an approximation to formula (11), since

$$\frac{1}{d} - \frac{1}{\delta} = \frac{1}{2} + \frac{i}{12} - \frac{i^2}{24} + \ldots .$$

These tables also use the formula

$$\bar{A}_x = A_x \left(1 + \frac{i}{2}\right), \tag{13}$$

which is an approximation to formula (9) or (10). When the approximation (13) is used in conjunction with the approximation (12), it will be found that the equation $1 = \bar{A}_x + i\mathring{a}_x$ holds exactly.

A USEFUL LEMMA

The present value of the additional benefit provided by a complete life annuity of one per annum and not by the corresponding curtate life annuity is given by

$$\mathring{a}_x^{(m)} - a_x^{(m)} = \frac{1-\bar{A}_x}{i^{(m)}} - \frac{1-A_x^{(m)}}{d^{(m)}} + \frac{1}{m} = \frac{A_x^{(m)}}{d^{(m)}} - \frac{\bar{A}_x}{i^{(m)}},$$

since

$$\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \ddot{a}_{\overline{\infty}|}^{(m)} - a_{\overline{\infty}|}^{(m)} = \frac{1}{m}.$$

This is a particular case of a more general theorem, which is stated in the following paragraph.

Let $\bar{A}$ denote the single premium for a benefit which consists of the payment of one unit immediately on the occurrence of a specified contingency. Let $A^{(m)}$ denote the single premium for a similar benefit, with the sole difference that the payment is no longer immediate, but is made on the $m$thly anniversary of the issuance of the contract next succeeding the occurrence of the specified contingency. Finally, let $I^{(m)}$ denote the present value of a benefit which consists of the payment, immediately on the occurrence of the specified contingency, of the amount given by the expression (2), t being the time elapsed between the last preceding $m$thly anniversary of the contract and the occurrence of the specified contingency. Then, we shall show that

$$I^{(m)} = \frac{A^{(m)}}{d^{(m)}} - \frac{\bar{A}}{i^{(m)}}. \tag{14}$$

Let the sum insured under $A^{(m)}$ be $1/d^{(m)}$ instead of unity, and let the sum insured under $\bar{A}$ be $1/i^{(m)}$. The single premiums then become the respective terms in the right member of equation (14). Now, the first contract is equivalent to the payment, immediately on the occurrence of the specified contingency, of

$$\frac{v^{(1/m)-t}}{d^{(m)}} = \frac{(1+i)^{t}}{i^{(m)}}.$$

Therefore, the difference between the two payments is exactly the expression (2), which is the benefit provided by the left member of equation (14). This establishes the lemma.

It will be noted that nothing in the proof requires the specified contingency to be one which must eventually occur. For example, the benefits might be payable on the failure of a given life only in the event he predeceases another life.

APPLICATION TO REVERSIONARY ANNUITIES

Spurgeon gives on page 322 and discusses at some length in the succeeding pages the following three examples of complete reversionary annuities payable m times a year to (x) after the death of (y):

1. An annuity to be entered upon by (x) immediately on the death of (y), the first payment $1/m$ to be made at the end of a period of one-mth of a year measured from the death of (y), later payments being made at the ends of successive periods similarly measured.
2. An annuity to be set up immediately, the periods, at the ends of which successive payments are to be made, being measured from the present time, the first full payment of $1/m$ to (x) being made at the end of the period of one-mth of a year, measured from the present time, in which (y) dies, so that the actual period of accrual is disregarded.
3. An annuity similar to (2) under which, however, the proportionate payment to the death of (y) is to be made to his estate, so that (x) will receive only the amount which has accrued since the death of (y).

It is instructive to consider these three cases from the point of view adopted in the present note. In the first case, the value is that of a complete annuity payable immediately on the death of (x), namely,

$$\int_0^\infty v^t {}_tp_{xy}\mu_{y+t}\mathring{a}_{x+t}^{(m)}dt = \int_0^\infty v^t {}_tp_{xy}\mu_{y+t}\frac{\delta}{i^{(m)}}\bar{a}_{x+t}dt$$

$$=\frac{\delta}{i^{(m)}}\bar{a}_{y|x} = \frac{\delta}{i^{(m)}}(\bar{a}_x - \bar{a}_{xy}) = \mathring{a}_x^{(m)} - \mathring{a}_{xy}^{(m)} .$$

In the third case, the value is clearly that of a complete annuity to (x) less a complete annuity during the joint lifetime of (x) and (y), and therefore identical with the result of the first case. Thus the distinguishing symbol ^ which Spurgeon uses in the first case appears to be unnecessary in the case of a complete annuity when compound interest is assumed.

With regard to the second case, we first note that a payment of $1/m$ at the end of the period of one-mth of a year, measured from the present time, in which (y) dies, is equivalent to a payment of $[(1+i)^t - 1]/i^{(m)}$ immediately on the death of (y) plus a further payment of $[(1+i)^{(1/m)-t} - 1]/i^{(m)}$ at the end of the period of one-mth of a year, where t is the fraction of a year elapsed between the beginning of this period and the death of (y). This follows from equation (4) and the remarks made in connection with it, from which it also follows that the value in the second case may be regarded as that for the third case plus an additional benefit of $[(1+i)^t - 1]/i^{(m)}$ payable immediately on the death of (y) if (y) predeceases

(x). Obtaining the value of the latter benefit from formula (14), we find for the total value

$$\mathring{a}_x^{(m)} - \mathring{a}_{xy}^{(m)} + \frac{\mathrm{A}_{x\overset{1}{y}}^{(m)}}{d^{(m)}} - \frac{\bar{\mathrm{A}}_{x\overset{1}{y}}}{i^{(m)}}.$$

If uniform distribution of deaths over each year of age is assumed,

$$\mathrm{A}_{x\overset{1}{y}}^{(m)} = \frac{\delta}{i^{(m)}} \bar{\mathrm{A}}_{x\overset{1}{y}},$$

and the preceding expression reduces to

$$\mathring{a}_x^{(m)} - \mathring{a}_{xy}^{(m)} + \frac{\bar{\mathrm{A}}_{x\overset{1}{y}}}{i^{(m)}} \left(\frac{\delta}{d^{(m)}} - 1 \right).$$

APPLICATION TO FAMILY INCOME BENEFITS

Cody has derived (*TASA* XLIX, 72) an expression for the single premium for a very general type of family income benefit on the assumption of uniform distribution of deaths over each year of age. In doing so, he found it convenient to consider the total benefits provided in three parts, of which only the first gave rise to a complicated expression. This involved the evaluation of a quantity ${}^kI^1_{x:\overline{n}|}$, which he defined as the single premium for \$1/12 monthly commencing at date of death to end of n years from issue with minimum of k years of payments, if death occurs within n years of issue. It was stipulated that interest was to be at the annual rate i before death and at the smaller annual rate i' after death.

We shall obtain the single premium on the assumption that a fractional annuity payment of $[(1+i)^t - 1]/i^{(12)}$ is made exactly at the expiration of the n-year period from issue, t being the fraction of a year then elapsed since the last regular payment, and compare the result with that obtained by Cody. Denoting by ${}^kJ^1_{x:\overline{n}|}$ the single premium for the modified benefit considered here, we have

$$\begin{aligned}
{}^kJ^1_{x:\overline{n}|} &= \int_0^{n-k} v^t {}_tp_x \mu_{x+t} \left(\mathring{a}'^{(12)}_{\overline{n-t}|} + \frac{1}{12} \right) dt + \ddot{a}'^{(12)}_{\overline{k}|}\, {}_{n-k|}\bar{\mathrm{A}}^1_{x:\overline{k}|} \\
&= \int_0^{n-k} v^t {}_tp_x \mu_{x+t} \left(\frac{1 - v'^{n-t}}{i'^{(12)}} + \frac{1}{12} \right) dt + \ddot{a}'^{(12)}_{\overline{k}|}\, {}_{n-k|}\bar{\mathrm{A}}^1_{x:\overline{k}|} \\
&= \left(\frac{1}{i'^{(12)}} + \frac{1}{12} \right) \int_0^{n-k} v^t {}_tp_x \mu_{x+t} dt - \frac{v'^n}{i'^{(12)}} \int_0^{n-k} v''^t {}_tp_x \mu_{x+t} dt \\
&\quad + \ddot{a}'^{(12)}_{\overline{k}|}\, {}_{n-k|}\bar{\mathrm{A}}^1_{x:\overline{k}|},
\end{aligned}$$

where $v'' = v/v'$, so that $1 + i'' = (1 + i)/(1 + i')$. Making use of the relations

$$\frac{1}{i'^{(12)}} + \frac{1}{12} = \frac{1}{d'^{(12)}} \qquad \text{and} \qquad \ddot{a}'^{(12)}_{\overline{k}|} = \frac{1 - v'^k}{d'^{(12)}},$$

we have then

$${}^kJ_{\underset{1}{x}:\overline{n}|} = \frac{1}{d'^{(12)}} \left[(1 - v'^k)\, \bar{A}_{\underset{1}{x}:\overline{n}|} + v'^k \bar{A}_{\underset{1}{x}:\overline{n-k}|} \right] - \frac{v'^n}{i'^{(12)}} \bar{A}''_{\underset{1}{x}:\overline{n-k}|}.$$

This expression differs only in the final term from that obtained by Cody. His final term can be reached by multiplying the one given here by $\delta'' d^{(12)} (1 + i')^{1/12} / \delta d''^{(12)}$. This is close to unity, and could probably be neglected in a practical situation. It is possible, however, by suitable algebraic manipulation, and using the expression previously given for the final payment, to obtain the present value of the additional benefit assumed, and to show that its deduction gives exactly Cody's expression.

A NOTE ON NOTATION

The use of the special sign ° in connection with the annuity certain serves not only to focus attention on the parallelism between the complete annuity certain and the complete life annuity but also to distinguish the value represented from other slightly different values which might be denoted by the annuity symbol without this sign. If the expiration of the term of an annuity certain is regarded as analogous to the termination of a life annuity by death, strict analogy would require that we define $a_{\overline{n+f}|} = a_{\overline{n}|}$, where n is an integer, and f a proper fraction. On the other hand, Hart (*Mathematics of Finance*, Third Edition, p. 285) has pointed out that if the symbol $a_{\overline{n+f}|}$ is used to denote $a_{\overline{n}|} + fv^{n+1}$, then the equation

$$P = Ra_{\overline{n}|},$$

where P is the amount of a debt subject to amortization and R the amount of the periodic amortization payment, holds exactly, whether n is integral or not. If it includes a proper fraction f, the amount of the final irregular amortization payment is fR. This suggested expression for $a_{\overline{n+f}|}$ is, of course, that which would be obtained by straight-line interpolation between $a_{\overline{n}|}$ and $a_{\overline{n+1}|}$.

It can also be shown that the amount of principal included in the rth amortization payment is given by

$$R\,(a_{\overline{n-r+1}|} - a_{\overline{n-r}|}),$$

whether n is integral or not.

A somewhat similar situation arises in connection with an installment refund annuity. Consider such an annuity of one per annum to a life aged x, and let A denote the gross single premium and L the amount of loading. If all annuities involved are assumed to be complete, we have, exactly,

$$A - L = \mathring{a}_{\overline{A}|} + \mathring{a}_x - \mathring{a}_{x:\overline{A}|} .$$

On the other hand, if curtate annuities are provided, we can still write

$$A - L = a_{\overline{A}|} + a_x - a_{x:\overline{A}|} ,$$

which is also exact if we take

$$a_{x:\overline{n+f}|} = a_{x:\overline{n}|} + f \; {}_{n+1}E_x ,$$

by analogy with Hart's definition for the annuity certain.