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DISCUSSION OF PRECEDING PAPER

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This interesting paper adds another link between compound interest theory and life insurance mathematics.

To my mind the basic idea of the paper is that $\delta\bar{a}_x$ is the present value of interest payments on a principal of 1, interest to be paid throughout the whole of the life of (x), on any basis that is equivalent to the force δ. The payments of interest could be made quite irregularly provided always that, in one way or another, interest for the whole term is accounted for on an exact basis. If $i^{(m)}$ and i are equivalent to δ, and interest is payable at the end of each mth of a year, with a final fractional payment of $(1+i)^t - 1$, where t is the portion of a year from the last payment to date of death, then

$$\delta\bar{a}_x = i^{(m)}\mathring{a}_x^{(m)}.$$

If $d^{(m)}$ is the rate of interest per year payable at the beginning of each mth of a year that is equivalent to δ, then

$$\delta\bar{a}_x = d^{(m)}\mathring{\ddot{a}}_x^{(m)}$$

where at death a payment of

$$\frac{d^{(m)}}{m}(1+i)^t - [(1+i)^t - 1] = 1 - (1+i)^t v^{1/m} = 1 - v^{1/m-t}$$

is *refunded*. The annuity-due here indicated is perhaps not too well described by the term *complete; apportionable* or *refund* may be better adjectives. The notion of a refund annuity-due is encountered in regard to premium payments for an insurance providing for a refund of premium paid beyond the date of death.

From the point of view of the preceding paragraph it is easy to see why the reversionary annuities 1 and 3 (of the paper) have the same present value. For an income of $i^{(m)}$ a year payable $m$thly, with appropriate fractional payments where necessary at the beginning and end of the annuity term, each of these annuities has present value $\delta\bar{a}_x - \delta\bar{a}_{xy}$.

It may be of interest to mention a formula for $\mathring{a}_x^{(m)}$ that we discuss in our classes at Michigan. In this instance $\mathring{a}_x^{(m)}$ is the present value of a complete annuity with final partial payment proportionate to the time elapsed. If we assume uniform distribution of deaths in the year of death, then the average value of the final fractional payment, accumulated to the end of the period of one-mth of a year in which death occurs, is

$$\frac{\int_0^{1/m} t\,(1+i)^{1/m-t}dt}{1/m}=\frac{i^{(m)}-\delta}{\delta^2}\,.$$

It follows that

$$\mathring{a}_x^{(m)}-a_x^{(m)}=\frac{i^{(m)}-\delta}{\delta^2}\,A_x^{(m)}$$

$$=\frac{i^{(m)}-\delta}{\delta^2}\,\frac{i}{i^{(m)}}\,A_x$$

$$=\left[\frac{1}{\delta}-\frac{1}{i^{(m)}}\right]\frac{i}{\delta}\,A_x$$

$$=\left[\frac{1}{\delta}-\frac{1}{i^{(m)}}\right]\bar{A}_x\,.$$

Under the assumptions stated, this formula is more exact than the usual formula

$$\mathring{a}_x^{(m)}-a_x^{(m)}=\frac{1}{2m}\,\bar{A}_x\,.$$

However, one could not expect the relation $1=i^{(m)}\mathring{a}_x^{(m)}+\bar{A}_x$ to hold exactly since our $\mathring{a}_x^{(m)}$ is not equivalent to the $\mathring{a}_x^{(m)}$ of the paper.