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DISCUSSION OF PAPERS PRESENTED AT THE SPRING MEETINGS

INSURANCE FOR FACE AMOUNT OR PAID-UP INSURANCE AMOUNT IF GREATER

CECIL J. NESBITT AND MARJORIE L. VAN EENAM

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DICKINSON C. DUFFIELD:

Dr. Nesbitt and Miss Van Eenam have provided still another fascinating quirk to the "Retirement Income" contract which has already received considerable attention in actuarial literature. As stated in the paper, however, this is a somewhat different approach from the Fassel approach. Under the latter, the amount of insurance becomes identical with the reserve in the later policy years coinciding in a curve under which both first and second differences are positive. Under the new scheme, the amount of insurance is also level in the early policy years, but after the paid-up amount exceeds the face the amount of insurance follows a curve which has positive first differences but negative second differences. The reserve does not equal the amount of insurance until the maturity date. As in the Fassel plan, however, the reserves in the two periods follow different curves.

In my discussion of "Analysis of Net Premium Formulas for the Income Endowment Policy" by Kermit Lang, (*RAIA* XXXII, 159), I studied the curves of the reserves on a "continuous" basis for the Fassel type contract both before and after the reserve crosses the face amount. The conclusion reached was that the two curves had equal values and first derivatives at the common point a, but that the initial curve had greater curvature at such point. I thought it might be interesting to apply a similar analysis to the new policy presented by Dr. Nesbitt and Miss Van Eenam. It is evident that the expressions for the reserve and its derivatives when only the face amount is payable are the same as for the Fassel case or, for that matter, for *any* level premium, level amount policy. For convenience the expressions are repeated below:

$$^{\infty}_{t}\overline{V}^{I}_{x}=\frac{^{\infty}\overline{P}\,(\overline{N}_{x}-\overline{N}_{x+t})-(\overline{M}_{x}-\overline{M}_{x+t})}{D_{x+t}}$$

$$\frac{d}{dt}\,^{\infty}_{t}\overline{V}^{I}_{x}={}^{\infty}\overline{P}+\delta\,^{\infty}_{t}\overline{V}^{I}_{x}-\mu_{x+t}\,(1-{}^{\infty}_{t}\overline{V}^{I}_{x})$$

$$\frac{d^{2}\,^{\infty}_{t}\overline{V}^{I}_{x}}{dt^{2}}=(\mu_{x+t}+\delta)\,\frac{d}{dt}\,^{\infty}_{t}\overline{V}^{I}_{x}-(1-{}^{\infty}_{t}\overline{V}^{I}_{x})\,\frac{d}{dt}\,\mu_{x+t}\,.$$

We now consider the formula and derivatives for the reserve when the paid-up insurance amount is payable. We may write

$$ {}^{\infty}_{\ t}\bar{V}^{III}_{x} = \bar{A}_{x+t:\overline{n-t|}}\ {}^{\infty}_{\ t}\bar{W}^{III}_{x} $$

where

$$ {}^{\infty}_{\ t}\bar{W}^{III}_{x} = 1 + {}^{\infty}\bar{P}\int_{b}^{t} (\bar{A}_{x+s:\overline{n-s|}})^{-1} ds $$

representing the amount of Paid-Up Endowment insurance in force at the instant t.

Differentiating we have:

$$ \frac{d\,{}^{\infty}_{\ t}\bar{V}^{III}_{x}}{dt} = {}^{\infty}\bar{P} + {}^{\infty}_{\ t}\bar{W}^{III}_{x}\,[\,\delta\bar{A}_{x+t:\overline{n-t|}} - \mu_{x+t}\,(1-\bar{A}_{x+t:\overline{n-t|}})\,] $$

$$ = {}^{\infty}\bar{P} + \delta\,{}^{\infty}_{\ t}\bar{V}^{III}_{x} - \mu_{x+t}\,({}^{\infty}_{\ t}\bar{W}^{III}_{x} - {}^{\infty}_{\ t}\bar{V}^{III}_{x}) $$

since

$$ \frac{d}{dt}\,{}^{\infty}_{\ t}\bar{W}^{III}_{x} = {}^{\infty}\bar{P}\,(\bar{A}_{x+t:\overline{n-t|}})^{-1}. $$

At the point $t = b$,

$$ {}^{\infty}_{\ b}\bar{W}^{III}_{x} = 1\ ; $$

so that

$$ {}^{\infty}_{\ b}\bar{V}^{I}_{x} = {}^{\infty}_{\ b}\bar{V}^{III}_{x} \qquad \text{and} \qquad \frac{d}{dt}\,{}^{\infty}_{\ t}\bar{V}^{I}_{x}\Big]_{t=b} = \frac{d}{dt}\,{}^{\infty}_{\ t}\bar{V}^{III}_{x}\Big]_{t=b}. $$

Differentiating again we have:

$$ \frac{d^2}{dt^2}\,{}^{\infty}_{\ t}\bar{V}^{III}_{x} = (\mu_{x+t}+\delta)\frac{d}{dt}\,{}^{\infty}_{\ t}\bar{V}^{III}_{x} - ({}^{\infty}_{\ t}\bar{W}^{III}_{x} - {}^{\infty}_{\ t}\bar{V}^{III}_{x})\frac{d}{dt}\,\mu_{x+t} $$

$$ -\,\mu_{x+t}\,{}^{\infty}\bar{P}\,(\bar{A}_{x+t:\overline{n-t|}})^{-1}. $$

At the point b we have

$$ \frac{d^2}{dt^2}\,{}^{\infty}_{\ t}\bar{V}^{III}_{x}\Big]_{t=b} = \frac{d^2}{dt^2}\,{}^{\infty}_{\ t}\bar{V}^{I}_{x}\Big]_{t=b} - \mu_{x+b}\,{}^{\infty}\bar{P}\,(\bar{A}_{x+b:\overline{n-b|}})^{-1}, $$

showing that the curvature of Curve III is less than that of Curve I by the force of mortality applied to the additional nominal annual amount of insurance bought by a year's premium at the instant b.

It is somewhat surprising at first glance that, although the reserves for the new plan exceed those for the corresponding Fassel plan at most durations, the reverse situation is true at the latest durations. On the annual premium basis, the maturity values and the death benefits in the final policy year are the same for both plans and hence the *initial* reserves at the beginning of such year are the same. However, since the new plan has a higher net premium, the *terminal* reserve in the penultimate year is higher for the Fassel plan. A similar effect would hold also for the continuous basis. The accompanying chart compares the new plan with the old on the continuous basis.

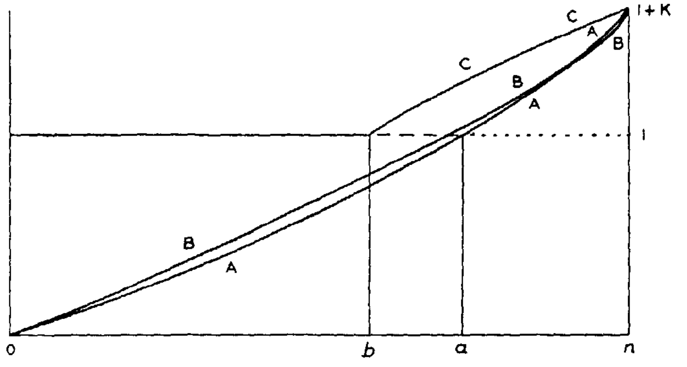

CHART 1

While the new plan is very interesting algebraically, it might as a practical matter have the following disadvantages as compared with the Fassel plan:

1. It increases the annual premium on a form which has probably already been subject to several increases because of steadily rising maturity values, which in turn have resulted from improved mortality among annuitants. Furthermore, the value of the additional benefits, as compared with the Fassel plan, might not be readily explainable by an agent in competition.
2. It does not appear to lend itself so readily to ease in valuation.

It would be a matter of individual company opinion as to whether these disadvantages are outweighed by the advantage of doing away with the annoying nonforfeiture problems presented by the Fassel plan.

JAMES E. HOSKINS:

The authors evidently had the Retirement Income type of policy in mind in developing their method. Mr. Duffield has suggested a reason why companies may hesitate to adopt it for that plan. It may be useful, however, on those plans in which the face amount increases to a higher level after a certain duration—*e.g.*, the familiar juvenile policy in which the insurance becomes five times the original amount when age 21 is reached. Here the paid-up value, but not the cash value, generally exceeds the initial insurance in the years just before age 21. The use of the proposed method permits a satisfactory relation between the insurance in force if premiums are continued and that in force if they are discontinued. By increasing the coverage gradually from the original toward the ultimate level, any antiselection that might occur on renewal when the face amount changes would be reduced.

HILARY L. SEAL:

It is well known (see, *e.g.*, Zwinggi, *Versicherungsmathematik*, Basle, 1945) that if the sum insured under a given plan of insurance may be written in the form

$$Q_t + k_t\,(Q_t - V_{t+1}) \qquad t = 0,\ 1,\ 2,\ \ldots$$

where V_t is the (reserve, surrender, asset-share) value of the policy at duration t, then that plan is actuarially equivalent to one under which the sum insured between durations t and $t+1$ is Q_t but the mortality rate between those durations has been changed from q_{x+t} to $(1+k_t)q_{x+t}$. The proof is straightforward.

In the present case we write

$$k_t = \begin{cases} 0 & t=0,\ 1,\ 2,\ \ldots\ b-1 \\ -\,(A_{x+t+1:\overline{n-t-1|}})^{-1} & t=b,\ b+1,\ \ldots\ n-1 \end{cases}$$

and

$$Q_t = \begin{cases} 1 & t=0,\ 1,\ 2,\ \ldots\ b-1 \\ 0 & t=b,\ b+1,\ \ldots\ n-1 \end{cases}$$

in order to reproduce the sum insured under the authors' plan of insurance, namely,

$$_{t+1}S = \begin{cases} 1 & t=0,\ 1,\ 2,\ \ldots\ b-1 \\ _{t+1}CV\,(A_{x+t+1:\overline{n-t-1|}})^{-1} & t=b,\ b+1,\ \ldots\ n-1 \end{cases}$$

The plan that is actuarially equivalent thereto is provided by

$$_{t+1}S' = \begin{cases} 1 & t=0,\ 1,\ 2,\ \ldots\ b-1 \\ 0 & t=b,\ b+1,\ \ldots\ n-1 \end{cases}$$

and

$$q'_{x+t} = \begin{cases} q_{x+t} & t=0,\ 1,\ 2,\ \ldots\ b-1 \\ [1-(A_{x+t+1:\overline{n-t-1|}})^{-1}]\,q_{x+t} & t=b,\ b+1,\ \ldots\ n-1 \end{cases}$$

Provided, therefore, that "special" mortality rates, depending on the rate of interest used, are substituted for the standard rates after b years, the authors' new plan may be replaced by a level term insurance for unit sum insured during the first b years *plus* a pure endowment of $1+k$ at the end of n years.

Writing

$$^{b}D_{x+t} = D_{x+b}\prod_{s=b}^{t-1}\left\{v\left(p_{x+s} + \frac{q_{x+s}}{A_{x+s+1:\overline{n-s-1|}}}\right)\right\} \qquad t = b+1,\ b+2,\ \ldots\ n$$

we obtain at once ($t = b + 1, b + 2, \ldots n$)

$$ {}^{b}\mathrm{D}_{x+t} \cdot {}_{t}\mathrm{CV} = {}^{C}\mathrm{P}\,({}^{b}\mathrm{N}_{x} - {}^{b}\mathrm{N}_{x+\min(m,t)}) - (\mathrm{M}_{x} - \mathrm{M}_{x+b}) - E \cdot \mathrm{D}_{x} \quad \text{(A)} $$

and, on putting $t = n$,

$$ {}^{C}\mathrm{P} = \frac{{}^{b}\mathrm{D}_{x+n}\,(1+k) + \mathrm{M}_{x} - \mathrm{M}_{x+b} + E \cdot \mathrm{D}_{x}}{{}^{b}\mathrm{N}_{x} - {}^{b}\mathrm{N}_{x+m}}. \quad \text{(B)} $$

Once b is determined the relations (A) and (B) provide a simple, self-checking method of calculating ${}^{C}\mathrm{P}$ and ${}_{t}\mathrm{CV}$ ($t = b + 1, b + 2, \ldots n$).

The first step in the calculation of b for each entry age x ($x = x_0, x_0 + 1, \ldots$) and maturity age y, is the preparation of a table of ${}_{y}\Delta_{z}$, where

$$ {}^{b_0}\mathrm{D}_{z} = \mathrm{D}_{x_0+b_0} \prod_{w=x_0+b_0}^{z-1} \left\{ v\left(p_w + \frac{q_w}{\mathrm{A}_{w+1:\overline{y-w-1}|}}\right)\right\} $$

$$ = \mathrm{D}_{x_0+b_0} \cdot {}_{y}\Delta_{z} \qquad z = x_0 + b_0 + 1,\; x_0 + b_0 + 2, \ldots y $$

and b_0 is the b corresponding to entry age x_0. Then, if $x + b$ increases monotonically with x (the modifications necessary in the converse case being obvious),

$$ {}^{b}\mathrm{D}_{z} = \mathrm{D}_{x+b} \prod_{w=x+b}^{z-1} \left\{ v\left(p_w + \frac{q_w}{\mathrm{A}_{w+1:\overline{y-w-1}|}}\right)\right\} $$

$$ = \frac{\mathrm{D}_{x+b}}{{}_{y}\Delta_{x+b}} \cdot {}_{y}\Delta_{z} \qquad z = x + b + 1, \qquad x + b + 2, \ldots y $$

and

$$ {}^{b}\mathrm{N}_{z} = \frac{\mathrm{D}_{x+b}}{{}_{y}\Delta_{x+b}} \sum_{w=z}^{y} {}_{y}\Delta_{w} = \frac{\mathrm{D}_{x+b}}{{}_{y}\Delta_{x+b}}\, {}_{y}\Gamma_{z}, \qquad \text{say.} $$

In order to calculate b for any particular entry age x, we note that it is the greatest integer satisfying

$$ 1 \geqslant (\mathrm{D}_{x+b}\mathrm{A}_{x+b:\overline{n-b}|})^{-1}\,[{}^{C}\mathrm{P}\,(\mathrm{N}_{x} - \mathrm{N}_{x+b}) - (\mathrm{M}_{x} - \mathrm{M}_{x+b}) - E \cdot \mathrm{D}_{x}] $$

i.e.,

$$ {}^{C}\mathrm{P} \leqslant \frac{\mathrm{D}_{x+b}\mathrm{A}_{x+b:\overline{n-b}|} + \mathrm{M}_{x} - \mathrm{M}_{x+b} + E \cdot \mathrm{D}_{x}}{\mathrm{N}_{x} - \mathrm{N}_{x+b}}. \quad \text{(C)} $$

It is thus a relatively simple procedure to calculate ${}^{C}\mathrm{P}$ on the basis of a trial value of b using the relation (B) and the ancillary columns ${}_{y}\Delta_{z}$ and ${}_{y}\Gamma_{z}$. The greatest trial value of b for which (C) is satisfied is the value required.

It is thought that the preceding analysis has a formal simplicity lacking in the joint authors' development. From the computational standpoint the preparation of the columns ${}_y\Delta_x$ and ${}_y\Gamma_x$ is a little more trouble than that of ${}_y\Lambda_x$. Likewise, the determination of b for each entry age x requires a few more arithmetical steps. However, relation (A) may be applied throughout the range of t to find ${}_t\mathrm{CV}$ and this is, perhaps, simpler than changing from retrospective to prospective formulas as the authors do in their relations (13).

ELGIN G. FASSEL:

My compliments are extended to the authors for an unusually well written paper. In view of their reference to my paper of twenty years ago "Insurance for Face Amount or Reserve if Greater," perhaps it will be in order to take stock of the experience which we have had in practice with the theoretical difficulty to which the authors make reference. This has to do with the reduced paid-up and the extended term nonforfeiture benefits at the later policy durations.

I think of the paid-up provision as exhibiting three phases. Phase 1 is regular paid-up as in ordinary endowment policies. Phase 2 is where the reserve on the paid-up starts at an amount less than the face but reaches a maturity amount greater than the face. Phase 3 is where the reserve on the paid-up is greater throughout than the face, and therefore the net amount at risk is zero.

For extended term insurance, Phase 1 is the same as for ordinary endowment policies. In Phases 2 and 3, extended term and reduced paid-up are identical.

The theoretical difficulty to which I referred is that Phases 2 and 3 are peculiar to this "insurance for face amount or reserve if greater" type of policy; and this is cured by the authors in their approach to the subject, their solution having the commendable result of eliminating Phases 2 and 3, with Phase 1 nonforfeiture benefits throughout the same as for ordinary endowment policies. This result, however, is accompanied by a corresponding increase in the cost to the policyholder.

In my company issuance of this "insurance for face amount or reserve if greater" type of policy was commenced in 1930. We have now issued a total of 228,000 such policies of which 180,000 were in force at the end of 1951. Of these the number outstanding in what I have described as Phase 2 is 102 policies and in Phase 3 is 200 policies. These 302 policies require seriatim determination of the annual dividends. In the year end valuation they are handled by a group method that determines the policy reserve approximately for Phase 2 and exactly for Phase 3. This special handling requirement for 302 policies out of an activity of 180,000 is quite insignificant.

In the policy we have evolved the language:

. . . except that the amount of such paid-up insurance shall not be in excess of the amount payable (before deducting indebtedness) had the death of the Insured occurred on the date of premium default, or the cash value of such paid-up insurance if larger.

Also our extended insurance provision automatically reverts to paid-up insurance after Phase 1 through the following language:

The excess value, if any, shall be used to purchase in like manner a nonparticipating paid-up pure endowment payable on the maturity date if the Insured is then living; provided that if the amount of such pure endowment should exceed the amount of the extended term insurance, then, in lieu of such term insurance and pure endowment, the provisions of section 14(d) shall automatically apply.

I have not been able to find any record or recollection in our office of any difficulty with policyholders or beneficiaries with regard to this matter.

I do wish to commend the authors on their excellent suggestion for avoiding the anomaly of an insurance contract with no amount at risk, at of course a price.

(AUTHORS' REVIEW OF DISCUSSION)

CECIL J. NESBITT AND MARJORIE L. VAN EENAM

We had been warned that our paper was too complete to admit discussion but it appears the warning was based on an underestimate of the ingenuity of our members.

Mr. Duffield's discussion of the curves for the amounts of insurance and for the reserves supplements the theory of our paper, and provided us with ready answers for questions asked in a recent class discussion.

Both Mr. Fassel and Mr. Duffield compare, on a practical basis, the plan suggested in our paper with the face amount or reserve if greater plan, and both point out the increased cost. We believe the main offsetting advantage of our plan is the increase in the amount of insurance provided. The very substantial experience of Mr. Fassel's company with the face amount or reserve if greater plan indicates that the supposed difficulty with nonforfeiture benefits is more imaginary than real.

We are grateful to Mr. Hoskins for indicating that the method of the paper might have application to other plans besides the retirement income form and, in particular, might be useful in connection with juvenile insurance. In reviewing his remark it occurred to us that there might be merit in a juvenile whole life policy with an initial amount of insurance \$1,000 and becoming paid-up at, say, age 25, for an amount of \$3,000. The method of the paper could easily be adapted to such a policy.

Mr. Seal's discussion suggests an interesting application of Cantelli's theory to the insurance for face amount or paid-up insurance amount if greater plan. Since this theory was not well-known to us, and possibly to others, it may be worth while to present the main idea in terms of the notation of our paper. If the amount of insurance ${}_{t+1}S$ in insurance year $t+1$ may be represented by ${}_{t+1}Q + k_{t+1}({}_{t+1}Q - {}_{t+1}\mathrm{CV})$ then formula (3) of our paper, namely

$$^{C}\mathrm{P} = {}_{t+1}S \cdot v\, q_{x+t} + {}_{t+1}\mathrm{CV} \cdot v\, p_{x+t} - {}_{t}\mathrm{CV}$$

becomes

$$\begin{aligned} ^{C}\mathrm{P} &= {}_{t+1}Q \cdot v\,(1 + k_{t+1})\, q_{x+t} + {}_{t+1}\mathrm{CV} \cdot v\,[1 - (1 + k_{t+1})\, q_{x+t}] - {}_{t}\mathrm{CV} \\ &= {}_{t+1}Q \cdot v\, q'_{x+t} + {}_{t+1}\mathrm{CV} \cdot v\, p'_{x+t} - {}_{t}\mathrm{CV} \end{aligned}$$

where $q'_{x+t} = (1 + k_{t+1})\, q_{x+t}$. This indicates that the original plan has the same cash value premium and cash values as a special plan based on the mortality rates q'_{x+t} and having ${}_{t+1}Q$ as the amount of insurance in year $t+1$. Mr. Seal shows that where the original plan is insurance for face or paid-up insurance amount then, by proper choice of ${}_{t+1}Q$ and k_{t+1}, the special plan is the same as the original plan for the first b years but converts to pure endowment for the last $(n-b)$ years and for those latter years is based on the special rates

$$q'_{x+t} = [1 - (\mathrm{A}_{x+t+1:\overline{n-t-1}|})^{-1}]\, q_{x+t}\,.$$

Evidently the special mortality rates are *negative;* nevertheless, they provide a practical basis for computation, as we have confirmed by checking against some of our original computations.

The formulas for the special plan are somewhat simpler than the formulas given in our paper, but they involve a computational problem since the special mortality rates depend upon the age at issue. An ingenious solution of this problem is offered by Mr. Seal. However, we are inclined to think that the premium-analysis method used in our paper is a more natural and meaningful approach than is given by the Cantelli theory.

A notational error has been called to our attention by Mr. Walter Steffen. On page 6, second line, the factor $\mathrm{A}_{x+b:\overline{n-b}|}$ should be $\mathrm{A}_{x+b:\overline{n-b}|}$.

The discussers, in their individual ways, have enlarged our thinking about the topic of the paper, and we thank them most heartily for their contributions.