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FUNDAMENTALS OF PENSION FUNDING

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CECIL J. NESBITT:

In this paper Mr. Trowbridge has done an excellent job of classifying and illustrating the various methods of pension funding. From time to time we have discussed similar ideas at Michigan but have never organized a complete analysis such as the author presents. We knew, for instance, that the contributions and fund under the aggregate method for a mature population would approach limits but we did not realize what is now fairly obvious, that those limits would be the contribution and fund for the entry age normal cost method.

Throughout the paper the author uses discrete functions which, of course, are convenient for the calculation of illustrations. For purposes of exploring the theory, continuous methods have some advantages. With a few changes in assumptions and notation it is easy to obtain continuous function formulas parallel to the discrete function formulas of the paper. For example, if we assume that the retirement benefit is $1 per year payable momently from age r, and let ${}^{A}C_t$ equal the annual rate of contribution at time t under the aggregate method, and ${}^{A}F_t$ the fund at time t, then corresponding to formulas of Demonstration II, we have

$$b=\int_a^r l_x \cdot {}_{r-x|}\bar{a}_x\,dx+\int_r^w l_x\bar{a}_x\,dx \equiv \frac{\mathrm{T}_r}{\delta}-\frac{1}{\delta}\,l_a \cdot {}_{r-a|}\bar{a}_a$$

$$y=\frac{\int_a^r l_x\bar{a}_{x:\overline{r-x}|}\,dx}{\int_a^r l_x\,dx} \equiv \frac{\frac{1}{\delta}\left[(\mathrm{T}_a-\mathrm{T}_r)-l_a\bar{a}_{a:\overline{a-r}|}\right]}{\mathrm{T}_a-\mathrm{T}_r}$$

$$p=\mathrm{T}_r$$

$${}^{A}C_t=\frac{b-{}^{A}F_t}{y}$$

$$d\left({}^{A}F_t\right)={}^{A}C_t\,dt+{}^{A}F_t\,\delta\,dt-p\,dt$$

or

$$\frac{d\,{}^{A}F_t}{dt}+{}^{A}F_t\left[\frac{1}{y}-\delta\right]=\left[\frac{b}{y}-p\right]$$

whence

$${}^{A}F_t = -\frac{b - py}{1 - \delta y}\, e^{-[(1/y)-\delta]t} + \frac{b - py}{1 - \delta y};$$

from which it may be shown that

$${}^{A}F_\infty = \frac{b - py}{1 - \delta y}.$$

Demonstration I is related to the "general average premium" concept discussed by Feraud in *Actuarial Technique and Financial Organization of Social Insurance,* page 28. By "general average premium" is meant that contribution which if paid in respect to all present and future participants would be sufficient to provide benefits for all present and future participants. If π^c denotes such a premium for a mature population whose members are to receive the momently benefit indicated above, and if π^c is payable momently, then

$$\pi^c \frac{1}{\delta}(\mathrm{T}_a - \mathrm{T}_r) = \frac{1}{\delta}\mathrm{T}_r$$

or

$$\pi^c = \frac{\mathrm{T}_r}{\mathrm{T}_a - \mathrm{T}_r}. \tag{1}$$

The premium π^c is independent of the interest rate and the method amounts to pay-as-you-go funding.

If the population is immature to the extent that it contains individuals up to age r only but is otherwise distributed according to the service table, and if the general average premium for this case is denoted by π^m, then

$$\pi^m \frac{1}{\delta}(\mathrm{T}_a - \mathrm{T}_r) = \frac{1}{\delta}(l_r \bar{a}_r)$$

or

$$\pi^m = \frac{l_r \bar{a}_r}{\mathrm{T}_a - \mathrm{T}_r}. \tag{2}$$

The justification for formula (2) is that at each moment dt in the future, benefits of value $l_r\bar{a}_r dt$ will be incurred and so the total present value of benefits for present and future participants will be $1/\delta(l_r\bar{a}_r)$. This is a terminal (or maturity) funding method but differs from the Class II funding illustration in Table IV in that the contribution remains level from year to year by reason of the assumed service table distribution below age r.

If the population is just commencing to be built up from l_a new entrants each year at age a, and π^n denotes the general average premium for this case, then

$$\pi^n \frac{1}{\delta}(l_a \bar{a}_{a:\overline{r-a}|}) = \frac{1}{\delta}(l_a \cdot {}_{r-a}|\,\bar{a}_a)$$

or

$$\pi^n = \frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a}|}}; \tag{3}$$

that is, π^n is the entry age normal cost.

For the current cost funding situation with general average premium π^c, the accrued liability, defined as the present value of benefits for all present and future participants less the present value of all future premiums from such participants, remains constant at 0. The accrued liability in regard to just present participants is

$$\int_a^r l_x \cdot {}_{r-x|}\bar{a}_x dx + \int_r^\infty l_x \bar{a}_x dx - \pi^c \int_a^r l_x \bar{a}_{x:\overline{r-x}|} dx$$

which may be reduced to

$$\frac{1}{\delta}(\pi^c - \pi^n)\, l_a \bar{a}_{a:\overline{r-a}|}\,. \tag{4}$$

The accrued liability for future new entrants, with value of benefits expressed in terms of normal cost, may be written as

$$\frac{1}{\delta}(\pi^n l_a \bar{a}_{a:\overline{r-a}|}) - \frac{1}{\delta}(\pi^c l_a \bar{a}_{a:\overline{r-a}|})$$

which is the negative of (4). Thus the accrued liability for present participants is balanced by an anticipated gain in regard to future new entrants to give a total accrued liability of 0.

For the maturity funding method with general average premium π^m, the accrued liability for retired participants ultimately becomes

$$\int_r^\infty l_x \bar{a}_x dx = \frac{1}{\delta}(\mathrm{T}_r - l_r \bar{a}_r)$$

$$= (\pi^c - \pi^m)\frac{1}{\delta}(\mathrm{T}_a - \mathrm{T}_r)\,. \tag{5a}$$

The accrued liability for active participants from ages a to r is

$$\int_a^r l_x \cdot {}_{r-x|}\bar{a}_x dx - \pi^m \int_a^r l_x \bar{a}_{x:\overline{r-x}|} dx$$

which reduces to

$$\frac{1}{\delta}(\pi^m - \pi^n)\, l_a \bar{a}_{a:\overline{r-a}|}\,. \tag{5b}$$

The gain in respect to future new entrants is also (5*b*); hence the total accrued liability for retired, active and future participants becomes just (5*a*), the liability for the retired group.

For the entry age normal cost funding situation with general average premium π^n, benefits and premiums for future new entrants exactly balance and the accrued liability for this group is always 0, and the total accrued liability is the liability for present participants (active or retired). It is convenient, however, in computing the total liability to calculate benefit values for both present and future participants and proceed similarly for the calculation of premium values. Thus, by the time the active group has grown, according to the service table, to include persons up to age r the total accrued liability is

$$\frac{1}{\delta}[l_r\bar{a}_r - \pi^n(\mathrm{T}_a - \mathrm{T}_r)] = \frac{1}{\delta}(\pi^m - \pi^n)(\mathrm{T}_a - \mathrm{T}_r). \qquad (6)$$

At time t years later the total accrued liability is

$$\frac{1}{\delta}[l_{r+t}\bar{a}_{r+t} + (\mathrm{T}_r - \mathrm{T}_{r+t}) - \pi^n(\mathrm{T}_a - \mathrm{T}_r)] \qquad (7)$$

and the ultimate total accrued liability is

$$\frac{1}{\delta}[\mathrm{T}_r - \pi^n(\mathrm{T}_a - \mathrm{T}_r)]$$

or

$$\frac{1}{\delta}(\pi^c - \pi^n)(\mathrm{T}_a - \mathrm{T}_r). \qquad (8)$$

In the foregoing discussion, gains in respect to future new entrants have been taken into account. It should not be inferred, however, that I favor the discounting of such gains in regard to actual pension plans; in fact, I usually take the opposite attitude. Whatever our attitude be toward that question, we should be willing to examine and understand the possibilities. The paper of Miles M. Dawson, "The Actuarial Basis of Compulsory Insurance," was a good step in that direction.

For some while Michigan students have been presented problems along the lines indicated in the above discussion. My thanks are due to the author for giving a more complete background for such problems.

W. RULON WILLIAMSON:

Mr. Trowbridge's tables on pages 32 and 36 suggest that present management is naturally considerate of future management's solvency.

Two Federal programs of pensions have had press attention lately, one because the concern seems too largely missing, the other because of an election.

1. The Federal Civil Service Retirement System has actuarial reports based on Mr. Trowbridge's Class IV-1 financing method. Last month Chairman

Ramspeck had two linked articles in the *Washington Post*. He said that there was \$4 billion in the fund, against accrued liabilities of \$9 billion. The \$4 billion assets fell short of the two requirements, liabilities on existing pensioners and the guaranteed return of contributions with interest to employees and their survivors. This left nothing toward the employer's liability to active working employees. Congressman Murray quoted the ratio of 12 to 1 for benefit payments and employee contribution for existing pensioners.

2. OASI is being presented as "the biggest pension plan," and "the biggest life insurance system."

On October 7 in Mississippi (press release of October 3) the Federal Security Administrator said the FSA serviced 157 million Americans for health, welfare and education. He said that 100 million of them had OASI wage records in Baltimore.

Also on October 3, Governor Stevenson said in Columbus, Ohio: "Today 65 million people have built up substantial equities in the Social Security system. When you and your wife reach the age of 65, your share in the retirement fund will amount to the equivalent of a \$15,000 annuity." The product of 65 million and \$15,000 is about a trillion dollars; of 100 million and \$15,000, \$1½ trillion, of 157 million and \$15,000, \$2¼ trillion. From age 18 (the end of "dependent childhood") to 65 (the age for "eligibility to OASI") is 47 years. Top pension now is \$85. Abject poverty is said to begin below \$2,000 a year. The monthly pension corresponding to \$2,000 is \$65. Using that as the average pension "expected" at 65, the yearly unit would be \$1.40 per year of presumptive work. Using only the trillion figure, and Mr. Trowbridge's unit method, Class III, a no-interest base would show about a half-trillion accrued liability. But using 2% interest, U.S. Life Table White Males, 1939–1941, pure annuities, and some rather ancient age distributions, might cut the accrued liability to \$150 billion. The present trust fund is about 10% of that. 2% interest on the accrued liability would take \$3 billion. \$280 billion of life insurance at the annual death rate of 6 per 1,000 would call for provision of \$1½ billions. The current liability for one unit of deferred annuity could run \$6½ billion. The annual load would reach \$11 billion. Current tax collection is about \$4 billion.

The Mississippi speech also said: "We are conducting the business of Social Security so efficiently that we have been able to expand the benefits. This month, with few exceptions, each check was larger than the previous one by at least \$5 or 12%, whichever was more. We had a little trouble getting the bill passed by Congress. But it went through." The prospect that biennially at each Congressional election \$20 billion additional accrued liability is to be accepted is an intriguing one.

The picture of my worried countenance in the first number of the new *Life Magazine*, beneath which was the claim that I would figure the lowest rates on Social Security, seems to have been prophetic.

CLARK T. FOSTER:

Mr. Trowbridge has been guilty of an understatement in the introduction to his valuable paper in describing the need for a text on pension funding methods. He points out that the beginner in the pension field, in his attempts to educate himself, must rely on the *Bulletin on Section 23(p)*, put out by the United States Treasury Department. Remembering the hopeless feeling I had when first studying that complex document, I have the feeling that anyone with no other means of learning about pension funding would remain a beginner all his life.

There are two points I would like to raise in connection with this paper; first, to introduce two additional funding classes, which might be referred to as $1\frac{1}{2}$ and $2\frac{1}{2}$, and second, to comment on several methods of combining two or more of the classes the author has described.

Class $1\frac{1}{2}$ belongs somewhere between Class 1, pay-as-you-go method, and Class 2, terminal funding. It has been used in a number of cases, particularly in some of the negotiated steel plans, as a means of leveling the cost in the first few years when the terminal funding cost is often quite high because of the large number of employees immediately eligible to retire. The present value of future pension benefits is paid into the fund in installments over a period of up to five years after each employee reaches retirement rather than in a lump sum at the time of retirement.

Class $2\frac{1}{2}$ lies between the terminal funding of Class 2 and the full funding of Class 3 or 4. In terminal funding, no contributions are made until an employee retires. Under Class 3 or Class 4 funding, a contribution is normally made each year for each employee covered under the plan. Under Class $2\frac{1}{2}$ funding, contributions are made only for employees who have reached a certain age or have completed a certain period of service, despite the fact that they are considered as being covered under the plan before satisfying such age or service requirements. The advantage of this method lies in the elimination of administrative records and actuarial calculations on the young or short-service employees who are most likely to terminate employment. It is particularly convenient to fund benefits only for employees age 35 or over if the plan provides benefits at age 65 based only on years of service up to a maximum of 30 years. Similarly, it is convenient to fund only for employees over age 40 if the maximum benefits are granted after 25 years of service.

In some cases, it is preferable to fund only for employees who have completed, say, two or three years of service. Upon an employee's completion of such a period of service, records are established, and his total estimated benefit is funded over his remaining years of employment.

Under any such program of funding, the annual cost for each employee for whom benefits are being funded is greater than if funding had started at employment, but since a large group of employees is not having any benefits funded, the reserves at any time are less than they would otherwise be. The pattern of contributions from year to year depends on the maturity of the group and its age distribution. In a new organization with a relatively low average age, the cost is likely to increase sharply as more and more employees pass the age at which funding commences.

This Class $2\frac{1}{2}$ funding becomes identical with Class 2, or terminal funding, if the age at which benefits are funded is the retirement age. Similarly, this Class $2\frac{1}{2}$ funding becomes identical with the Class 3 or Class 4 full funding methods if benefits are funded for employees as soon as they are eligible for coverage under the plan.

There are a number of ways by which the various funding methods are frequently combined. It is common, for instance, in a plan providing normal benefits in accordance with a percentage formula, but in which benefits are subject to a certain minimum, to fund the percentage accruals on a Class 3 unit credit method and to fund any additional benefit required by the minimum on a Class 2 terminal funding basis at retirement. Similarly, in a plan allowing retirement at any time after age 65 with additional benefits accruing as a result of service after 65, it is convenient to assume that each employee will retire at 65 and fund such benefits on a Class 3 or a Class 4 program, funding any additional benefits resulting from service after age 65 on a terminal funding arrangement at the end of each year of service after age 65. The cost of such additional benefits is normally offset by the savings resulting from payments that would otherwise have been made to the employee during his period of postponement.

Another frequently used combination of methods is to establish a past service liability on the Class 3 unit credit method and to fund the future service benefits on a Class 4 individual level premium method. Occasionally, this combination might be further complicated by the use of Class 2 terminal funding for the purchase of disability benefits.

Just as the funding methods themselves may be combined, the various methods of handling actuarial gains and losses may also be combined. Frequently, future service gains are immediately used as a credit against a plan's normal cost, whereas past service gains are temporarily ignored, serving to shorten the period over which the past service liability is funded. This arrangement is possible as long as a corporation's total contributions for any year fall within the Internal Revenue Bureau's specified maximum. Occasionally, certain types of gains from either past or future service are taken immediately while others are spread over a period of years. For example, a loss resulting from salary increases in a plan involving an assumed salary scale may be spread over the period to an employee's retirement, while all other gains or losses are immediately recognized. Alternatively, the loss from salary changes might be allocated between past and future service, with the future service loss recognized at once and the past service loss spread over the past service funding period.

ROBERT F. LINK:

One can visualize the population of a group as an organism which passes through a period of growth, a period of maturity, and finally senescence. This is a rather idealistic description since the characteristics of growth, maturity and senescence are usually obscured by such extraneous factors as ups and downs of the economic cycle, changes in the characteristics of the particular industry, etc. The theory of most pension funding methods is most easily examined on the assumption that one has a stable group which can be expected to remain stable for a number of future years. However, this idea is realized so infrequently in practice that the examination and comparison of funding methods on the basis of a stable population may create or promote misconceptions rather than otherwise. Thus, any mathematical theory of pension funding must take as its laboratory a group population which is assumed to be subject to change in its composition as to ages, salaries and so forth.

Mr. Trowbridge is to be congratulated for a paper which (within the limited range of my own reading) appears to be the first attempt to state the definitions, axioms, and theorems of a true science of pension funding methods. His concept of the immature group gets the science of pension funding off immediately on the right foot. His descriptions of various funding methods and their operation in the context of a simple group population should probably be required reading for students of pension funding; they might well be adopted as the foundation for any future developments along these lines.

I am sure that Mr. Trowbridge will not be offended if I suggest that his paper has barely scratched the surface of a great body of potential scientific knowledge of various funding methods. Further points to be examined (and which have been intentionally avoided by Mr. Trowbridge) are such matters as:

a) The effect of differences between the actuarial assumptions and the true experience of the particular group;

b) Extension of his theories to cover the more realistic situation of multiple entry ages;

c) Examination of various methods of estimating pension costs with respect to their appropriateness in predicting future costs;

d) Miscellaneous matters such as the choice of correct and meaningful turnover rates, salary scales, etc.;

e) Special problems arising from the introduction of unusual benefits or employee contributions (in particular those arising from superimposition of an alternate benefit formula on an existing scale of benefits).

In describing the trend of normal costs under various classes of funding, Mr. Trowbridge has tended to give further documentation to what I believe to be an overworked thesis. This thesis is that rising costs under plans which are funded by the unit credit cost method are due mainly to the increase in the average premium age of the group. Occasionally an employer has requested that we explain the reason for rising costs under a deferred annuity plan and predict the trend of these costs for the future. We have found that of the total rise in cost only a small part was usually attributable to increasing average premium ages. The rest was due to such factors as:

a) The tendency, as a group progresses toward maturity, for a higher percentage of the total lives in the group to find themselves in the group of eligible employees;

b) Generally rising salaries (which have an intensified effect under an approximately integrated unit plan);

c) Amendments of the group annuity contract with respect to the rate basis of purchase.

The attempt to analyze trends of cost in terms of an initially immature group population leads to a rather interesting result. One tends to think of the asymptotic approach from the l'_x distribution of Mr. Trowbridge's immature population to the l_x distribution of his stationary population as a smooth progression of uniform direction. Actually this asymptotic approach looks more like a decreasing sine wave. This is somewhat evident from Mr. Trowbridge's illustrations; the terminal funding amounts shown in Table 4 rise until the 35th year, drop again to the 50th year, and reach an ultimate level higher than that of the 50th year. It can easily be seen that if lives leave a group only by retirement (there being no deaths or withdrawals at all) the population would tend to repeat itself on a cycle of $r-a$ years. The decrements have a damping effect on this tendency. This wave motion makes it just a little trickier to draw conclusions from numerical illustrations.

The Equation of Maturity can also be written in such a way as to exclude the liability for retired lives (this is equivalent to paying the benefit in a lump sum at retirement age). In this alternate form, the equation looks like this:

$$C + dF = vR\ddot{a}_r$$

where R is the total annual income for new retirements.

For certain purposes, there seems to be some merit in extending Mr. Trowbridge's notation to embrace two variables, entry age and attained age. This leads to a set of select functions which can be identified by the subscript x, y (x representing attained age and y representing entry age). If one assumes a constant percentage distribution of entry ages for each generation of new entrants, one should ultimately come up with a stationary population expressed by a distribution consisting of the values of $l_{x,y}$. Mr. Trowbridge's algebraic identities based on the Equation of Equilibrium will still apply for the unit credit cost method and the entry age normal cost method, since these equations can be expressed for the double variable case in terms of the sums of various items for each entry age.

A little caution is needed, however, when approaching the problem of ultimate cost under an aggregate funding method. The Equation of Maturity can be written with the ultimate cost per life expressed as an unknown (using the alternate form of the Equation of Maturity), as follows:

$$v\sum_{y} l_{r,y} B_y \ddot{a}_r = {}^{A}\text{CPL}_\infty \sum_{x,y} l_{x,y}$$
$$+ d\Big[\sum_{x,y} l_{x,y} B_y \cdot {}_{r-x|}\ddot{a}_{x,y} - {}^{A}\text{CPL}_\infty \sum_{x,y} l_{x,y} \ddot{a}_{x,y:\overline{r-x|}}\Big]$$

where CPL is the ultimate normal cost per life and B_y is the annual rate of retirement income for an entrant at age y. If we solve this equation, the normal cost per life turns out to be

$${}^{A}\text{CPL}_\infty = \frac{\sum_{y} l_{y,y} B_y \cdot {}_{r-y|}\ddot{a}_{y,y}}{\sum_{y} l_{y,y} \ddot{a}_{y,y:\overline{r-y|}}}$$

and the normal cost turns out to be

$${}^{A}C_\infty = {}^{A}\text{CPL}_\infty \sum_{x,y} l_{x,y} .$$

Put in words, the ultimate normal cost under any aggregate funding method is a constant amount paid each year for each active life, such constant amount to be the same as the amount which, if paid over the future lifetime of each of the entrants of one calendar year, would provide the benefits for these entrants. This result, in retrospect, is not particularly surprising. This normal cost is not precisely the same as that under individual entry age funding, since the latter would be expressed as follows:

$$^{\text{EAN}}C=\sum_{x,y} l_{x,y} B_y \frac{_{r-y|}\ddot{a}_{y,y}}{\ddot{a}_{y,y:\overline{r-y|}}}.$$

The difference between $^{A}C_{\infty}$ and $^{\text{EAN}}C$ is that $^{A}C_{\infty}$ "socializes" the cost as between entrants who, in $^{\text{EAN}}C$, have different level premium costs. There is an actual numerical difference between the two, which is probably unimportant in most cases. The difference ought to be recognized, however, in any theoretical discussion. The two are identical when $B_y \cdot {}_{r-y|}\ddot{a}_{y,y} \div \ddot{a}_{y,y:\overline{r-y|}}$ is constant for all values of y.

It may be felt by some that this type of analysis of funding methods is of little practical value. As a young actuary struggling with practical questions of funding, I believe that many practical questions which have caused me great difficulty in the past can be answered by this paper and its sequelae.

HILARY L. SEAL:

The author has divided methods of funding pension plans into six classes, one of these being further subdivided into four different methods. He has thus specified nine different funding methods. However, by introducing the concept of alternative "immediate" or "spread" adjustments on account of the gains or losses that can occur in seven of these methods, he has effectively provided us with *sixteen* different ways of funding a pension plan. How many of these would be acceptable to the Treasury for tax deduction purposes?

Judging from the opinions expressed in their *Bulletin* of June 1945 the "spread" method of adjusting for *gains* would not be acceptable if the funding was based on (i) unit credit or (ii) individual level premium methods. On the other hand, *losses* could be made subject to some degree of "spread" by using the Treasury's "Special 10% base" in conjunction with the unit credit method. Further, the aggregate method as described in the *Bulletin* automatically uses the "spread" method of adjustment, though it is likely that the "immediate" type of adjustment could be adopted for *gains*. Naturally, the methods classified by the author as V and VI would not be acceptable for tax deduction.

The net result of these tax considerations is to reduce the author's sixteen funding methods to the nine employed in practice, namely:

(1) Pay-as-you-go
(2) Terminal
(3) Unit credit with immediate gains adjustment

(4) Entry age normal with immediate gains adjustment
(5) Entry age normal with spread gains adjustment (frozen initial liability)
(6) Individual level premium with immediate gains adjustment
(7) Aggregate with spread gains adjustment
(8) Attained age with immediate gains adjustment
(9) Attained age with spread gains adjustment

A comparison of the last seven of these from the income tax viewpoint was made by the speaker in the recently published *Proceedings of the Conference of Actuaries in Public Practice*, Vol. ii, 1952. It will be observed from that paper that, contrary to Mr. Trowbridge's opinion, the Treasury's published views on spreading gains are clear and, with one exception, reasonably consistent.

I mention, in conclusion, that methods (3), (8) and (9) above suffer from a serious limitation: they can be applied only where the benefits for each employee may be regarded as accruing each year on a level, or gradually changing, scale. The methods are difficult to apply, for example, to plans with a lump-sum death benefit or where the pensionable earnings base is the average of the five years' earnings preceding retirement.

CHALMERS L. WEAVER:

This discussion is offered to supplement this excellent paper with the answer to a question that occurs on reading the paper. The author demonstrates that if the tabular assumptions as to mortality and interest prevail in the mature state the entry age normal and aggregate methods lead to identical ultimate funds and annual contributions. The question is how these two methods compare at maturity when the population mortality and interest earnings are not tabular. It is to be demonstrated that if the tabular assumptions as to mortality and interest are conservative the aggregate method produces a larger fund and smaller annual contribution at maturity than does the entry age normal method. The converse is true if the tabular assumptions as to mortality and interest are liberal.

The equations at maturity in this more general state are symbolically the same as those in the paper. The difference is that now the annuities in all the formulas given involve tabular assumptions as to mortality and interest that may differ from the mortality indicated by the l's of the population and the actual interest rate earned. The latter are combined with these annuities in the formulas. These more general conditions hold in the symbolic definitions of b, y, and p given in the paper, and in

$$T = \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}} \sum_a^{r-1} l_x .$$

We have, as in the paper,

$$ {}^{\mathrm{EAN}}F_\infty = b - Ty $$

$$ {}^{\mathrm{EAN}}C_\infty = p - d\,{}^{\mathrm{EAN}}F_\infty = p - db + dTv $$

$$ {}^{\mathrm{A}}F_\infty = b - {}^{\mathrm{A}}C_\infty y = \frac{b - py}{1 - dy} $$

$$ {}^{\mathrm{A}}C_\infty = \frac{b - {}^{\mathrm{A}}F_\infty}{y} = \frac{p - db}{1 - dy} = p - d\,{}^{\mathrm{A}}F_\infty $$

$$ = p - db + d\,{}^{\mathrm{A}}C_\infty y\ . $$

From these we obtain

$$ {}^{\mathrm{A}}F_\infty = {}^{\mathrm{EAN}}F_\infty + (T - {}^{\mathrm{A}}C_\infty)\, y $$

$$ {}^{\mathrm{A}}C_\infty = {}^{\mathrm{EAN}}C_\infty - d\,(T - {}^{\mathrm{A}}C_\infty)\, y\ . $$

Note that if tabular assumptions are realized

$$ {}^{\mathrm{EAN}}C_\infty = T\ , \qquad \text{and then} \qquad {}^{\mathrm{EAN}}C_\infty = {}^{\mathrm{A}}C_\infty = T\ . $$

If tabular assumptions are conservative, it is obvious that under the entry age normal method

$$ {}^{\mathrm{EAN}}C_\infty = T - vG = T - (1 - d)G\ , $$

where G is the gain that would turn up at the end of the year if T were the contribution.

$$ vG = T\,(1 - dy) + db - p\ . $$

Then

$$ {}^{\mathrm{A}}C_\infty = T - (1 - d)G - dTy + d\,{}^{\mathrm{A}}C_\infty y $$

$$ {}^{\mathrm{A}}C_\infty = T - \frac{1 - d}{1 - dy}G $$

$$ {}^{\mathrm{A}}F_\infty = {}^{\mathrm{EAN}}F_\infty + \frac{1 - d}{1 - dy}Gy\ . $$

Since in practice $dy < 1$ we have
if $G > 0$

$$ {}^{\mathrm{A}}F_\infty > {}^{\mathrm{EAN}}F_\infty $$

$$ {}^{\mathrm{A}}C_\infty < {}^{\mathrm{EAN}}C_\infty\ , $$

and if $G < 0$

$$ {}^{\mathrm{A}}F_\infty < {}^{\mathrm{EAN}}F_\infty $$

$$ {}^{\mathrm{A}}C_\infty > {}^{\mathrm{EAN}}C_\infty\ . $$

The aggregate method has a fundamental weakness. If the tabular assumptions are modestly on the conservative side the fund will grow to a materially higher limit than the funds that would be held by the entry age normal method. The converse is also true. The method is very sensitive to the tabular assumptions and should not be used unless there is frequent readjustment of these assumptions to realistic values. Most actuaries would rather use modestly conservative tabular assumptions, check these less frequently, and use a less sensitive method in the meantime.

Conclusion.—If tabular assumptions are conservative, then under any funding method the immediate adjustment of gains would still leave a fund that would be adequate. The spread method of adjustment would build up additional funds. If tabular assumptions are liberal, then under any funding method the immediate adjustment of losses would still leave a fund that would be inadequate. The spread method of adjustment would draw down the fund to a lower level.

A numerical illustration has been prepared. The fund and annual contribution at maturity have been computed for *active* lives for a population with entry age 35 and retirement age 65. The population has CSO mortality, and turnover of 5% at ages under 50 graded to no turnover at ages 60 and over. The earned rate of interest is set at three values, 2%, 2½% and 3%. The tabular assumptions are CSO mortality, no turnover, and 2½% interest. The figures for the unit credit method are also included.

		Unit Credit	Entry Age Normal	Aggregate
2% Interest	*F*	$932,000	$1,114,000	$1,607,000
	C	49,400	45,800	36,100
2½% Interest	*F*	$932,000	$1,114,000	$1,768,000
	C	44,600	40,200	24,200
3% Interest	*F*	$932,000	$1,114,000	$1,961,000
	C	39,800	34,600	9,900

GEORGE E. IMMERWAHR:

Mr. Trowbridge's paper answers a long-standing need in actuarial literature for a description and analysis of the pension funding methods commonly used in the United States.

In discussing the Treasury rules relating to limitations applying to deductions for pension contributions, Mr. Trowbridge states that the Treasury position on spread adjustment for gains is not too clear, but that approval of spread adjustment is implied by its description of the aggregate, the attained age normal, and the frozen initial liability methods in the

June 1945 *Bulletin on Section 23 (p)* of the Internal Revenue Code. The Treasury regulations on the matter, as revised in November 1948, state that "in determining the costs and limitations an adjustment shall be made on account of any experience more favorable than that assumed in the basis of limitations for prior years, and, unless such adjustments are consistently made every year by reducing the limitations otherwise determined by any decrease in liability or cost arising from experience in the next preceding taxable year more favorable than the assumed experience on which the costs and limitations were based, the adjustment shall be made by some other method approved by the Commissioner."[1] These regulations would imply the acceptability of adjustments made by the methods set forth in the 1945 *Bulletin,* since that bulletin has not been revoked, or by various other methods satisfactory to the Commissioner. While no amplification of the 1945 *Bulletin* has been published, it would appear, from the types of adjustment regularly employed by a number of consulting actuaries and insurance companies, that the following practices would be found satisfactory.

1. For plans where "spread adjustment" is implicit in the funding method, *e.g.*, in the aggregate method or frozen initial liability method as described in Mr. Trowbridge's paper, full contributions determined in accordance with the method would be deductible provided the following conditions (designed to prevent initial overfunding) are met:

a) adequate allowance is made for withdrawals and mortality, and all other assumptions are reasonable; and

b) no substantial proportion of the contributions is paid on behalf of employees who are unlikely to receive any benefits.

Condition (*b*) can perhaps best be satisfied for the typical plan by eliminating from coverage (at least for the purpose of computing contributions) those employees who fall below a specified age, such as 30, or those with less than a given number of years of service, such as 3 for salaried employees or 5 for hourly-paid employees, or those who fail to meet some appropriate combination requirement of age and years of service. Allowance for withdrawals may sometimes be omitted from condition (*a*) where no salary scale is assumed and where broad enough elimination from coverage is made under condition (*b*).

2. For plans funded by the entry age normal (nonfrozen) method or the unit credit method, a method may be used under which (rather than requiring the full immediate adjustment each year described in Part VIII of the 1945 *Bulletin*) the amount of deductible contribution in any year is

[1] Section 29. 23(p)—4 of Regulations 111, as revised by T. D. 5666.

based upon the revised costs of the plan as they would have been currently determined less any excess of (*a*) amounts of contributions actually taken as deductions in past years, over (*b*) the amounts which would have been deducted based on such currently redetermined cost. An illustration of the application of this method to a typical group annuity is shown in the accompanying table; the application of the method to a self-insured plan funded by the entry age normal method would be somewhat different but would follow from the same principle.

ILLUSTRATION OF MAXIMUM DEDUCTIBLE CONTRIBUTIONS UNDER CONVENTIONAL NONCONTRIBUTORY GROUP ANNUITY PLAN
(Minor interest adjustments ignored)

	YEAR OF PLAN		
	First	Second	Third
1. Initial past service cost as determined at inception of plan	$100,000	....	....
2. Reductions in initial past service cost due to withdrawals in year (other than deaths or ill-health terminations), whether past service annuities had been purchased for withdrawing members or not	6,000	$ 7,000	$ 2,000
3. Initial past service cost as redetermined at beginning of year	100,000	94,000	87,000
4. Gross current service costs in year	15,000	16,000	18,000
5. Cost credits allowed against current service contributions in year, arising from			
a) Cancellation of past service annuities already purchased	0	2,000	500
b) Cancellation of current service annuities already purchased	0	500	900
c) Total	0	2,500	1,400
6. Total contributions paid in year	30,000	30,000	30,000
7. Portion of contributions deductible (and deducted) under method referred to in this discussion			
a) Gross current service costs less current service cost credits (4 less 5*b*)	15,000	15,500	17,100
b) 10% of redetermined initial past service cost (10% of 3)	10,000	9,400	8,700
c) Cumulative deductions for previous years (sum of 7*g* for all previous years)	....	25,000	49,300
d) Sum of 7*a* for all previous years	....	15,000	30,500
e) Current year's 7*b* multiplied by number of previous years	6	9,400	17,400
f) Adjustment = sum of previous years' actual deductions less deduction on redetermined basis (7*c* less sum of 7*d* and 7*e*)	....	600	1,400
g) Deductible contribution for current year (7*a* plus 7*b* less 7*f*)	25,000	24,300	24,400

WILLIAM M. RAE:

It is somewhat difficult to visualize the relationship between the various, apparently unrelated, funding methods. Mr. Trowbridge has very ably demonstrated their relationship by the algebra pertaining to a mature population. I have also found the following general reasoning approach to be quite helpful. It is a prospective approach.

Every funding method calls for determining the group to be valued. This is usually all pensioners and present employees, or all except those not yet meeting certain minimum age and service requirements.

Having determined the group to be valued, every funding method can be viewed as calling for the calculation of the present value of all future benefits for the valuation group *as a closed group*. This present value, less funds on hand, is then split into two parts, (*a*) unfunded accrued liability and (*b*) present value of future normal costs. The split is dictated by the funding method chosen. Each part is then amortized over a period of years in the manner dictated by the particular funding method. Under some methods (*e.g.*, aggregate method, individual level premium method) the amortization scheme is the same for both parts. The different amortization schemes of the various funding methods cause the different incidence of annual cost between the methods.

The amortization scheme, in dollars, for (*b*) above can be level *as to an individual* or increasing as to an individual (*e.g.*, entry age normal with salary scale method, unit credit method), but will decrease in the aggregate as the closed valuation group is assumed to retire, die or withdraw.

The valuation process in subsequent years can be viewed in exactly the same fashion, subject to whatever adjustment for gains and losses is called for by the particular funding method. In subsequent years we will again be valuing a closed group, but the composition of the closed group will be different from that of the preceding year. New lives will have been added. These, broadly speaking, counterbalance the exits of the previous year. As a consequence the total normal cost will not actually decrease from year to year as might be inferred from the preceding paragraph.

It is theoretically possible to value an open group rather than a closed group, making assumptions as to new entrants in future years. Mr. Trowbridge does this in his Demonstration I. In practice it is rarely, if ever, done for private pension plans.

FRANK L. GRIFFIN, JR.:

The author is to be complimented on a clear exposition of the nature of various methods of budgeting pension costs. While the paper deals with matters largely theoretical, and therefore does not lend itself to a discussion from the standpoint of the strictly practical problems faced by consulting actuaries, nonetheless his general approach, omitting the mathematical symbolism, is sometimes found useful by consultants in dispelling for their clients the "technical mysteries" of different methods. Furthermore, an extension of the principles set forth in the paper makes possible an appraisal of the results obtained by using various methods in an actuarial valuation of costs, considering both the nature of the employee group and the purpose to be served by the particular valuation.

For his classification of funding methods, the author has made use of the "Equation of Maturity," $C + dF = B$, in which only the size of the ultimate contribution (C) and fund (F) will vary according to the method. Using the so-called mature population concept, he determines the ultimate C and F, by means of which the funding methods are classified in a logical order—namely, in ascending order of F (descending order of C). Omitting Classes V and VI, which were included in the paper for theoretical reasons only, the remaining classes are: (I) pay as you go, (II) terminal funding, and two classes (III and IV) of funding in advance of retirement. The separation of funding in advance of retirement into Classes III and IV was necessitated on the basis of the ultimate contribution required, a point on which further comment will be made later.

One or two comments relative to the mature population concept which forms the basis of the author's presentation may be in order. A "mature age distribution" and a "stationary population" are not one and the same for purposes of a pension forecast, since the size of an employee group may remain stationary indefinitely without its having reached a mature age distribution. The difference, of course, can be brought about by a varying number of new hires each year or by hirings at many different ages, rather than a uniform number of hires each year at the youngest age of the service table which is the unrealistic assumption inherent in the conventional maturity concept.

In the case of a well established organization, the assumption of a constant work force moving toward maturity in its age distribution is probably as defensible as any other approach. However, the "mature age distribution" which the group might be considered to reach would not be of the form usually assumed, namely, proportionate at all ages to the l_x^s column of the service table. The latter would be true, as indicated in the preceding paragraph, only if all new entrants came into service at the youngest age of the service table. If, for example, a constant number of annual new entrants were distributed in fixed ratios at each age from 20 to 40, the ultimate "mature age distribution" would be in proportion to the l_x^s column only at ages 40 and over. Below age 40, the distribution would be in proportion to $l_x^s \sum_{y=a}^{y=x} (H_y \div l_y^s)$ where H_y is the percentage of total hirings at age y.

In practice, none of the "ideal" conditions of a mature population (either initially or in the ultimate) will ever be found. Notwithstanding this fact, the concept may serve a useful purpose as a limiting value in a pension projection. For example, if the actuary wishes to compare the results of a valuation by any particular cost method, with a projection of pension payouts considering future new entrants, the reasonableness of his results for a "going concern" or the relative trend of costs by different methods may be made apparent.

Since the actuary is confronted, not with a mature group, but with a group of unknown future age distribution and size, practical considerations usually dictate that any valuation he makes (Class III or IV) be limited to the group of employees existing on the date of valuation, without allowance on any empirical basis for any new entrants of the future. Depending on the actuarial cost method, the resulting costs may or may not reasonably approximate the long range requirements, even in a case where it is thought that the work force will remain constant in the future; and a projection of payouts (Class I) or terminal funding requirements (Class II), taking into account future new entrants on a reasonable basis for maintaining the work force, may help to establish the relative merits or deficiencies of different Class III or IV valuation methods for a "going concern."

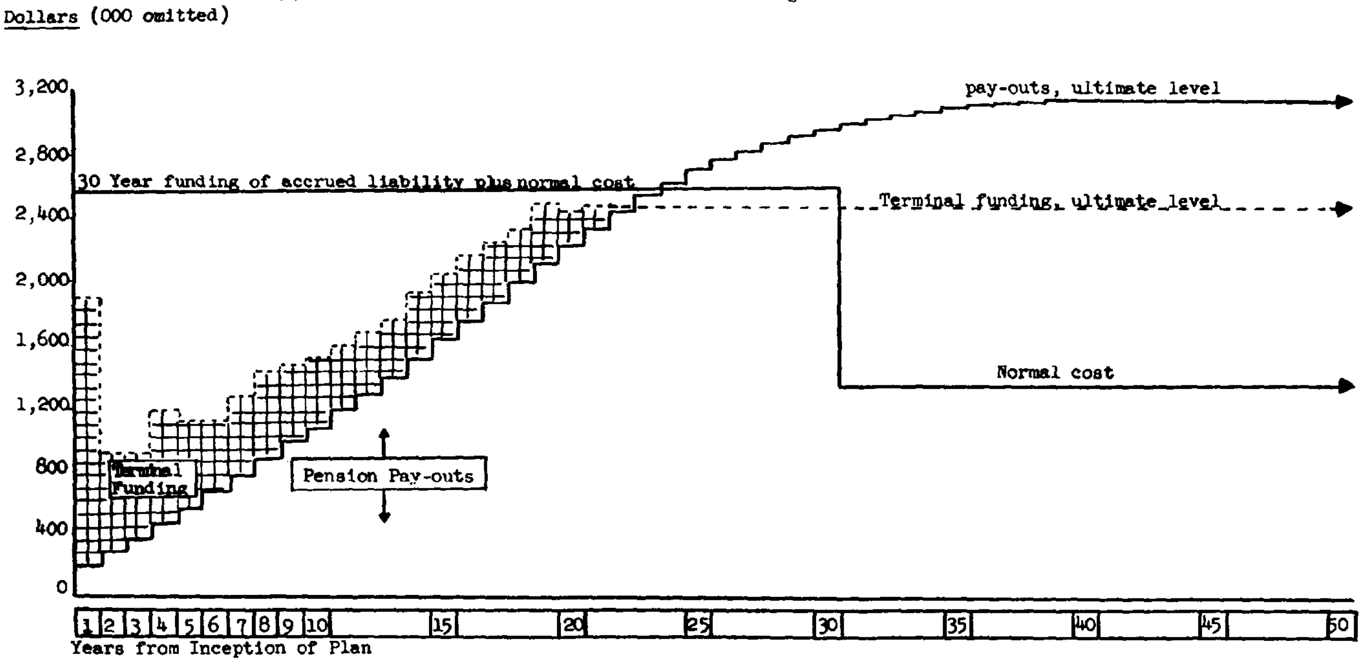
Obviously, if we were in a position to predict the new entrants of the future with any accuracy, the projected requirements by Class I or II would be exactly equivalent financially to the contributions developed, in turn, by the initial and successive future valuations of the plan, by any valuation method selected. Therefore, the result obtained by a particular method in a single valuation, measured against a long range projection of disbursements, affords an indication of the reliability of such result in relation to future requirements, or, what is the same thing, the relative trends which contributions determined by different valuation methods will follow in future years.

The accompanying chart, prepared for a large organization, sets out the projected payouts and terminal funding requirements against the indicated annual contributions determined from an initial valuation by the entry age normal method. In this chart, the discounted value of payouts

CHART 1

PROJECTION OF PENSION CONTRIBUTIONS, PAYOUTS, AND TERMINAL FUNDING REQUIREMENTS

Assuming: (1) Constant Work Force Supported by New Hires with Identical Entry Ages as the Original Group
(2) Mortality, Disability, Withdrawal, and Interest as Assumed in Valuation and
(3) Indefinite Continuation of the Plan without Change



into perpetuity, considering new entrants, is practically identical to the discounted value of contributions into perpetuity, *if such contributions determined in respect of the present employee group only were to remain at their originally determined level.* The propriety of the valuation method for a continuing plan and a "going concern" is thus reasonably well established. In contrast, if the "unit purchase" method had been employed in this case, the indicated level of contribution (initially determined amount) would have been much less, leading to the conclusion that contributions by such a method would increase in the future if the group were to maintain its size.

Other actuarial assumptions being the same, the entry age normal method always produces a higher accrued or past service liability than the unit purchase method. The relative size of the normal (or current service) costs, however, will depend on the existing age distribution on date of valuation. Examples of comparative figures by the two methods, derived from other cases, are as follows:

	Past Service Liability	Normal (Current Service) Cost
Case A: Entry age normal	$37,908,000	$2,461,000
Unit purchase	21,895,000	2,401,000
Case B: Entry age normal	$ 820,000	$ 76,000
Unit purchase	601,000	71,000

The wide difference in the results of initial valuation by the two methods, when it is a certainty that all methods must produce the same capitalized value of contributions, points up the absurdity of trying to compare such results without recognizing the different purposes they are intended to serve. The difference in purpose, implicit in the author's separate treatment of Class III and IV methods, may be stated briefly as follows. From one viewpoint (that of a going concern and a continuing plan), the funding requirements developed by the valuation should take into account not only the past but also the future requirements on a basis which will tend to equalize long range trends in the age distribution. The entry age normal method does this to the maximum extent possible for a group assumed to be stationary in size. From another viewpoint (that of establishing liquidation values under a terminating plan, *i.e.*, accrued liabilities without regard to the future), the requirements developed by the valuation will take into account only the past. The unit purchase method is the only one which provides this particular answer, and it will do no more.

The structure of Class IV methods, adapting them to the requirements of a "going concern," is therefore such as to strike a balance at a given moment of time between (*a*) the existing funds and anticipated future income, and (*b*) anticipated future disbursements. The Class III method (unit purchase) omits all consideration of future income, and of future disbursements arising from pension credits for service after the valuation date.

Of the Class IV methods, the entry age normal undoubtedly has more to commend it in the usual case than the others which the author has mentioned. It may be of interest, therefore, to illustrate the manner in which (and the conditions under which) this particular method will afford a reasonable representation of the long range requirements for an organization expecting to continue in business indefinitely. The following valuation results are presented on the basis of a conventional valuation, and the present values of both benefits and normal costs are with respect to present employees only, without allowance for any future new entrants.

1. Present Value of Future Benefit Payments	$42,000,000
2. Present Value of Future Normal Costs	$16,000,000
3. Balance = Gross Accrued Liability	$26,000,000
4. Funds Accumulated	$ 7,000,000
5. Balance = Unfunded Accrued Liability	$19,000,000
6. Normal cost (in addition to any payments toward unfunded accrued liability), used in determination of Item (2)	$ 1,300,000

Put in the form of a balance sheet, the asset items would be items (4), (2) and (5), usually in that order, and the balancing liabilities would be represented by item (1).

To complete the illustration, *if future new entrants were introduced in such a manner as to maintain a constant normal cost in future years*, the balance between the asset and liability figures would not be disturbed. For example, assuming new hires sufficient to maintain a constant work force and at the same entry ages as the group being replaced each year (one of several possible assumptions), the normal cost developed by the initial valuation would be paid in perpetuity, and the benefits ultimately payable to new entrants of the future would (on the actuarial assumptions) be exactly met by their normal costs. This being the case, items (1) and (2) of the above table would be increased by exactly the same amount, leaving all other figures unaffected. Thus, one of the virtues of the entry age normal method is that, even though future new entrants are not specifically considered in a valuation, the result will be as good an approximation to the long range level of costs for a well established continuing organization as it is possible to furnish.

Because the concept fits naturally into the author's classification of funding methods by means of an ultimate "Equation of Maturity," considerable stress has been given in the preceding paragraphs to the "going concern." This concept implies the use of a cost method which, for a constant work force, would develop an initial normal cost as consistent as possible with the normal costs developed in the ultimate situation. My remarks do not in any way bear upon the aptness of the experience assumptions selected by the actuary, nor upon widely divergent philosophies as to the timing of pension contributions over a long period of time. Obviously, there may be situations where even Class I or Class II funding can be considered appropriate, at least for temporary periods. Class III represents a big step toward funding in advance of retirement, and the fact that this method is not designed on a "going concern" basis does not destroy its usefulness for purposes other than the establishment of termination values. Class IV represents the practical ultimate in advance funding.

Mr. Trowbridge's paper provides a logical system for the classification of actuarial cost methods and its clear and concise presentation will undoubtedly be found helpful by many students of the subject.

(AUTHOR'S REVIEW OF DISCUSSION)

CHARLES L. TROWBRIDGE:

Mr. Link's analysis puts certain limitations on the ultimate identity of the aggregate and entry age normal methods. He grants the conclusion reached by the paper where a single entry age is assumed, but proves that if the stationary population arises from entrants at several ages, ${}^{A}C_{\infty}$ and ${}^{EAN}C_{\infty}$ are no longer identical, although the numerical difference may be unimportant.

The aggregate method, in effect, views the multi-entry age stationary group as if it all entered at the youngest possible entry age, with later entrants treated like negative withdrawals. The resulting normal cost is, as Mr. Link says, a "socialization" of the individual normal costs at exact entry ages.

Mr. Link is also quite correct that the initially immature group approaches the ultimate mature state asymptotically from both sides. Under the conditions stated for Table IV, the group passes from badly immature to somewhat overmature, then back to slightly immature, etc. This came as somewhat of a surprise to the author, as it did to Mr. Link.

Mr. Foster's Classes 1½ and 2½ can be presented algebraically and numerically in the same fashion as the other methods have been analyzed in the paper. The following table presents the formulas for normal cost and ultimate fund for the initially stationary population, assuming that in Class 1½ the funding period beyond retirement is uniformly five years, and assuming in Class 2½ a "waiting period" of five years.

Class 2½ naturally breaks into two subclasses, depending on whether Class III or Class IV funding becomes subject to the waiting period. Perhaps Mr. Foster would permit me to call these III minus and IV minus. The numerical illustration can be considered an addition to Table II, and to the limiting situation in Table IV.

CLASS	NORMAL COST	ACCRUED LIABILITY (ULTIMATE FUND)
	Algebraically	
1½ (or II minus)	$\frac{\ddot{a}_r}{\ddot{a}_{r:\overline{5\vert}}}\sum_{r}^{r+4} l_x$	$\sum_{r+1}^{\omega} l_x\ddot{a}_x - \frac{\ddot{a}_r}{\ddot{a}_{r:\overline{5\vert}}}\sum_{r+1}^{r+4} l_x\ddot{a}_{x:\overline{r-x+5\vert}}$
2½ (or III minus)	$\frac{1}{r-a-5}\sum_{a+5}^{r-1} l_x \cdot {}_{r-x\vert}\ddot{a}_x$	$\frac{1}{r-a-5}\sum_{a+5}^{r-1}(x-a-5)\, l_x \cdot {}_{r-x\vert}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x$
2½ (or IV minus)	$\frac{{}_{r-a-5\vert}\ddot{a}_{a+5}}{\ddot{a}_{a+5:\overline{r-a-5\vert}}}\sum_{a+5}^{r-1} l_x$	$\sum_{a+5}^{r-1} l_x \cdot {}_{r-x\vert}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x - \frac{{}_{r-a-5\vert}\ddot{a}_{a+5}}{\ddot{a}_{a+5:\overline{r-a-5\vert}}}\sum_{a+5}^{r-1} l_x\ddot{a}_{x:\overline{r-x\vert}}$
	Numerically	
1½ (or II minus)	\$53,206	\$ 401,542
2½ (or III minus)	\$35,410	\$1,131,208
2½ (or IV minus)	\$30,978	\$1,312,888

For practical work involving Class 2½ it is not uncommon to offset the cheapening of the funding method due to exclusion of certain lives from the funding by overconservative assumptions with respect to the included lives. This tends to "remove the minus" and bring the funding closer to Classes III or IV.

Mr. Rae analyzes the various funding methods by means of the concept of an "ever changing closed group." This analysis is particularly appealing because it follows exactly the kind of group actually employed in practical valuations. The "open group" approach to which he refers is of considerable theoretical interest, even though the necessity for assumptions as to future new entrants eliminates it for most practical work.

Dr. Nesbitt's (and Mr. Feraud's) "general average premium" is of course a result of the "open group" approach. The general average premium π^c in the stationary population assumed is equivalent to the pay-as-you-go contribution, which is in turn equivalent to π^n, or what the paper refers to as Class IV normal cost, plus interest on the Class IV accrued liability. If we now shift our frame of reference and think of π^c instead of π^n as the "normal cost," the corresponding "accrued liability" becomes 0. It is under these latter definitions that the anticipated gains from future new entrants offset the shortage of funding in respect to the initial group. Dr. Nesbitt states that in general he does not advocate the discounting of such gains; I assume this means that he ordinarily recommends that the accrued liability (in the sense used in the paper) should eventually be funded.

Mr. Griffin views Class III funding as essentially retrospective, looking back at benefits accrued. On the other hand he thinks of the Class IV methods as fundamentally prospective, and points out that under certain conditions the initial normal cost is representative of the ultimate cost. These conditions involve among others an unchanging average entry age.

Suppose, however, that present hiring policies indicate a different entry age for future new entrants than the average for initial participants. In such cases the above characteristic of Class IV funding can be preserved only by assuming for the initial group the same pattern of hiring ages as is indicated for the future. If it is important that future normal costs remain relatively constant, this modification of the usual exact entry age method would seem to be appropriate.

Dr. Seal attempts to put a practical emphasis on what is essentially a theoretical paper. After examining the theoretical possibilities to determine which are acceptable to the Treasury, he produces a list of nine "practical" methods, three of which he limits to certain types of benefit formulas.

Dr. Seal's inclusion of the immediate adjustment form of attained age normal in a list of practical methods is rather surprising, since immediate adjustment for gains or losses is as difficult to make in this method as it is in the aggregate method. Perhaps Dr. Seal is thinking of immediate adjustment in respect to the past service portion (this appears to be feasible), but with a spread of gains or losses arising from the future service portion.

Evidently Dr. Seal finds something not apparent to the author in the *Bulletin on 23(p)*, leading him to the conclusion that spread adjustment for gains is not acceptable under unit credit funding. True, the *Bulletin* does not specifically permit the practice in question; nor does it rule against it. The same situation exists in regard to the technique described by Mr. Immerwahr, which he has found to be acceptable despite its non-inclusion in the *Bulletin*.

Mr. Immerwahr's remarks center around the Treasury regulations with respect to adjustment for actuarial gains. His two conditions under which the spread adjustment technique is acceptable appear to be essentially the same. If I understand him correctly he states that spread adjustment is acceptable provided turnover is adequately recognized—either by a realistic withdrawal assumption, or by sufficient elimination of short service employees from the funding.

It is interesting to note that he has found acceptable a modification of immediate adjustment. This modification appears to amount to the spreading of gains arising within the initial accrued liability over the minimum funding period for such liability, even though greater gains may occur in a particular year.

Mr. Weaver reaches the conclusion that if gains predominate the spread adjustment form of any funding method produces a higher fund than the corresponding immediate adjustment form; but conversely a lower fund is produced by spread adjustment if assumptions are unconservative and losses prevail. The validity of Mr. Weaver's conclusion can be demonstrated rather easily by simple general reasoning. If there is no change in assumptions (and under these conditions a change would seem to be appropriate) spread adjustment tends to exaggerate the overfunding arising from assumptions that prove to be too conservative, and also tends to accentuate any underfunding arising from too liberal assumptions.

The reader of Mr. Weaver's discussion should realize that the comparison there being drawn is between the aggregate method (spread adjustment technique) and the immediate adjustment form of entry age normal. The "frozen initial liability" form of entry age normal produces the same eventual fund and same ultimate contribution as aggregate, even if tabular assumptions are not realized (subject to Mr. Link's exception as to multiple entry ages).

The author does not feel particularly qualified to comment on Mr. Williamson's observations regarding the funding of the Federal Civil Service Retirement System and OASI. Mr. Williamson's comments bring to mind, however, an earlier study of pension funding which might well be brought to the attention of those interested in this subject. I refer to

Actuarial Study No. 10 of the Office of the Actuary, Social Security Board, entitled "Various Methods of Financing Old-Age Pension Plans." Mr. Williamson, Mr. R. J. Myers, and Mr. E. A. Rasor were the authors of this pamphlet, which is an excellent primer on funding method, written in 1938 at the time of the controversy over reserve financing of OASI.

Since the publication of the paper the Treasury position with respect to maximum deductions under individual level premium funding has been changed with the Commissioner's acquiescence in the *Saalfield* decision. It now appears that the contributions called for by individual level premium funding are fully deductible, even if in excess of the "normal cost plus 10%" maximum for entry age normal.

The author wants to thank the several persons who participated in the discussion of the paper, each of whom have in one way or another added to published knowledge regarding methods of pension funding. Even so the author would like to echo Mr. Link's statement to the effect that there is a long way yet to go.