Risk-Adjusted Underwriting Performance Measurement

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Abstract

To measure economic profits generated by an insurance policy during its lifetime, we compare the terminal assets of the policy account with certain break-even values. Policies with multi-year loss payments and income tax payments are studied. The break-even value of terminal assets is given in closed form, and shown to be an increasing function of the claims risk and the asset investment risk. Profits from underwriting and from capital investment are measured separately. Simple equations are found that link the cost of capital to the risk-adjusted loss discount rate. Methods developed in the paper are also useful for fair premium calculations.

Keywords

Risk-adjusted performance measure, risk-adjusted discount rate, policy-year profit, cost of capital, EVA, fair premium

1 Introduction

A typical property-casualty insurance policy covers one accident year, but its claims stay open for many years. The ultimate profitability of the policy cannot be determined until all claims are settled. In the interim, profits may be estimated by projecting future investment gains and loss and expense payments. At the moment the premium is collected, a policy account is established, whose initial asset is the premium net of acquisition expenses. The value of the asset increases as investment incomes accumulate, and decreases as losses, expenses and taxes are paid. At the end of the policy life, the terminal asset of the policy account is the ultimate profit generated by the policy. A policy is considered profitable, if its terminal asset exceeds a certain break-even value. A main goal of the paper is to calculate this break-even value.

A number of favorable factors allow a policy to generate positive profits. These include higher premiums collected, lower losses and expenses paid, payments made later rather than sooner and extraordinary investment gains. If all these factors are at their fair or expected values, the resulting terminal asset is the break-even terminal asset. Modern finance teaches that the expected return of an asset investment is in direct proportion to the investment's risk. Likewise, the break-even terminal asset will be shown directly related to the claims risk and the asset investment risk.

For investment portfolios, there are many risk-adjusted performance measures, including the well-known Sharpe ratio, the Treynor ratio (ratio-based tests) and Jensen's alpha (a value-based test), all thoroughly discussed in Part 7 of Bodie et al. (2002). In insurance, the risk-adjusted return on capital (RAROC) and the economic value added (EVA) have become popular. All these measures, however, are designed for testing performances in one time period. It is much harder to construct a performance test for a real insurance policy, whose claim payments span across multiple years. The RAROC and the EVA are usually applied on the calendar-year basis, thus cannot answer the question whether a policy (or a policy year) is ultimately profitable.[1] The internal rate of return (IRR) of equity flows is a valid policy-year metric. Yet it cannot be called a risk-adjusted measure unless it is explicitly linked to the underwriting and investment risks. Such a link will be discussed in this paper. Further, the RAROC, IRR and EVA all measure total profits from both underwriting and investment operations. It is useful for the underwriting managers to know if the underwriting operation alone, for a particular policy or in a particular year, is successful. Our approach will address this issue directly.

Few research papers have centered on the economic performance of a multi-year property-casualty (P&C) policy. Among them Schirmacher and Feldblum (2006) is a noticeable one. It uses a numerical example to examine how profits emerge over time. It shows that calendar-year profits depend on the accounting system. Our concern here is on the ultimate economic profit, which is independent of the accounting system used. In computing the EVA, Schirmacher and Feldblum (2006) assumes that a cost of capital (COC) is given extraneously. As just mentioned, we will relate it to internal risk metrics.

Profit measurement is intertwined with fair premium determination. The

[1] Some one-year tests have been adapted for use in a multi-year framework and on the policy-year basis, see Goldfarb (2006). But these are only tentative solutions and lack solid theoretical support.

break-even level of profit is produced by assuming that the premium is at its fair level, and investment returns and loss and expense payments are all at their expected values. Conversely, the fair premium may be determined by setting to zero the market value of the policy account terminal asset. Our study thus may be found useful for premium calculation.

The paper is organized as follows. Section 2 starts with a discussion of the risk-adjusted loss discount rate. The discount rate is used to quantify a policy's risk. Then, for a single-year model, the break-even value of terminal assets is derived. The break-even value is an increasing function of asset and claims risks. This result is generalized to a multi-year model in Section 3. Here a numerical example is introduced that will be used throughout the paper to illustrate calculations. The main results of the paper are stated in Section 4. Income tax is brought into the model. Closed-form formulas are derived for the fair premium and break-even terminal assets. To obtain tractable results, tax rules are simplified. In Section 5, we give equations linking the break-even terminal assets to the COC. A consequence of the relationship is that the COC is an increasing function of the claims risk, and a decreasing function of the initial capital. In Section 6, we show how to define the EVA for the policy account only, and separately measure profits from underwriting and from capital investment. Section 7 further shows that, since the loss discount rate and the COC both characterize the internal risk of the company (assuming investments are risk-free), each can be derived from the other with simple equations. The fair premium for a policy may be calculated by selecting either a loss discount rate or a target COC. Section 8 concludes the paper.

2 Benchmark of Underwriting Profit

Issuing an insurance policy establishes a mini-bank, which we will call a policy account. The starting asset of a policy account is premium minus acquisition expenses. The value of the asset then changes in time. Investment gains increase the value, and loss and expense payments decrease it. The remaining assets right after the last payment is made are called the terminal assets. Actuaries that have done analysis on finite reinsurance should be familiar with the calculation of terminal assets. For a policy to be profitable over its lifetime, the terminal assets must be positive and sufficiently high. Intuitively, the more risky the policy, the greater the terminal assets need to be. The purpose of this paper is to derive a benchmark for terminal assets in the framework of modern finance.

2.1 Discount Rate for Liabilities

Pricing actuaries use various "loadings" to quantify claims risk of a policy. Loadings may be additive or multiplicative, or may be calculated by risk-adjusted discounting. One approach may be easier to apply than another in a given situation, but they are all equivalent mathematically. The most convenient way for presenting out results is risk-adjusted discounting.

The following example illustrates how the risk-adjusted discount rate reflects the risk level of a policy. Assume a policy has only one loss, which will be paid in one year, and the expected payment is $100. The risk-free interest rate is 4 percent. The fair premium for the policy is the sum of market values of future claims, expenses and capital costs.[2] If the claim amount is $100 certain, its market value is the risk-free present value $100/1.04 = 96.15$. Market value of a claim increases with riskiness of the claim, whereas the risk-adjusted discount rate decreases. Table 1 shows the risk-adjusted rates implied in the market value, with different levels of claims risk.

Table 1: Risk-Adjusted Discount Rate

Riskiness of Claim (1)	Expected Claim (2)	Risk-free Present Value (3)	Market Value (4)	Risk-Adjusted Discount Rate (5)
No Risk	100.00	96.15	96.15	4%
Low Risk	100.00	96.15	97.09	3%
High Risk	100.00	96.15	200.00	-50%

(5) = (2)/(4) − (1)

In general, let r_f be the risk-free rate, L the random claim amount, and r_l the claim's risk-adjusted discount rate. Then the risk-free present value $\text{PV}(L) = E[L]/(1 + r_f)$, and the market value $\text{MV}(L) = E[L]/(1 + r_l)$. For a risky claim, $\text{MV}(L) > \text{PV}(L)$, the difference being the claim's risk margin.[3] This implies

[2] This fair premium formula is derived from the net present value principle in finance. A well-known early paper that advocates it is Myers and Cohn (1987). It is now a standard assumption in theoretical research. Its use in actual pricing is somewhat limited by lack of method for calculating market value of future claims.

[3] Although it is intuitively clear that if a claim is risky, its value should contain a positive risk margin, empirical evidence that supports the assertion is sparse. Most authors assume the risk margin is positive (Myers and Cohn 1987, Bingham 2000). But Feldblum (2006) suggests that

$r_l < r_f$. If risk is low, r_l is close to r_f. For a highly risky policy, r_l approaches -1.

2.2 Break-even Terminal Assets

Let p be a policy premium net of acquisition expenses. Then p is the starting balance of the policy account. Assume the balance is invested in a financial security whose annual return is a random variable R_a; the policy losses L, paid one year later, has a risk-adjusted discount rate of r_l. So the expected return of assets is $r_a = E[R_a]$, and the market value of loss is $\mathrm{MV}(L) = E[L]/(1 + r_l)$. Capital cost is not considered in this section.

If the fair premium is charged, then $p = \mathrm{MV}(L) = E[L]/(1 + r_l)$. At the end of the year the assets grow to $\mathrm{MV}(L)(1 + R_a) = E[L](1 + R_a)/(1 + r_l)$. After loss L is paid, the terminal assets[4] are

$$A_1 = \mathrm{MV}(L)(1 + R_a) - L$$

The expected value of terminal assets is

$$a_1 = E[A_1] = \mathrm{MV}(L)(1 + r_a) - E[L] = \mathrm{MV}(L)(r_a - r_l) = \frac{E[L]}{1 + r_l}(r_a - r_l) \quad (2.1)$$

Since $r_a \geq r_f$ and $r_l \leq r_f$, $a_1 \geq 0$. A more risky security has a greater r_a, and a more risky liability has a smaller r_l. So the spread $r_a - r_l$ is a measure of total risk of the policy account. a_1 is in proportion to the total risk.

a_1 is the risk-adjusted break-even value of terminal assets. If the actual realized terminal assets are greater than a_1, the company makes money on the policy. If the terminal assets are only slightly positive, but less than a_1, the company appears to make money, but actually does not make enough to compensate for risk. If the claim or the investment is very risky, the terminal assets need to be very high.

Comparing the actual terminal assets with a_1 gives us a value-based test for underwriting performance. It is interesting to compare this with the combined ratio, a ratio-based underwriting performance measure. The undiscounted combined ratio is used most often. But it has an obvious shortcoming—it does not reflect the time value of money. Two lines of business may have the same combined ratio, but the longer tailed line pays out losses more slowly, generates more investment income along the way and is more profitable. The economic combined ratio (ECR)

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most P&C liabilities have no systematic risk (i.e., uncorrelated with the market return), thus the risk margin equals zero. The disagreement can only be settled with future empirical research.

[4] Terminal assets are profits generated by the policy over its lifetime. I choose to use "assets" rather than "profits" because, for a multi-year model, I will keep track of values of assets from year to year.

is introduced to correct this problem. In the calculation of ECR all underwriting cash flows are discounted, at the risk-free rate, to the time of policy inception. The ECR is strongly advocated in Swiss Re (2006), which claims that the ECR of 100 percent "truly indicates the watershed between profit and loss" (page 24). When investments and claims are risky, however, the risk-adjusted break-even ECR is not 100 percent, but something lower. Table 2 gives the break-even ECR and the break-even terminal assets for the three policies in Table 1, with the following additional assumptions: First, fair premium is charged and there is no expense, i.e., $p = \mathrm{MV}(L)$; and second, investment is riskless, i.e., $r_a = r_f = 4\%$.

Table 2: Break-even ECR and Break-even Terminal Assets

Riskiness of Claim (1)	$E[L]$ (2)	r_l (3)	$p = \mathrm{MV}(L)$ (4)	$\mathrm{PV}(L)$ (5)	Break-even ECR (6)	a_1 (7)
No Risk	100.00	4%	96.15	96.15	100.0%	0.00
Low Risk	100.00	3%	97.09	96.15	99.0%	0.97
High Risk	100.00	-50%	200.00	96.15	48.1%	108.00

(6) = (5)/(4)

(7) = (4) × (4% − (3))

The ECRs in Table 2 are computed with the fair premium in the denominator. So they are the break-even ECRs. The table shows if claims risk is high, the break-even ECR is much lower than 100 percent, and a_1 is very large compared to $E[L]$.

3 Multi-Year Underwriting Profit Measure

3.1 Break-even Terminal Assets a_n

A typical P&C policy has a multi-year payout pattern. Assume a policy is written at time 0, and the loss payments are random variables $L_1, \ldots, L_n$, L_i paid at time i. The nominal total loss is $L = L_1 + \ldots + L_n$. We will derive a break-even value for terminal assets at time n, after all losses are paid. Assume a loss discount rate r_l, constant throughout the n years, can be found that correctly reflects the risk of the payments L_i. Then the market value, at time 0, of the payments is

$$\mathrm{MV}_0(L) = \frac{E[L_1]}{1+r_l} + \ldots + \frac{E[L_n]}{(1+r_l)^n} = \sum_{i=1}^{n} \frac{E[L_i]}{(1+r_l)^i} \tag{3.1}$$

The fair premium of the policy, net of expenses, equals this market value: $p = \mathrm{MV}_0(L)$. Let the premium be invested risk free, and r_f be the constant risk-free rate.[5] The following formulas give the expected net assets at each time i, after loss L_i is paid

$$\begin{aligned}
a_1 &= p(1+r_f) - E[L_1] \\
a_2 &= p(1+r_f)^2 - E[L_1](1+r_f) - E[L_2] \\
&\vdots \\
a_n &= p(1+r_f)^n - E[L_1](1+r_f)^{n-1} - \ldots - E[L_n]
\end{aligned}$$

a_n is the expected (benchmark) terminal assets of the policy. Substituting equation (3.1) for p in the last formula, we get

$$\begin{aligned}
a_n &= (1+r_f)^n \sum_{i=1}^{n} E[L_i]\left((1+r_l)^{-i} - (1+r_f)^{-i}\right) && (3.2) \\
&= (1+r_f)^n \left(\mathrm{MV}_0(L) - \mathrm{PV}_0(L)\right) && (3.3)
\end{aligned}$$

$\mathrm{MV}_0(L) - \mathrm{PV}_0(L)$ is the risk margin of the policy losses, which is greater than 0 if $r_l < r_f$. A less risky policy has a smaller spread $r_f - r_l$, and a relatively smaller a_n. A very risky policy can have a negative r_l, and a very large a_n.

a_n is a function of expected values of loss payments, investment rates and discount rates. These expected values are generally forecasted at the beginning of the policy term, when an initial a_n can be computed. a_n may be revised later as new information about the claims, markets and general economy comes in. But it should not be affected by normal fluctuations in loss payments and investment returns. Final assessment of a policy's profitability has to wait until the end of year n, when the actual terminal assets can be calculated. But an interim estimate may be performed by projecting future actual loss payments and investment returns. A number of beneficial factors can render a policy profitable: higher (than expected) premium, smaller losses, slower loss payments or higher investment returns. Among them the premium is what a company has the most control over.

3.2 A Numerical Example

I will use a multi-year numerical example to illustrate the calculations in this paper. The example is borrowed from Schirmacher and Feldblum (2006), so we

[5] For simplicity, results on multi-year models in this paper are stated for risk-free investment returns. With little additional work, they can be generalized to random (risky) investments, as long as the investment returns in different time-periods are independent.

can compare their methods with ours. Assume a policy is issued on Dec. 31, 20XX, for accident year 20XX+1. The underwriting cash flows are as follows. On Dec. 31, 20XX (time 0), a premium of \$1,000 is collected and acquisition expenses of \$275 paid. General expenses of \$150 are paid six months later (time 0.5). The policyholder has one accident in the year and will receive one payment of \$650 on Dec. 31, 20XX+3 (time 3).

Schirmacher and Feldblum (2006) choose a surplus requirement of 25 percent of the unearned premium reserve plus 15 percent of the loss reserve. The risk-free rate is 8 percent per year compounded semi-annually (4 percent per half year). The basic policy cash flows are summarized in Table 3.

Table 3: Policy Account Assets - No Tax

Time (1)	Premium (2)	Expense (3)	Loss (4)	Investment Income (5)	Policy Account Assets (6)
0.0	1000.00	275.00	0.00	0.00	725.00
0.5	0.00	150.00	0.00	29.00	604.00
1.0	0.00	0.00	0.00	24.16	628.16
1.5	0.00	0.00	0.00	25.13	653.29
2.0	0.00	0.00	0.00	26.13	679.42
2.5	0.00	0.00	0.00	27.18	706.59
3.0	0.00	0.00	650.00	28.26	84.86
Sum	1000.00	425.00	650.00	159.86	
PV	1000.00	419.23	513.70		

$(6)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0}$
For $i > 0.0$, $(5)_i = (6)_{i-0.5} \times r_f$
For $i > 0.0$, $(6)_i = (6)_{i-0.5} + (2)_i - (3)_i - (4)_i + (5)_i$

The nominal losses (L) and expenses (X) add up to \$1,075, and the premium (p) is \$1,000. The underwriting profit is -\$75 and the combined ratio is $1,075/1,000 = 1.075\%$. The risk-free present value of losses and expenses $\text{PV}_0(L + X) = 932.94$, and the ECR equals $\text{PV}_0(L + X)/p = 93.29\%$. By the ECR standard (Swiss Re 2006, as proposed in) the policy is profitable, but the ECR standard incorrectly ignores risk.

To compute the risk-adjusted break-even value $a_{3.0}$, we need a few more assumptions. Assume $r_l = 3\%$ per half year, and the only loss payment of \$650 at time 3 is both the expected and the actual loss. By (3.2), the break-even net assets at time 3 are $(1+0.04)^6 \cdot 650 \cdot \big((1+0.03)^{-6} - (1+0.04)^{-6}\big) = 38.80$. Since the actual terminal assets are \$84.86 (column 6 of Table 3, last entry), greater than \$38.80, the policy is profitable under the risk-adjusted measure.[6]

The economic value added by the policy is $84.86 - 38.80 = 46.06$. However, this assessment overstates the quality of the policy, because the $\$1,000$ premium includes a provision for income taxes, but taxes are omitted so far. Taxes are a significant cost, which I will discuss in the following sections.

4 After-Tax Profit Measures

Insurance companies have a greater tax burden than non-financial companies. In addition to taxes on profits from underwriting and premium investment, a company has to pay taxes on gains from capital investment. This cost is dubbed double taxation—gains from capital investment are taxed twice, first at the corporate level and then at the personal level. All income taxes, including those from capital investment, should be covered by premium, so that the investor who contributes capital to the company does not lose out compared with an investor who directly buys securities on the market. Income taxes are generally a fixed percentage of the pre-tax income. But the precise IRS tax codes are complex. I will make simplifying assumptions to obtain closed-form, trackable results.

4.1 Single-year model

Let c be initial capital contributed by shareholders. The capital serves two purposes. First, the company invests the capital to earn income. Second, with the safety margin provided by the capital, the company is able to issue insurance policies. Assume the company issues policies and collects premium p (net of expenses), and invests the total cash $c + p$ in securities. The policy loss L is paid one year later, and the remaining assets are returned to shareholders.

Assume there is one tax rate, denoted by t, for both underwriting and investment profits. The pretax operating income, from both the policy account and the capital investment, is $(p - L) + (p + c)R_a = p(1 + R_a) - L + cR_a$. The total income

[6] In this illustration, for simplicity, I assume that the actual loss payments, investment returns and discount rates are equal to their expected counterparts. But these two sets of numbers are usually different.

tax, paid at time 1, is $t(p(1+R_a)-L)+tcR_a$.[7] The whole tax payment should be deducted from the policy account. The policy account's after-tax net assets are

$$\begin{aligned} A_1 &= p(1+R_a) - L - t(p(1+R_a) - L) - tcR_a \\ &= (1-t)\left(p(1+R_a) - L - \frac{tc}{1-t}R_a\right) \end{aligned} \tag{4.1}$$

Policy premium p is considered fair if it makes the market value of terminal assets zero. Setting $\mathrm{MV}(A_1) = 0$, and noting $\mathrm{MV}(1+R_a) = 1$ and $\mathrm{MV}(R_a) = r_f/(1+r_f)$, we have

$$p = \mathrm{MV}(L) + \frac{tcr_f}{(1-t)(1+r_f)} \tag{4.2}$$

The second term is the amount of premium needed to cover taxes on investment income of capital, which is in direct proportion to c. So too much capital hurts the company in price competition. Also note that the fair premium is not affected by how the premium and capital are invested. To derive the expected terminal assets for the policy, we substitute (4.2) into (4.1), and calculate the expected values

$$\begin{aligned} a_1 &= (1-t)\left(\left(\mathrm{MV}(L) + \frac{tcr_f}{(1-t)(1+r_f)}\right)(1+E[R_a]) - E[L]\right) - tcE[R_a] \\ &= (1-t)\mathrm{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) \\ &= (1-t)\frac{E[L]}{1+r_l}(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) \end{aligned} \tag{4.3}$$

This is the risk-adjusted break-even value for the after-tax terminal assets. In (4.3) the first term is essentially the after-tax version of the break-even value (2.1), and the second term reflects tax on capital. A policy generates a profit if and only if its after-tax terminal assets are greater than a_1.

Consider a special case where the investment is risk free, i.e., $r_a = r_f$. Formula (4.3) reduces to

$$a_1 = (1-t)\frac{E[L]}{1+r_l}(r_f - r_l) \tag{4.4}$$

Remember $r_l \leq r_f$, and the riskier the policy, the smaller the r_l. So a riskier policy has a greater break-even value (4.4). Note that capital c does not appear in (4.4). This is because, as the investment is risk free, the amount of tax on capital gain is certain, and is exactly covered by the second component of fair premium (4.2).

[7] In practice, when taxable income is negative, the company may not be able to receive the full tax refund in the current year. But we will ignore this complication here.

4.2 Multi-year model

The main goal of the paper is to derive the break-even value of terminal assets in the most general setting—multi-year, with income tax. This I will do in the remainder of the paper. To obtain trackable results I will make simplified assumptions on the timing and amount of tax payments. Assume taxes are paid at time 1, 2, ... (The interval between $i-1$ and i need not be one year. In the example in Table 3, each time period is one half year.) Tax paid at time i equals tax rate (a constant) times taxable income earned between times $i-1$ and i. For the purpose of income calculation, I assume loss reserves are discounted at the risk-adjusted rate (also a constant). The simplified tax rules are summarized below

$$\begin{aligned}
\text{Tax Paid}_i &= t \times (\text{Underwriting Gain}_i + \text{Investment Gain}_i) \\
\text{Investment Gain}_i &= r_f \times \text{Investible Assets}_{i-1} \\
\text{Underwriting Gain}_1 &= p - L_1 - \text{Loss Reserve}_1 \\
\text{Underwriting Gain}_i &= -L_i - \text{Loss Reserve}_i + \text{Loss Reserve}_{i-1} \\
\text{Loss Reserve}_i &= \text{MV}_i(\text{Unpaid Loss}_i)
\end{aligned}$$

where Unpaid Loss_i means the future payments $L_{i+1}, L_{i+2}, \ldots$, and $\text{MV}_i(\text{Unpaid Loss}_i)$ is the present value of this flow discounted at r_l. These rules will be used to derive a break-even value for after-tax terminal assets. A test on profitability is to compare actual terminal assets, which follows the IRS tax rules, with this break-even value. Deviation of the simplified tax rules from the IRS rules would create some distortion, which I hope is not material.

Derivation of results in a multi-year model is inevitably complicated. I will state the results here, and present their proofs in appendices. Again use A_i, $i = 0, 1, \ldots, n-1$, to denote the (random) assets of the policy account at time i, after loss L_i is paid. ($A_0 = p$.) Let c_i, $i = 0, 1, \ldots, n-1$, be the amount of capital held, so that $A_i + c_i$ is the total investable assets of the company at time i. c_i might be an amount required by regulators (as assumed in Schirmacher and Feldblum 2006), or desired by the company management. Again we assume assets are invested risk free, and the risk-free rate r_f is constant for all years.

The fair premium (net of expenses) p satisfies the equation $\text{MV}_0(A_n) = 0$. The following theorem gives a concise, closed-form formula for the fair premium.

Theorem 1 The fair premium p is given by

$$p = \text{MV}_0(L) + \frac{tr_f}{(1-t)(1+r_f)} \left(c_0 + \frac{c_1}{1+(1-t)r_f} + \ldots + \frac{c_{n-1}}{(1+(1-t)r_f)^{n-1}} \right) \tag{4.5}$$

Premium calculation will be discussed in depth in Section 7. To derive the break-even value for the after-tax terminal assets, we start from time 0 with premium (4.5), and successively compute underwriting, investment, tax cash flows and the net policy account assets A_i. The result is also a simple closed-form formula.

Theorem 2 The break-even value for the after-tax terminal assets in the policy account is given by

$$\begin{aligned} a_n &= \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l} \sum_{i=1}^{n} E[L_i] \left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i} \right) \\ &= \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l} (\mathrm{MV}_0(L) - \mathrm{PV}_0^{\mathrm{tax}}(L)) \end{aligned} \quad (4.6)$$

where $\mathrm{PV}_0^{\mathrm{tax}}(L)$ stands for the present value discounted with the after-tax interest rate $(1-t)r_f$.

Formula (4.6) does not involve taxes on capital investments (c_i does not appear in the formula). This is because the tax component in fair premium (4.5) exactly covers all those taxes. (As in the single-year model, if investments are risky, the break-even terminal assets will depend on c_i.)

These formulas are easy to apply. In Table 4, a tax column is added to Table 3, and the policy account assets at each time are recalculated by deducting taxes. (Note that the tax column contains all taxes, including that on capital investments.) The last entry of column 7 gives the policy's after-tax terminal assets of \$33.55. Given the rates $r_f = 4\%$, $r_l = 3\%$ and $t = 35\%$, we compute $\mathrm{MV}_0(L) = 650/(1+0.03)^6 = 544.36$, and $\mathrm{PV}_0^{\mathrm{tax}}(L) = 650/(1+(1-0.35)\times 0.04)^6 = 557.22$. Substituting these figures into (4.6), we get the break-even terminal assets $a_{3.0} = 24.37$. Since this is less than the actual terminal assets of \$33.55, the policy is profitable. The value added by the policy is $33.55 - 24.37 = 9.18$.

Calculation of the break-even value a_n, formula (4.6), does not require the values of c_i. But c_i are needed in the fair premium formula (4.5).

5 Linking Break-even Terminal Assets to Cost of Capital

The cost of capital (COC) is the rate of return on capital required by shareholders. Shareholder return is a random variable, and the COC is the expected value of this return. According to modern finance, the COC is in direct proportion to the riskiness of the return. For the insurance models under consideration, risk of the

Table 4: Policy Account Assets - After Tax

Time (1)	Premium (2)	Expense (3)	Loss (4)	Tax (5)	Investment Income (6)	Policy Account Assets (7)
0.0	1,000.00	275.00	0.00	-26.25	0.00	751.25
0.5	0.00	150.00	0.00	32.45	30.05	598.86
1.0	0.00	0.00	0.00	29.39	23.95	593.42
1.5	0.00	0.00	0.00	8.13	23.74	609.03
2.0	0.00	0.00	0.00	7.97	24.36	625.43
2.5	0.00	0.00	0.00	-3.57	25.02	654.01
3.0	0.00	0.00	650.00	-3.38	26.16	33.55
Sum	1,000.00	425.00	650.00	44.73	153.28	
PV	1,000.00	419.23	513.70	40.55		

Column (5) from Table 2 in Schirmacher and Feldblum (2006).
$(7)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0} - (5)_{0.0}$
For $i > 0.0$, $(6)_i = (7)_{i-0.5} \times r_f$
For $i > 0.0$, $(7)_i = (7)_{i-0.5} + (2)_i - (3)_i - (4)_i - (5)_i + (6)_i$

shareholder return comes from two sources—volatilities in investment gains and claim payments. In the preceding sections, I have shown that the expected values of claims and investment gains determine an expected value of terminal assets in the policy account. The total return to shareholders is the sum of policy account terminal assets and the investment return on capital. Therefore, the COC is a simple function of the break-even terminal assets and the expected investment return on capital. This relationship may also be used reversely: If the COC is obtained through stock analysis, a required level of terminal assets can be inferred, which may lead to useful premium calculation.

5.1 Single-year model

The expected value of policy account terminal assets is given in equation (4.3). The expected value of capital investment at time 1 is $c(1 + r_a)$. Therefore, the total expected after-tax net assets at time 1 are

$$(1-t)\mathrm{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) + c(1+r_a) \tag{5.1}$$

The expected rate of return on capital is

$$\mathrm{COC} = \frac{(1-t)\mathrm{MV}(L)}{c}(r_a - r_l) - \frac{t}{1+r_f}(r_a - r_f) + r_a \tag{5.2}$$

By (5.2), the COC is the sum of three terms: the investment rate of return r_a; the after-tax spread $(1-t)(r_a - r_l)$ times the "leverage ratio" $\mathrm{MV}(L)/c$; and a term related to taxes on capital investment, which vanishes if the investment is risk free. The following factors would cause the COC to increase (i.e., shareholders require a greater return): riskier investments (greater r_a), more volatile claims (smaller r_l), or a higher leverage ratio. Increasing the amount of capital would reduce the COC.

In Appendix B, I will explain that formula (5.2) is consistent with the Capital Asset Pricing Model (CAPM). In the CAPM world, an asset's expected return is in direct proportion to its β, which is a measure of the asset's systematic risk. I will use the given asset rate r_a and liability rate r_l to determine β of the shareholder return, and show the expected value of the return—the COC—is exactly given by (5.2).

5.2 Multi-year model

In a multi-year model, shareholders contribute an initial capital c_0 and establish a capital account. The capital account then earns investment income and pays out dividends (releases capital). Let c_i be the amount of capital held at time i. Then the total assets at time i are $A_i + c_i$.[8] Assume the interim dividends are released entirely from the capital account; the policy account only distributes its profit at time n.[9] Therefore, the dividend at time i is simply c_{i-1} plus the investment income in the year minus c_i. If each c_i is invested at the constant risk-free rate r_f, then the dividend flows out of the capital account are $-c_0$, $c_0(1+r_f) - c_1$, $c_1(1+r_f) - c_2$, …, $c_{n-1}(1+r_f)$. Obviously, the internal rate of return (IRR) of these flows is r_f.

[8] The asset in this paper corresponds to the income-producing asset in Schirmacher and Feldblum (2006). Non-income-producing assets, like the deferred tax assets (DTA), are not considered.

[9] This distinction between the policy account and the capital account does not affect profit measurement of the company as a whole. But it is important for measuring the policy account profit separately from the capital account.

The expected terminal assets a_n of the policy account is given in (4.6). Thus the expected total dividend flows are $-c_0$, $c_0(1+r_f)-c_1$, $c_1(1+r_f)-c_2$, $\ldots$, $c_{n-1}(1+r_f)+a_n$. The IRR of the total dividend flows is given by the following equation

$$c_0 = \frac{c_0(1+r_f)-c_1}{1+\text{IRR}} + \frac{c_1(1+r_f)-c_2}{(1+\text{IRR})^2} + \ldots + \frac{c_{n-1}(1+r_f)+a_n}{(1+\text{IRR})^n} \tag{5.3}$$

This IRR is the average—over n years—cost of capital of the company. After a policy has run its course, we can compute the IRR of the actual capital flows. If the IRR is greater than (less than) the average COC given by (5.3), the company's overall operation is profitable (unprofitable). It is worth noting that, since $a_n > 0$, the COC is greater than the expected asset rate of return. A greater claims risk implies a greater spread $r_f - r_l$, thus a greater a_n and a greater COC.

Back to the example of Table 4. In their paper, Schirmacher and Feldblum (2006) assume the required capital is 25 percent of the unearned premium reserve plus 15 percent of the loss reserve. They then compute the required assets at each time i and the corresponding dividend flows. Table 5 shows the dividend flows out of the capital account (column 6), the total dividend flows (column 7) and the break-even flows (column 8). These three columns only differ in their last entry. The IRR for column 6 equals the asset rate of return 4 percent, as expected. The IRR for column 7 is 6.18% (obtained also in Schirmacher and Feldblum 2006). The IRR for column 8, 5.62%, is the COC. Since the IRR of total dividend flows is greater than the COC, the company creates value for shareholders.

6 Decomposing the EVA

Shareholders invest capital in a company expecting to earn the cost of capital. If they earn more than (less than) the COC, then the investment adds (destroys) value. The economic value added is defined as (see, e.g., Schirmacher and Feldblum 2006)

$$\text{EVA} = \text{After-tax Net Income} - \text{COC} \times \text{Capital Held}$$

The second term in the formula, COC $\times$ Capital Held, is the break-even value of after-tax income. So this scheme of measuring profits is similar to what we developed in sections 2 to 4. The difference is that, the EVA measures the total profits, while our method addresses profitability of the policy account. From our discussion so far, it is straightforward to decompose this EVA measurement into one for the policy account and another for the capital account.

Table 5: Dividend Flows

Time (1)	Policy Account Assets (2)	Capital Held (3)	Total Assets (4)	Investment Income on Capital (5)	Capital Account Dividend Flow (6)	Total Dividend Flow (7)	Break-even Dividend Flow (8)
0.0	751.25	428.75	1,180.00	0.00	-428.75	-428.75	-428.75
0.5	598.86	362.62	961.48	17.15	83.28	83.28	83.28
1.0	593.42	149.53	742.95	14.50	227.60	227.60	227.60
1.5	609.03	122.54	731.58	5.98	32.97	32.97	32.97
2.0	625.43	94.77	720.20	4.90	32.67	32.67	32.67
2.5	654.01	79.84	733.85	3.79	18.73	18.73	18.73
3.0	33.55	0.00	33.55	3.19	83.03	116.58	107.40
IRR					4.00%	6.18%	5.62%

Column (2) is column (7) in Table 4.

Column (4) from Table 7 in Schirmacher and Feldblum (2006).

For $i > 0.0$, $(5)_i = (3)_{i-0.5} \times r_f$

For $i > 0.0$, $(6)_i = (2)_{i-0.5} + (5)_i - (2)_i$

For $i < 3.0$, $(7)_i = (6)_i$; $(7)_{3.0} = (6)_{3.0} + (7)_{3.0}$ in Table 4

For $i < 3.0$, $(8)_i = (6)_i$; $(8)_{3.0} = (6)_{3.0} + a_{3.0}$ ($a_{3.0}$ calculated in Section 4.2)

6.1 Single-year model

For the single-year model, the COC is given in formula (5.2), which can be split into two parts. The last term, r_a, is the hurdle rate for the capital account. If the actual return on capital is greater than r_a, then the investment operation adds value. By Section 4.1, the first two terms of (5.2) gives the hurdle rate of the policy account. This leads to the following definition of EVA for the two accounts separately

$$\text{EVA}_c = c \times (\text{Actual Investment Rate} - r_a) \tag{6.1}$$

$$\text{EVA}_p = \text{Actual After-tax Terminal Assets} - (1-t)\text{MV}(L)(r_a - r_l) + \frac{tc}{1+r_f}(r_a - r_f) \tag{6.2}$$

In (6.2) the actual assets in the policy account are after all taxes, including those on capital gains. Readers familiar with investment portfolio analysis may recognize that the rate spread, Actual Investment Rate $- r_a$, in (6.1) is Jensen's alpha for the asset portfolio. Obviously, $\text{EVA} = \text{EVA}_c + \text{EVA}_p$. If one wishes to review the past underwriting performance and make price changes, EVA_p would provide more accurate information than the EVA.

There are previous efforts on finding methods to separately measure underwriting and investment activities. Bingham (2004) proposes to allocate capital between underwriting and asset investment, find a cost of capital for each of the functions, and separately calculate their value creation. In practice, some companies build models to calculate the underwriting ROE, the investment ROE, and ROEs at various policy group or investment portfolio levels. Our method has some unique features. First, it emphasizes that income tax on capital investment should be deducted from policy account profits. Second, it treats the capital account no different than other investment portfolios, and tests it with the established Jensen's alpha. Third, it is consistent with the CAPM (see Appendix B), so is theoretically solid.

The policy account itself consists of two activities, underwriting (collecting premiums and paying losses) and investment of premium. But these two activities are more intertwined and we cannot measure them separately. An increase in premium is an achievement of the underwriting department. The resulting gain in profit should be credited entirely to underwriting, not to investment. But the additional premium generates an additional investment income, which cannot be cleanly attributed to either underwriting or investment. Also, the policy account covers income tax on capital investments. It is not clear whether this tax should be covered by underwriting profits or by investment income. Even in our method, performances of the capital account and the policy account are not completely independent. If capital investment generates a higher return, the corresponding income tax increases, which reduces EVA_p.

6.2 Multi-year model

In general, the EVA is calculated annually based on that year's income. For an n-year model, this means a stream of n EVAs that depends on how loss reserves are set in a particular accounting system. Schirmacher and Feldblum (2006) compute the EVA stream in two accounting systems, the net present value (NPV) and the IRR. I will not deal with accounting rules here, but only discuss the measurement of economic profits at the end of the policy life.

The IRR of the total dividend flows, denoted by IRR_{tot}, is a standard profit measure of shareholders' investment. The cost of capital is the break-even value of IRR_{tot}. As shown in Section 5.2, the total dividend flows are the sum of two component flows: a single flow at time n—the terminal assets—from the policy account, and a stream of dividends from the capital account. Methods of evaluating the two component flows have essentially been derived in previous sections, which are summarized below.

For the policy account, we define

$$\text{EVA}_p = \text{Actual After-tax Terminal Assets} - a_n \tag{6.3}$$

EVA_p is the ultimate cash value added by the policy. For the example in Table 5, $\text{EVA}_p = 33.55 - 24.37 = 9.18$. The dividends that flow out of the capital account are most conveniently measured by the IRR, denoted by IRR_c. The break-even value of IRR_c is the expected investment return r_a (or r_f, if capitals are invested risk free). Clearly, if $\text{EVA}_p > 0$ and $\text{IRR}_c > r_a$, then $\text{IRR}_{tot} > \text{COC}$; conversely, if $\text{EVA}_p < 0$ and $\text{IRR}_c < r_a$, then $\text{IRR}_{tot} < \text{COC}$. Note that EVA_p, IRR_c and IRR_{tot} are all independent of the accounting system. EVA_p provides more useful information to underwriting management than IRR_{tot}.

Here is an extreme situation that in a profitable company the underwriting operation is very unprofitable. An unexpected large loss may exhaust policy account assets before claims are settled. That is, $A_i \leq 0$ for some $i < n$. When this happens, the policy account assets stay negative for all later years. The negative EVA_p would correctly indicate that the policy is unprofitable. However, if the capital investment generates large returns, it is possible that IRR_{tot} still exceeds the COC, indicating a profitable overall operation. IRR_{tot} here says nothing about the underwriting performance.

7 Comparing Direct and Indirect Pricing Methods

Insurance pricing methods are broadly divided into two types. The *direct* methods are represented by Myers and Cohn (1987), formula (3.4). In these methods, risk of future claims is quantified by the risk load—in this paper, the risk-adjusted discount rate. With an *indirect* method, a target return on capital is first chosen, which is to match the total risk of claims and investments. Then the premium is back-solved to achieve this target. If both methods produce the same fair premium, they are economically equivalent. The formulas derived in sections 4 and 5 give us a mathematical relationship between the two methods.

Formula (4.5) is a direct method for computing p. r_l is the key parameter that captures the risk of claims.[10] A corresponding indirect method works via these steps: Choose a COC, substitute it into (5.3) to get a_n, solve (4.6) (using a numerical method like Goal Seek or Solver in Excel) for r_l, and then compute p with (4.5). This also shows how r_l and the COC, the two variables that characterize risk, uniquely determine each other.

I will demonstrate these calculations with the same example. In Section 4.2, we selected $r_l = 3\%$ and calculated $\text{MV}_0(L) = 544.36$. Substituting this $\text{MV}_0(L)$ and the c_is in column 3 of Table 5 into (4.5), yields $p = 569.08$. Loading in the present value of expenses, \$419.23, we get the full policy premium of \$988.31. (This break-even value is less than the actual charge of \$1,000. Thus the policy is a good deal to begin with.) To illustrate the algorithm from the COC to r_l and p, we adopt the assumption in Schirmacher and Feldblum (2006) that the COC is 5 percent. Plugging this IRR into the denominators of (5.3), and the c_is in column 3 of Table 5 into the numerators, we obtain $a_n = \$14.76$. Now use equation (4.6). Since there is only one loss payment of \$650 at time 3, $\text{MV}_0(L) = 650/(1 + r_l)^6$, where r_l is an unknown variable. Also $\text{PV}_0^{\text{tax}}(L) = 557.22$. Plugging these and all known parameters into (4.6) and solving for r_l, we have $r_l = 3.39\%$. Using this r_l in (4.5) gives $p = 556.98$. Adding in \$419.23 for expenses we obtain the total policy premium of \$976.21. To sum up, $r_l = 3\%$ corresponds to a COC of 5.62% (column 8 of Table 5), and $r_l = 3.39\%$ corresponds to a COC of 5 percent; the first scenario is more risky, and has a higher premium of \$988.31 (vs. \$976.21 for the second scenario).

To use these approaches in a pricing project, it is imperative to determine the capitals (required or desired) $c_0, c_1, \ldots, c_{n-1}$. In a multi-line company, the company-wide capital needs to be allocated at each time i. More research is needed on these issues.

8 Conclusions

In this paper, we studied risk-adjusted performance measures. We focused on the measurement of terminal assets of the policy account. By comparing the terminal asset with a break-even value, we determine the amount of profit created by the policy over its lifetime. The main results of the paper are the two theorems in Section 4.2, in which the break-even terminal asset is formulated in closed form

[10] Although we have been addressing calculating $\text{MV}_0(L)$ with the risk-adjusted discount rate, formula (4.5) would still apply if $\text{MV}_0(L)$ can be obtained with another method.

in terms of the rates of investment return and loss discounting. In contrast to the familiar risk-adjusted performance measures, the RAROC and the EVA, our approach addresses the underwriting profits separately from capital investment results. The mathematical relationship between the terminal asset, the cost of capital and the EVA is also discussed.

A key input for calculating the fair premium and the break-even terminal assets, formulas (4.5) and (4.6), is the loss discount rate r_l. It characterizes the underwriting risk of the policy. A link between r_l and the COC, which reflects the total risk of the company, is discussed in Section 7. How to select either r_l or the COC to correctly reflect risk is a challenge in application of the formulas. Determination of the stream of capitals c_i also needs further research.

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Appendices

A Proof of the Theorems

A.1 Proof of Theorem 1

Formula (4.5) can be written as

$$p = \sum_{i=1}^{n} \left(\mathrm{MV}_0(L_i) + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \right) \tag{A.1}$$

We only need to prove the theorem for each i, that is, for a policy with a single loss payment L_i at time i, and with a single nonzero capital c_{i-1} at time $i-1$ (considered beginning of year i). The fair premium is given by

$$p = \mathrm{MV}_0(L_i) + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \tag{A.2}$$

If (A.2) holds for all i, then (A.1) is true simply by additivity.

If time i is the only time losses and capital account taxes are paid, then at all other times j, $j = 1, \dots i-1, i+1, \dots n$, the only payments are taxes on policy account profits. These profits need to be carefully calculated according to the rules stated in Section 4.2.

Let $\mathrm{MV}_j(L_i)$ be the market value of L_i at time $j < i$. The value of $\mathrm{MV}_j(L_i)$ will not be known until time j. Therefore, viewed at time 0, $\mathrm{MV}_j(L_i)$ is a random variable.[11] To simplify notations, let $\mathrm{MV}_j(L_i)$ be denoted by V_j. The loss reserve at time $j < i$ is V_j, and loss reserves are zero after time i. In the following derivation, I start by calculating the net assets at time 1, move forward in time, and end up with terminal assets at time n. I then set the market value of the terminal assets to zero, and solve for the fair premium p.

At time 1, the underwriting gain is $p - V_1$ and the investment gain is pr_f. Then the tax is $t(p(1+r_f) - V_1)$. The net assets at time 1 are

$$A_1 = p(1+r_f) - t(p(1+r_f) - V_1) = p(1+r_f)(1-t) + tV_1$$

At time 2, the underwriting gain is $V_1 - V_2$ and the investment gain is $A_1 r_f$. Then the tax is $t(V_1 - V_2 + A_1 r_f)$. The net assets at time 2 are

$$\begin{aligned} A_2 &= A_1(1+r_f) - t(V_1 - V_2 + A_1 r_f) \\ &= p(1+r_f)(1-t)(1+(1-t)r_f) + tV_2 + r_f t(1-t)V_1 \end{aligned}$$

[11] Rigorously, the market values $\mathrm{MV}_0(L_i)$, $\mathrm{MV}_1(L_i)$, $\dots$, $\mathrm{MV}_{i-1}(L_i)$, L_i, are a stochastic process adapted to a filtration indexed by time j.

In general, the tax at any time $j < i$ is $t(V_{j-1} - V_j + A_{j-1}r_f)$, and it is not hard to prove by induction that the net assets at j is given by the following formula

$$\begin{aligned} A_j &= p(1+r_f)(1-t)(1+(1-t)r_f)^{j-1} + tV_j + r_f t(1-t)\Big(V_{j-1} \\ &\quad +(1+(1-t)r_f)V_{j-2} + \ldots + (1+(1-t)r_f)^{j-2}V_1\Big) \end{aligned} \tag{A.3}$$

Now this formula holds for A_{i-1}. At time i, there are two additional payments, loss L_i and tax on capital investment $tr_f c_{i-1}$, and no further loss reserves. So the total tax is $t(V_{i-1} - L_i + A_{i-1}r_f) + tr_f c_{i-1}$, and the net assets are

$$\begin{aligned} A_i &= A_{i-1}(1+r_f) - t(V_{i-1} - L_i + A_{i-1}r_f) - tr_f c_{i-1} \\ &= p(1+r_f)(1-t)(1+(1-t)r_f)^{i-1} - (1-t)L_i + r_f t(1-t)\Big(V_{i-1} \\ &\quad +(1+(1-t)r_f)V_{i-2} + \ldots + (1+(1-t)r_f)^{i-2}V_1\Big) - r_f t c_{i-1} \end{aligned} \tag{A.4}$$

After time i, every year the assets A_i are reinvested and taxes on the investment gains paid. The after-tax investment rate of return is $(1-t)r_f$. So, the terminal assets at time n are

$$A_n = A_i(1+(1-t)r_f)^{n-i} \tag{A.5}$$

The fair premium p is so defined as to make the market value of A_n zero. By (A.5), $\mathrm{MV}_0(A_n) = 0$ if and only if $\mathrm{MV}_0(A_i) = 0$. So we need to calculate the market value of each term in (A.4).

The first and the last term in (A.4) are nonrandom constants. The market value, at time 0, of a constant is the constant divided by $(1+r_f)^i$. The market values of other terms in (A.4) are obtained using the following formula.

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(L_i)/(1+r_f)^{i-j} \tag{A.6}$$

I will use this formula now to complete the proof. The formula itself will be proved later in the section.

$$
\begin{aligned}
& \mathrm{MV}_0(A_i) \\
= \; & \mathrm{MV}_0(p(1+r_f)(1-t)(1+(1-t)r_f)^{i-1}) - (1-t)\mathrm{MV}_0(L_i) + r_f t(1-t)\Big(\mathrm{MV}_0(V_{i-1}) \\
& +(1+(1-t)r_f)\mathrm{MV}_0(V_{i-2}) + \ldots + (1+(1-t)r_f)^{i-2}\mathrm{MV}_0(V_1)\Big) - \mathrm{MV}_0(r_f t c_{i-1}) \\
= \; & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
& + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 + \frac{1+(1-t)r_f}{1+r_f} + \ldots + \frac{(1+(1-t)r_f)^{i-2}}{(1+r_f)^{i-2}}\right) - \frac{r_f t c_{i-1}}{(1+r_f)^i} \\
= \; & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
& + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\right) \div \left(1 - \frac{1+(1-t)r_f}{1+r_f}\right) - \frac{r_f t c_{i-1}}{(1+r_f)^i} \\
= \; & \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - \frac{(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\mathrm{MV}_0(L_i) - \frac{r_f t c_{i-1}}{(1+r_f)^i}
\end{aligned}
$$

Setting $\mathrm{MV}_0(A_i) = 0$ and solving for p, we get the formula (A.2). This proves Theorem 1.

Note that in the derivation of (4.5) we do not need the assumption that there is a constant risk-adjusted discount rate r_l. Therefore, (4.5) can be used to calculate the fair premium whenever the market value $\mathrm{MV}_0(L)$ can be reasonably estimated.

A.2 Proof of formula (A.6)

In formula (A.4), A_i is a random variable conditioned on all information up to time i. This conditioning statement is important when computing the market value of V_j. $V_j = \mathrm{MV}_j(L_i)$ is a random variable viewed at any point $j' < j$, but is nonrandom at any $j' > j$. So it is easy to first discount V_j to time j,

$$\mathrm{MV}_j(V_j) = V_j/(1+r_f)^{i-j}$$

Then, further discounting the above to time 0, we get

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(\mathrm{MV}_j(L_i))/(1+r_f)^{i-j} = \mathrm{MV}_0(L_i)/(1+r_f)^{i-j}$$

which proves (A.6).[12]

A.3 Proof of Theorem 2

Theorem 2 says if a policy charges premium (4.5), then its expected terminal assets at time n have the form (4.6). I will again prove the theorem by splitting it

[12] A rigorous proof of the formula may be stated with stochastic discount factors. The technique is standard in asset pricing theory.

into n simpler components. For any $i < n$, assume that a subpolicy i has premium (A.2), makes only one loss payment L_i at time i, and is supported by one nonzero capital c_{i-1} at time $i-1$. I will prove that the expected terminal assets of the subpolicy, at time n, are given by the following formula

$$a_{n,i} = \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l} E[L_i] \left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i} \right) \tag{A.7}$$

Obviously, the expected terminal assets of the original policy is the sum of these $a_{n,i}$s. This will prove Theorem 2.

Substituting (A.2) into the righthand side of (A.4), we have

$$\begin{aligned} A_i \;=\; & \left(V_0 + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \right) \\ & \times (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} - (1-t)L_i \\ & + r_f t(1-t)\Big(V_{i-1} + (1+(1-t)r_f)V_{i-2} + \ldots + (1+(1-t)r_f)^{i-2} V_1 \Big) - r_f t c_{i-1} \end{aligned}$$

The assumption that there is a constant loss discount rate r_l implies that $E[V_j] = E[L_i]/(1+r_l)^{i-j}$. Noting that the c_{i-1} terms cancel out, we have

$$\begin{aligned} E[A_i] \;=\; & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} \frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\ & + \frac{r_f t(1-t)}{1+r_l} E[L_i] \left(1 + \frac{1+(1-t)r_f}{1+r_l} + \ldots + \frac{(1+(1-t)r_f)^{i-2}}{(1+r_l)^{i-2}} \right) \\ =\; & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} \frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\ & + \frac{r_f t(1-t)}{1+r_l} E[L_i] \left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}} \right) \div \left(1 - \frac{1+(1-t)r_f}{1+r_l} \right) \\ =\; & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} \frac{E[L_i]}{(1+r_l)^i} - (1-t)E[L_i] \\ & + \frac{r_f t(1-t)}{r_l - (1-t)r_f} E[L_i] \left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}} \right) \\ =\; & \left((1+r_f)(1-t)(1+(1-t)r_f)^{i-1} \frac{E[L_i]}{(1+r_l)^i} - \frac{r_f t(1-t)}{r_l - (1-t)r_f} E[L_i] \frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}} \right) \\ & + \left(-(1-t)E[L_i] + \frac{r_f t(1-t)}{r_l - (1-t)r_f} E[L_i] \right) \\ =\; & \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^i}{(1-t)r_f - r_l} \frac{E[L_i]}{(1+r_l)^i} - \frac{(1-t)(r_f - r_l)}{(1-t)r_f - r_l} E[L_i] \end{aligned}$$

A_i is the net asset at time i. After time i, the asset grows at the after-tax investment yield $(1-t)r_f$. So the terminal net assets at time n is $A_{n,i} = A_i(1+$

$(1-t)r_f)^{n-i}$. Therefore,

$$\begin{aligned} a_{n,i} &= E[A_{n,i}] = E[A_i](1+(1-t)r_f)^{n-i} \\ &= \frac{(1-t)(r_f-r_l)(1+(1-t)r_f)^n}{(1-t)r_f-r_l} E[L_i] \left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i} \right) \end{aligned}$$

This proves (A.7), thus completes the proof of Theorem 2.

B Formula (5.2) is Consistent with the CAPM

Assume our insurance company exists in a CAPM world. That is, the invested assets, the claims and the shareholders' capital all satisfy the CAPM. For the invested assets we have

$$E[R_a] = r_a, \quad r_a - r_f = \beta_a m \tag{B.1}$$

where m is the market risk premium. For a policy with premium p (net of expenses) and future claim L, we define its "return" to be $R_l = (L-p)/p$, and assume that

$$E[R_l] = r_l, \quad r_l - r_f = \beta_l m \tag{B.2}$$

This liability CAPM is a natural extension of the standard (investment) CAPM, and has been proposed by many authors, see, for example, Sherris (2003). $r_l \leq r_f$ implies $\beta_l \leq 0$. An implicit assumption in this CAPM framework is that all assets are traded at the market value, and all policies are charged the fair premium, which equals the market value of claims.

Formula (4.2) provides a policy's fair premium p. Substituting it into formula (4.1) gives the after-tax policy account net assets at time 1. Adding in the value of the capital amount we have the following total net assets

$$C = (1-t)\mathrm{MV}(L)(R_a - R_l) - \frac{tc}{1+r_f}(R_a - r_f) + c(1+R_a) \tag{B.3}$$

The return on capital, $R_c = (C-c)/c$, is

$$\begin{aligned} R_c &= \frac{(1-t)\mathrm{MV}(L)}{c}(R_a - R_l) - \frac{t}{1+r_f}(R_a - r_f) + R_a \\ &= \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right) R_a - \frac{(1-t)\mathrm{MV}(L)}{c} R_l \\ &\quad + \frac{t}{1+r_f} r_f \end{aligned} \tag{B.4}$$

Based on formula (B.4), the shareholder investment C can be replicated by the following three investments: short-selling an asset, whose return is R_l, and receiving cash $(1-t)\mathrm{MV}(L)$; lending an amount $tc/(1+r_f)$ at the risk-free rate r_f; and buying an asset, whose return is R_a, with the net cash $c+(1-t)\mathrm{MV}(L)-tc/(1+r_f)$. Thus, the β of the investment C is given by the weighted average of β's of the three investments

$$\beta_c = \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right)\beta_a - \frac{(1-t)\mathrm{MV}(L)}{c}\beta_l \tag{B.5}$$

If C satisfies the CAPM, then $E[R_c]-r_f = \beta_c m$. Using (B.1) and (B.2), we obtain

$$\begin{aligned} E[R_c] &= \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right) r_a - \frac{(1-t)\mathrm{MV}(L)}{c} r_l + \frac{t}{1+r_f} r_f \\ &= r_a + \frac{(1-t)\mathrm{MV}(L)}{c}(r_a - r_l) - \frac{t}{1+r_f}(r_a - r_f) \end{aligned} \tag{B.6}$$

This is exactly the COC in formula (5.2).