A Structural Model of Sovereign and Bank Credit Risk

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1. Introduction

In our research we address a significant challenge of today's risk management theory and practice: determining the credit outlook of sovereign entities and incorporating this information into a coherent risk framework alongside other asset classes. The issue has been evading the main focus of industry and academic interest for a long time, overshadowed by the preponderance of credit risk in the private sector. With shifted world dynamics and news of sovereign entities making and breaking the financial markets, management of risks stemming from sovereigns and, by extension, large financial institutions, is now of primary importance in every corner of the economy. We would like to present an economically intuitive and tractable approach to address this task.

We first establish a logical basis for the development of our methodology: an accepted structural model of credit risk of firms. We further lay out the characteristics of sovereign entities, paying attention to their difference from corporations. We then bridge the gaps, so that the structural risk methodology becomes applicable to sovereign entities, by translating microeconomic arguments into macroeconomic ones, expanding the analysis to higher dimensions, and incorporating additional variables pertinent only to sovereign credit risk. Consequently, we define three categories of sovereign entities, which require distinct treatment within the model. Alongside our exposition we point out the key improvements of the model over prior works that have applied a structural credit framework to government debt. We offer a case study of an implementation of the model and apply it to a number of countries—an opportunity to see how the theory behind the model produces intuitive results that agree with market metrics. Our journey from basic principles to a complete model is a step-by-step proof of the inextricable connection between sovereign risk and enterprise risk management of large financial institutions, exposing these asset classes' true complexity. We will conclude with a discussion of how the model's approach puts the relationship of government and bank debt, with all of its complexity, within the reach and best practices of multiasset class risk management.

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2. Structural Credit Risk

In this section we make a quick retrospective of the concepts of structural credit risk as applied in the corporate arena. We quickly develop and extend this framework with some important relationships that will serve as reference points onward in our analysis. The retrospective is needed to lay out both the building blocks for our sovereign risk model as well as a summary of the concepts applied to a key ingredient of the model—the risk of financial firms and banks.

The structural credit risk approach to corporate credit, pioneered by Merton (1974) and separately by Leland and Toft (1996), has gained wide acceptance since its introduction. The main reason for its success is the intuition it inspires, as it builds on the fundamental economics driving the financial sustainability of a debt issuer. The structural credit model puts in a rigorous framework the common sense understanding that the more assets and the more stable they are, and the less debt, the more creditworthy a debtor is. These three fundamental variables determine the financial capacity of the borrower to meet his obligations in the future.

The theory views a corporation as a pair of options. First, stockholders hold an implicit call option that the bondholders are short, with a strike price equal to the level of the firm debt, and with an underlying asset represented by the total of the firm's assets. Provided that traditionally the stockholders' claim in a corporation is referred to as "equity" or "stock," the market value of this call option is the market capitalization of the company. The put-call parity further implies that for the stockholders, who are the owners of the portfolio consisting of the firm's assets and a short position in the firm debt, there is a corresponding put option, the exercise of which is, essentially, the event of default. Such a view of default is dictated by the rational loss-minimizing choice of the stockholders and thus sometimes referred to as "ruthless default."

As a basis of any further inquest into the structural credit model relationships, we derive the relationship between the return of a firm's bond and the return of the firm's stock. To do that we use some simple fundamental relations between the pair of options and their underlying, namely,

Credit Risky Bond Value = Riskless Bond Value − Default Put Option Value,

assuming constant riskless bond value (i.e., no changes in time value of money) over a period of time:

Change in Credit Risky Bond Value = −(Change in Default Put Option Value),

Return on Credit Risk Bond = −(Change in Default Put Option Value)/Credit Risky Bond Value,

or separately:

Option Delta of Put = Change in Default Put Option Value/Change in Firm's Assets,

Change in Default Put Option Value = Option Delta of Put × Change in Firm's Assets.

Given that a firm's assets do not have observable market value, but a firm's equity does, we further use that

Change in Firm's Assets = Change in Call Option Value/Option Delta of Call

and that

Stock Return = Change in Call Option Value/Call Option Value.

As a result we derive a linear approximation of corporate bond's return as a function of the company's stock return. The relation is

Bond Return = −(E/B) × *(delta put/delta call)* × *Stock Return,* (1)

where *E* is the market capitalization of the firm, *B* is the market value of the firm's debt, and the put and call are calculated with respect of the time horizon of the particular bond tranche. The implicit assumption regarding the time of default "exercise" is that if a firm defaults, it will default on all of its debt, which is consistent with conventional bond indentures.

This result appeals to intuition. First, it states that, everything else the same, the closer the firm is to default the higher the delta of the put will be relative to the delta of the call. Given that an option gamma (the change of an option's delta with a unit change in the value of the underlying) is the same for puts as for calls, the approach to junk status will tend to proportionately increase the ratio of two deltas more than it will decrease the ratio *E/B* per unit of decline in the firm asset value. That will make the bond's return more similar to that of the stock, which reflects the common understanding that junk bonds behave like equities.

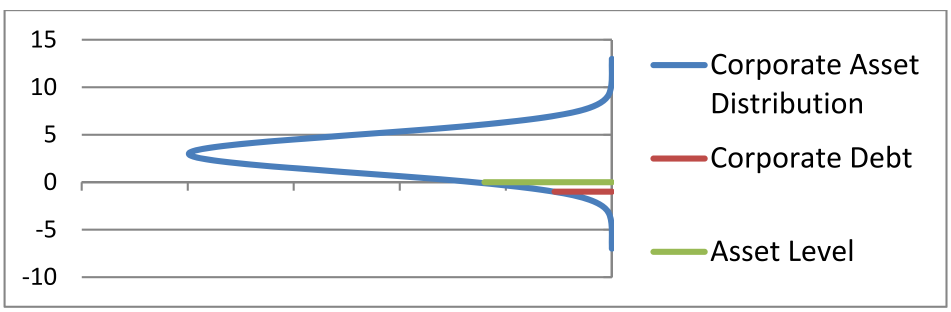

Figure 1

To gain further intuition, we change expression (1) as a function of a firm's return on assets:

Bond Return = −(delta put) × *(Firm's Assets/B)* × *Firm's Asset Return.* (2)

The factor −*(delta put)* has a special meaning in the context of the firm's asset distribution (see Fig. 1). It gives an explicit expression for the expected salvage asset value in the event of default as a fraction of a firm's present assets.

A bond's value will not respond to projected realizations that bring a firm's asset's above that of a firm's debt, as the bond's payoff is capped by its face value and coupon rate. In contrast, a bond's value will be responsive to those projected realizations where firm's assets are in the default zone (the region lower than the debt "strike" point on the diagram). Consequently, keeping the value of the riskless component of a bond's value constant, we have

Bond Return = $(\frac{1}{B}) \times \int_{Distress\ region} \Delta A \times\ q(A)\ dA$,

where q is the risk-neutral density used for asset valuation, and ΔA is the change in a particular projected realization of the assets over a period of time $t \in [T, T + \Delta \mathrm{t}]$ with a corresponding change in the firm's present level of assets in the period $t \in [0, \Delta t]$, where Δt is chosen sufficiently small. Provided that the assumed process for the asset realizations is geometric, we can write

$$\Delta A = A \times\ Return\ on\ Firm's\ Present\ Assets,$$

where A is a future realization of the firm's asset level in the asset value distribution at time $t = T$. Then we obtain

$$\text{Bond Return} = \left(\frac{1}{B}\right) \times Firm's\ Asset\ Return_{t \in [0, \Delta t]} \times \int_{Distress\ region} A \times q(A)\ dA. \qquad (3)$$

Combining (2) and (3) we obtain

$-$*(delta put)* × *(Firm's Assets/B)* × *Firm's Asset Return*

$= \left(\frac{1}{B}\right) \times Firm's Asset\ Return \times \int_{Distress\ region} A \times q(A)\ dA,$

which yields

$-$*delta put* × *Firm's Assets* $= \int_{Distress\ region} A \times\ q(A)\ dA$.

In essence, the multiplier *(−delta put)* helps us approximately integrate the distressed portion of a firm's asset distribution.

Whether we use the approximate integration with a multiplicative scalar or explicit integration techniques to obtain the expected salvage value, we will be just a step away from deriving another important metric of credit risk. Given the expected salvage value, and a value for the default put option, we can estimate of the probability of default (PD) of a company's debt. The PD value is the likelihood of default that makes an investor indifferent to holding one *risky bond* with a value equal to a riskless bond's present value minus the value of the default put, versus a *portfolio of two riskless bonds,* one having a weight of (1 − PD) and a full face amount *F* and another riskless bond having a weight of PD and a face amount equal to future expected salvage value. Alternatively, instead of using a default put option calculated using the structural credit model, it is possible to use an option adjusted spread (OAS). In that case, OAS enters as an increment over the risk-free rate to derive the present value of the risky bond.

Adding ammunition for our joint analysis of sovereign entities and banks, we can extend all of the above analysis to more than one issuer's debt. If we examine the joint distribution of the two issuers' assets, observe the structural credit risk option strike for both, and integrate over the appropriate portion of the distribution, we could estimate both the Probability of Default (*PD*) and expected Loss Given Default (*LGD*) of joint default events:

$$\text{Joint } PD_{ij} = \int_{-\infty}^{Di}\int_{-\infty}^{Dj} fj(xj|xi) \times f(xi)\partial xj\, \partial xi$$

and

$$\text{Joint } LGD_{ij} = \int_{-\infty}^{Di}\int_{-\infty}^{Dj}(Dj - xj) \times fj(xj|xi) \times (Di - xi) \times f(xi)\partial xj\, \partial xi. \qquad (4)$$

By using various configuration of any individual bonds and "portfolios" of two or more of the remaining bonds in any chosen universe, we will be able to assign the probabilities of default of any combination of two or more corporate defaults in the chosen group of companies and their obligations. Thus we can construct a full distribution of joint default occurrences. Using the same toolset and amount of effort, we also will be able to generate a distribution of Loss Given Default across the chosen debt portfolio. This distribution will play a central role in modeling the dynamics of sovereign credit.

3. Defining Characteristics of a Sovereign Credit Entity

Following our review of the theoretical background, we move into putting together a working definition of the main player in our analysis—the sovereign government. To have any practical value, the definition we pursue should yield itself to the structural credit framework. With this consideration in mind we can make the following observations:

1. A sovereign entity can be viewed as a collective fund enterprise of taxpayers who assign a management body—the government—to disburse those funds to vendors of social goods, and in effect ensure the social stability and infrastructure for progress. Although the benefits procured by government spending are nonmonetary, the assets of such enterprise—a stream of taxes, fees, and all government receipts—are monetary, and the value of those assets, net of government expenditure, is what stands as a guarantee of the debt obligations of the social goods fund. Consequently, the value of those assets in broad terms depends on three factors: the size of the economy and national income, the proportion of national income that shareholders in the fund contribute to the fund (generally a fixed proportion or percent tax), and the social policy followed by the government regarding the extent of expenditure out of revenues.

2. Governments of countries having their own fiat currency can generate more currency and monetize the debt existing in this national currency. They can do that in some cases on the grounds that the increased money supply will be matched with future expansion of the economy, in other cases that the inflation generated by a mismatch of the money supply and the real size of the economy is the price to pay to avert an economic collapse, or, in some cases, ruthlessly, without inflationary considerations.

3. The nature of "ruthless default" of sovereigns is different from corporations. While it is possible for a government to abandon its obligations in the face of depleting assets and revenues, they are unlikely to give up national assets for liquidation or appropriation. The typical scenario is that the government will renegotiate the terms and amount of its outstanding debt. The rational sovereign loss-minimizing choice for the new debt deal is that the terms will reflect the new diminished capacity of the government to make debt payments, which basically entails "haircuts" in the balance and interest payments. Moreover, when in default, a government will likely choose to continue funding expenditure instead of paying off debt, due to social and political priorities (governments are elected by taxpayers, not creditors), and also due to economic priorities, as follows.

4. When a country is hit by economic and financial turmoil, a responsible government will grant priority to ensuring the survival of the vessels of capital and credit flows—the banks and other systemic financial institutions. This need is inherent in nonbarter, specialized producer economies: temporal mismatches in traded delivery of goods and services require credit, and credit requires guarantees. Without credit flows, trade is impeded and the economy comes to a halt.

Financial institutions that are sufficiently large are systemically important credit depots, and a government will almost certainly try to salvage them during a crisis. The amounts involved will be in excess of deposit insurance systems like the FDIC, as the government will attempt to keep those institutions operational, not simply to alleviate the loss of creditors and depositors.

This situation is epitomized by the balance sheet of a bank. Extreme leverage ratio of debt to equity of 10 to 20 times of banks are the norm. The small equity fraction of bank capital is sufficient only for a bank to finance the least risky loans and projects to keep its own risk in check, which will make any bank unable to fulfill any systemic function. Given that banks do, and are encouraged to, engage in projects of higher risk, this would make financial sense only if there is the tacit expectation of government intervening if loss of bank capital occurs, and this extra level of surety is an implicit bank balance sheet item.

5. When a government faces the dilemma of saving banks and providing Keynesian support with government spending on one hand versus meeting its debt obligations on the other hand, it should be seen as a logical choice that it will support its financial institutions and economy first and pay debt second. If priorities were in reverse, the sovereign entity will have put its economy—its funds-generating mechanism—in a categorically more difficult place, which will preclude any avenues of meeting current debt obligations and rolling over its debt in the future.

4. Sovereign Credit and the Structural Model

The previous section demonstrates that sovereign entities have a general similarity with corporate entities, but also substantial differences. The similarity has to do with that the ability of the entity to meet its financial obligations depends on the size of the financial obligations, as well as the performance and volatility of its assets. The main differences have to do with the role of the government as a guarantor of banks, and the general lack of recourse of creditors of a sovereign government in the event of default, apart from retracting future credit.

Conceptually, the implicit guarantees to the banking system are not difficult to incorporate in the structural model. They are just an additional drain on the assets available to meet sovereign obligations. If we are able to estimate the value of those guarantees, we on our way to estimating the payoff profile of sovereign assets.

As a consequence of the specifics of sovereign government behavior regarding credit, we need to adapt the structural credit model to agree with the sovereign framework. In this section we elaborate on this modification, as part of the model inputs.

In general terms, structural credit, being an option pricing model, requires three inputs: underlying price, underlying volatility, and a strike price. For a sovereign entity, the strike price is relatively well defined, represented by the market value of the government debt. The underlying sovereign assets, however, are a more abstract concept than in the corporate model. Theoretically, the assets of a sovereign can be derived from the explicit liabilities on the government balance sheet, but that will be a subjective estimate. Apart from countable and appraisable items like reserves and fixed property, the remainder (and major portion of) of assets is implied from the government liability side, notably consisting of currency in circulation. The currency in circulation is only as much a liability *of* the government as it offsets the holder's own liability *to* the government—a tax liability or a direct liability to the government's bank (i.e., credit originated by the central bank). The assumption in calling currency a liability is then that all of the currency will circulate back to the government for an unrestricted amount of time through tax revenues and credit service of the private sector to the central bank. This view, however, does not incorporate (a) the possibility that the amount time of this currency cycle is infinite, (b) the time value of money, and (c) the fact that the currency is almost immediately put back in circulation by the government, through expenditure and new central bank credit issuance. In general, (a) and (b) will tend to diminish the level of assets of the government in comparison with the assets from the "naively" derived government balance sheet described previously, and (c) will tend to boost the ability of the government to meet future temporal debt obligations in comparison with what the "naive" government balance sheet would have us believe.

In contrast, we shall postulate that the asset level of a sovereign entity is the present value of the stream of all future government receipts—taxes, fees, tariffs, exploration rights, etc., net of the present value of all future government expenditures. Even if thought of as perpetuities, these cash flow streams have to have a finite value, as cash flows very far into the future (e.g., 200 years) discount to be of infinitesimal present value. Given that cash these flows grow in the future (e.g., revenue grows due to nominal GDP growth), we can assume the Gordon growth model:

Present Value = Periodic Cash Flow/(Discount Rate − Cash Flow Growth Rate).

The discount rate is chosen so that it reflects the equilibrium demand-supply market rate for the given level of uncertainty of the cash flow. For this reason the discount rate reflects the highest growth rate attainable in the economy, as expected by the market for the given level of volatility. That is why the denominator cannot become negative, as it is impossible for the growth rate of the economy and hence government revenues to be consistently higher than the best attainable rate of return in the market. The growth in expenditure also cannot be higher than the best attainable growth rate in the economy. As expenditure is tightly connected to the growth of the population, and population being one of the factors of production—labor—will necessarily be growing at best as the economy would be growing, provided that the other factors of production—capital, technology, materials—have non-negative growth. We present a full discussion and a novel approach to the essential question of choosing an appropriate discount rate in Appendix A. To obtain the total of a government's assets we need to add all current assets in the form of cash at hand, foreign currency reserves, bank deposits and receivables, commodities reserves, and so on.

Next, we proceed with estimating the volatility of sovereign assets. In that respect, we shall recognize that provided an assumption that sovereign net assets follow a geometric process, a proportional change in the numerator (a single cash flow) of the Gordon growth model will precipitate a commensurate proportional change of the present value of the full cash flow stream. This is true for all the possible realizations of the single cash flow in a period $t \in [0,1]$, so by the nature of the calculation of the second moment of a return distribution, the standard deviation, which we shall accept as a measure of volatility, will be equivalent for the return on net asset present value and the return on net asset cash flow. The task then simplifies to estimating the standard deviation of the proportional changes of government revenue and government expenditure. We can readily make this estimate, provided times series of government expenditures and revenues available from, for example, the World Bank Data Bank.

Here we shall draw a quick parallel of how prior applications of the structural credit framework to sovereign credit have dealt with the issue of estimating asset volatility. As a notable example, the work by Gray et al. (2007) addresses the issue in the following way. First, it assumes that the level of net fiscal assets (the present value of revenues minus expenditures) and assets are unobservable. It views local currency debt and base currency as a claim similar to "equity" in the sovereign entity, in the sense that they represent claims into government receipts and national goods and services, and are generally subject to dilution (by printing more currency and issuing more local currency debt) just like corporate common stock is. It views foreign currency debt as a Merton option's strike price. Then using information from the interest rate and foreign exchange derivative markets the system of variables is solved, producing a solution for the net asset values and volatility.

The reason we sought a different path in our model was to minimize the need for an assumption of correctness of the market's forecasts in the estimation of asset and volatility parameters. Any values implied from traded market quotes (in this case from the interest rate derivative and foreign exchange market) makes the implicit assumption that (a) a prior model of these parameters already exists, (b) that the model is correct, and (c) it is applied by the marginal trade player in the market. A more relaxed version of this assumption is that the market knows better than any individual player, by employing the collective wisdom of all pricing models in the market, which, even if not perfect, equilibrate prices at the "correct" level, or risk being punished the most when model predictions turn out wrong . The bids for financial assets that cross are necessarily the bravest, and it could be assumed that the higher market risk is a sufficient motivator for the players to reduce model risk by producing correct models. The speed with which market yields and market prices shifted during the European Credit Crisis (European Central Bank 2012) indicates that the prior models employed by the market are off by a wide margin. In the absence of natural disasters or war, no country's fundamentals could have changed with that speed in a matter of months, let alone this happening for a number of countries at a time. The reason for the credit risk models' poor performance might have been the long-term historical "stability" of government debt in the developed world, diminishing the perception of model risk. One could also even make the argument that it was the self-perpetuating trend of the model->implied inputs->model that created the perception of stability, but proving this assertion is beyond the scope of our effort. It was the pursuit of a "prior model" that instigated our interest to employ explicit input variables that avoid feeding back potentially inaccurate implied estimates into the sovereign credit modeling process.

It would be appropriate to make another distinction between our approach and previous models. This concerns the treatment of debt renegotiation. Unlike other models, notably Jeanneret (2008), and separately François et al. (2011), we do not pursue a separate model for the renegotiation of debt based on a game strategy (e.g., Nash or Rubenstein equilibrium). The estimation of the parameters of gamed debt renegotiation, taken out of the direct context of the long-term financial sustainability of a country, can be based only on empirical data of past sovereign debt deals, for which there has been a limited number of observation to reliably estimate the level of bargaining power or forecast the impact of foreign sanctions that enter as inputs into the strategy renegotiation model. The alternatives are to base these parameters on judgment, or calibrate them to market credit spreads, for example, which would increase the dependence of the model on market-implied inputs from variables that are themselves possible outputs of the model (like credit spreads)—a practice of self-validation of in-sample results that we aim to avoid.

Our view is that any rational negotiation will be based on the future sustained capacity of a country to bear the burden of a new (renegotiated) debt deal. In that, it is no different from determining the probability and expected magnitude of a net fiscal assets shortfall below the debt level, where the present value of any new debt deal gravitates toward the new diminished net fiscal asset value. Even if the government has the bargaining power to muscle into a current debt deal below this level, rational lenders will demand higher rollover debt costs when debt comes for renewal to compensate for their own bargaining disadvantage, creating a drain on future assets commensurate with this current negotiated gain. In that sense the strategy advantage will be short lived.

In light of our prior discussion of the interconnection between banks and governments, net asset volatility has another aspect that we must address to be able to develop an integrated approach to sovereign and bank risk. This is the correlation between the assets of a sovereign and those of banks. Factor models of the market risk of banks have been in existence for quite some time and usually take the form

$$R_{bank\ stock} = \sum \beta_{factor\ i} \times R_{risk\ factor\ i} + \varepsilon, \tag{5}$$

where R is return, factor i is a common risk driver in the market, and ε is residual return component unexplained by the common factors. Using a leverage ratio or, alternatively, a risk model for a bank's debt of the same form as expression (5), and recognizing that banks assets are the sum of debt and equity, we can construct a similar expression for bank asset return:

$$R_{bank\ asset} = \sum \beta_i^{asset} \times R_{risk\ factor\ i} + \varepsilon^{asset}. \quad (6)$$

Among many other reasons, factor models are important because they allow the estimation of the joint return distribution of any number of assets, provided that all of the assets are spun by the same number of common factors. With this consideration, our task then is to express the return of sovereign assets in terms of a factor risk model.

A broad market index supplies a convenient reference point for solving this task. Let us revisit the fact that, in broad terms, the contributors to the government finances (taxpayers) allocate a fixed proportion of their income. By the basic arithmetic of return calculation, where the same tax rate gets factored out in both the numerator and denominator, and therefore cancels out, we can easily see why the volatility of the present value of the national income σ_{Nat}, which is the second moment of the distribution of such return realizations, will be approximately the same as the volatility of the proportional change of the present value of government assets:

$$\sigma_{Sov} \approx \sigma_{Nat}.$$

A country's economy clusters around corporate activity, and the volatility of the corporations' contributions to sovereign receipts will be a possible gauge for the volatility of the sovereign assets themselves. Corporate contributions to sovereign receipts, on the other hand, have volatility commensurate with that of market capitalization of all taxable corporations, as long as those corporations pay approximately a fixed proportion of their income as tax.

To the extent that individuals' tax receipt streams' present value—the major component of government receipts (Williams 2011)—is not changing commensurately with that of corporate tax, we can at least be certain the corporate volatility estimate will be an upper bound estimate of overall national product volatility. The reasons for this are two. First, corporations are, generally, product differentiators and operate at marginal revenue greater than marginal cost, which helps them absorb a larger portion of national income when the economy is booming. On the other hand labor wage levels, and any proportional tax collections based on them, are sticky in the downward direction when the economy is in decline. Hence, the volatility of the market value of corporations is an upward bound of the volatility of the present value of the individual's income. Second, a surge of economic profits on the macrolevel entails a higher utilization of labor input as well as higher consumption by individuals, which is synonymous with higher individual income, and higher individual sourced tax receipts. This means that on aggregate corporate profits and individual income will tend to move synchronously. In combination, these two reasons indicate that the broad market and the sovereign assets will be significantly correlated, and the magnitude of the volatility of the sovereign assets will be lower than the volatility of the market. This hypothesis is confirmed by our empirical findings presented in the Case Study. We also find significant negative correlation between expenditure proportional changes and the broad market, even if sizewise the amount volatility explained by the broad market is low. We attribute this finding to the fact that improvements and, respectively, deterioration of an economy's prospects respectively decrease or increase the need of the government to expend welfare, which is not one of the sizable government budget items (Blodget 2012).

The significance of the relationship between the sovereign assets and the broad market is that we can now relate net sovereign assets to the same risk factors as those explaining the returns of the broad market. The procedure is similar to return-based style analysis first proposed by Sharpe (1992) to determine strategy weights of funds and later applied by DiBartolomeo (2007) to factor risk models.

The particular procedure for sovereign net assets is an ordinary least squares regression applied separately on the proportional changes in revenue and proportional changes in expenditure. The resulting regression coefficient is then multiplied by the risk factor betas of the broad market index to derive risk factor betas of two separate positions—revenue and expenditure and expenditure streams. The factor betas of the broad market are readily estimable as a linear combination of the factor betas of the broad market index constituents, which in turn are the output of the estimation of any generic risk factor model. As a result we combine the risk factor betas of the revenue and expenditure streams in similar form as expression (6):

$$R_{sovereign\ assets} = w_{revenue\ stream}\left(\sum \beta_i^{revenue\ stream} \times R_{risk\ factor\ i} + \varepsilon^{revenue\ stream}\right)$$

$$+ w_{expenditure\ stream}\left(\sum \beta_i^{expenditure\ stream} \times R_{risk\ factor\ i} + \varepsilon^{expenditure\ stream}\right)$$

$$= \sum \beta^{asset} \times R_{risk\ factor\ i} + \varepsilon^{asset}, \qquad (7)$$

where

$$w_{revenue\ stream} = \frac{Value\ Revenue\ Stream}{Value\ Revenue\ Stream - Value\ Expenditure\ Stream},$$

$$w_{expenditure\ stream} = 1 - w_{revenue\ stream.}$$

Given that the factor risk model is broad enough to span both the broad market index as well as bank assets, result (7) puts us in a position to produce a joint distribution of the sovereign and bank assets, which is key in order to project any joint default behavior.

To gain intuition how result (7) can be useful, we investigate the portfolio of sovereign and bank assets. The portfolio is applicable to an investor who is long government debt and simultaneously short the put option not only on the sovereign assets but also joint default events of banks. Given that risk factor exposures combine linearly for a portfolio of combined assets to produce combined asset values and volatility, we can think of the sovereign plus bank assets portfolio as similar to that of a "holding" company. The event of default of *any* (the union of defaults) of the underlying entities will be estimable using a Merton-style option with a strike equal to the sum of the debt of all underlying entities. The same type of analysis will be applicable to any two of the underlying entities. Given that, for any two the joint probability of default is equal to

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Using as well the prior result from the estimation of the (risk-neutral) probability of default, which in this case will be the union of two defaults, we would be able to estimate the (risk-neutral) joint probability of default. Using the superset of configurations of entities (1-, 2-, 3-, … *N*), number of entities, and some Boolean algebra we will be to estimate the joint probability of default of any configuration. Similarly, using the joint assets and strike we will be able to estimate the joint Loss Given Default of the particular configuration, which will be equal to the value of the default put for that configuration of entities.

Consequently, assuming a portfolio of the sovereign assets and the assets of the *N* largest banks, we can build a distribution of the Loss Given Default, using a table of all default configurations similar to Table1.

Table 1

Bank A Default	Bank B Default	Bank C Default	Bank D Default	Bank E Default	Sovereign Assets Shortfall under Sovereign Debt	*PD*	*LGD*
N	N	N	N	N	**Y**	P1	a
N	N	N	N	**Y**	N	P2	b
N	N	N	N	**Y**	**Y**	P3	c
N	N	N	**Y**	N	N	P4	d
…	…	…	…	…	…	…	…

We recognize that the values P1 through PN are risk-neutral probabilities, which can be converted to real world probabilities using

$$PD_{real\ world} = F_{r+Asset\ Expected\ Return,\sigma}\ (F_{r,\sigma}^{-1}(PD_{risk-neutral}),$$

where $F_{A,B}$ and $F_{A,B}^{-1}$ are, respectively, the quantile and inverse-quantile functions of the lognormal distribution of the sovereign-banks asset portfolio with location and scatter parameter of *A* and *B*, respectively. Following that we can estimate traditional risk metrics like Value-at-Risk, Expected Tail Loss, as well as an explicit distribution of the government debt return from which to estimate the standard deviation of return.

It should be noted that while this procedure is possible, it might represent practical difficulties. If the number of reference entities becomes sufficiently large, constructing and estimating the resulting configurations of entities can produce a convoluted and tedious solution process in practice. That is why we would like to pursue a more efficient solution technique. We will return to this issue once we have paid close attention to one last but crucial sovereign characteristic.

5. The Monetary Argument

In a prior section, we determined that governments differ in respect to their ability to control the national currency in circulation. For sovereigns like the United States or the United Kingdom this capability is a given. For others that are part of a monetary union (like the Euro zone), this is not so, especially in regard to smaller member states with little influence over the union's central bank activities. Furthermore, for the countries that have control over the currency they use, there are those that have the expertise and motivation to anticipate the long-term effects of changes in the money supply, and, separately, those countries that do not have this expertise or motivation.

Consequently, we differentiate between three types of sovereign entities depending on their behavior, provided a breach of the net sovereign assets into the default zone. The model treatment of each type is based on the same approach, but incorporates adjustments in consideration of the differences between the types.

The first type consist of sovereigns with no capability to influence directly the amount of currency their country uses. This is the "pure play" type, and as such it is, intuitively, a convenient starting point. The characteristics of these sovereign entities are that they hold a general long-term view of popular economic prosperity, a disciplined behavior in the face of financial difficulties, a discipline mostly supported by fiscal means, taxation, and spending, and observing the macroeconomic priorities laid out previously. For brevity, we shall refer to these entities as "fiscally responsive sovereigns."

The next type encompasses governments in control of the amount of their national currency. All characteristics of this group overlap with the previous type but have one additional important tool at their disposal in times of crisis: expansionary monetary policy. Not only is monetary policy a course of action prescribed by certain economic schools of thought, but also the political avenue of least resistance, as it is solves immediate budget problems in subtle ways that do not have a visible impact as changes in the tax code and its costs to the public that are not completely apparent. For these reasons such sovereign entities with the power to do so should be invariably expected to pursue the monetization of their debt, to the extent that they backstop the free fall of their credit standing and the economy that supports it. We shall refer to these sovereigns as "monetarily responsive."

The third category includes governments that are usually in control of their own currency and do not hold any long-term views of the prosperity of their people. Those governments would generally print as much money as needed to meet their budget needs without any concern about the longer-term economic impact of such actions. The impact generally consists of a vicious cycle of hyperinflation and economic deterioration brought about by a lack of credit in the medium of exchange. We will refer to such entities as rogue sovereigns. Admittedly, there may be some rogue governments that are not in control of their active currency; that is, the country's use of U.S. dollars has made it the de facto official currency. Historically, such governments have spring up as consequence of sudden and violent events—juntas, military coups, and civil unrest—without developing the means to introduce a national currency. As such, they generally do not warrant any credit analysis, because they would not inspire public interest in public financing in the first place.

The factors that differentiate between responsible and rogue monetarily flexible governments are as follows. To classify a government as rogue or responsible, one ought to consider all factors in conjunction:

- The degree of democratization of the government that ensures the checks and balances between personal and short-term motives with long-term social well-being goals. The long-term sustainability of credit is an indelible part of long-term economic progress. Governments, on the other hand, are subject to short-term personal motives and lobbying group influence. The more democratic a country, the better the balance.
- The degree of the government's economic sophistication. The complexity of the feedback mechanisms in a macroeconomic system, including the management of government credit, requires in-depth expertise. The lack of such expertise makes the economy an easy victim to political myopia.
- Existence of social unrest. Governments might be subject to undue influences to rational economic thinking or pressured to follow social priorities that require funds to be diverted from paying off financial obligations.
- The degree of dependence of the sovereign entity on other countries for its well-being in multinational affairs. One type of dependence will be the need for military or humanitarian support. Another type of dependence will be a large proportion of national income and spending attributable to exports and imports. If the government credit fails, the banking system would be undermined with it, and imports as well as exports will cease due to lack of trade credit.

6. Fiscally Responsive Sovereign Entities

Previously we have laid out a procedure whereby we can estimate the distribution of default losses in the absence of monetary intervention—the "pure play" government response to sovereign asset distress. In this section using a factor risk model and a multinomial lattice (a multidimensional version of a binomial tree) we modify the numerical procedure to make it tractable and efficient.

First, we can observe that the return of the assets of any of the debt issuers in our case can be decomposed in a risk-neutral world as[1]

$R^* = r + \beta R_M + R_S,$

where R_M is the return on the market index and R_S is an issuer specific return (idiosyncratic) component. As a start we consider the case of the joint default put option on two entities, for example, a sovereign

[1] The risk-neutral probabilities are the ones applicable to pricing cash flows, and as such the pertinent probability measure for valuation of the default put option.

and a bank. On arbitrage-free grounds, we can state a general expression for the value of the default put as follows:

$$P = e^{-rt} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \max[0, (D - S_0 e^{r+\beta R_M + R_S})] f_s(R_s|R_M) f_M(R_M) \partial R_s \, \partial R_M.$$

This simply states that the value of the option is the expectation, under the risk-neutral probability density $f(\cdot)$, of the discounted cash flows existent at the states of the world where the joint-underlying option is exercisable.

We can extend this to any number of credits as follows:

$$P_{Sov\_FISC} = e^{-rt} \int_{-\infty}^{+\infty} \int \ldots \int_{-\infty}^{+\infty} \sum_{i=0}^{n} \max[0, (D_i - S_0 e^{r+\beta_i R_M + R_{Si}})] \prod_{i=0}^{n} f_s(R_s|R_{s\ j>i}, R_M) f_M(R_M) \, \partial R_{si} \ldots \partial R_{sn} \, \partial R_M. \quad (8)$$

In this case we have used a risk model where the factor is the market index, but it is important to note that any risk factor model could be used as long as it is designed to explain the return on assets of the entities in consideration. If we have more than one risk factor, obviously we would have more than a single dimension apart from the idiosyncratic component of return $R_{si}$of the individual credit entities.

One practical approach to solving expression (8) will be using a multinomial lattice (tree), each dimension based on, for example, the Ross-Cox-Rubenstein model, having orthogonal dimensions for the market factor, the sovereign assets, and bank portfolio–specific return variables. The advantage of this approach is that it will eliminate the need for calibration of the tree, as long as the current value of each entity's assets is used as the starting point of diffusion of the tree when setting out the probabilities of each additional dimension. If we choose to apply a multifactor risk factor model, we will need to create separate dimensions for each risk factor, which might be correlated with the rest of the risk factors. In that case, the solution would be to express the correlated factors (and the factor exposures of the assets to the factors) through the factors' orthogonal principal components. Apart from capturing the multidimensionality of the default process, the lattice will also allow us to extend our approach from one time horizon to many, allowing for a continuous possibility of exercise (similar to an American option), which is the most realistic case considering the actual possibility of default.

7. Monetarily Responsive Sovereign Entities

Having estimated the value of the default put option for fiscally responsive sovereign entities, the estimation of the value of the default put for responsible monetary sovereigns is straightforward. Whatever the shortfall in government assets determined with the methods in the previous section, generating new currency in the amount of this shortfall will share the default loss to the investor with all users of that currency. If we attribute the currency purchasing power loss to a particular sovereign debt investor to his holdings in the government debt, then, at worst, the loss will be equal to the level of inflation in the market for goods and services, and financial assets sold in that currency invoked by the money creation due to a fiscal default. The loss will be projected on the full debt holding dollar amount, and the inflation generated will be equal to the monetized fiscal loss divided by the supply of money. Hence,

$$P_{Sov} = (P_{Sov\_FISC} / (MS_o)) \times Debt_{Sov\_FISC}\ , \qquad (9)$$

where MS_o is the level of money supply in broad terms immediately preceding the fiscal default. Given that the monetization of debt involves a credit transaction (even if it is credit from a central bank) the increase of the money supply should be categorized as broad money, rather than narrower measures like M1.

To the extent that government debt is categorically smaller than the broad money in the economy, the default option on a monetarily responsive sovereign entity's assets will have a significantly lower value than that of a fiscally responsive sovereign entity.

8. Rogue Sovereign Entities

Rogue sovereigns are similar to responsible monetary entities in the ability to control their money supply, but differ in one very important aspect. They manipulate the money supply without consideration for its long-term effects. The reasons for this are economic short-sightedness, the disinterest of the government in the social well-being of the population, or both.

The distinctive phenomenon related to rogue monetary governments is hyperinflation. As the rogue government prints money to pay off tranches of debt with prior maturity and ongoing expenses, it creates expectations that it might haphazardly continue to do so if future need arises. In turn all vendors of goods and services, including vendors of goods and services to the government, will require price increases in excess of the inflation generated with the current increase in the money supply. As the sovereign entity adjusts money in circulation to meet new spending levels, it creates a level of price increase higher than the previous increase. Market participants observe a dynamic that indicates an increase in the rate of inflation, which is enough to urge them to act as to propagate this trend: raise their prices even more, induce larger money creation, etc. This time trend is accelerated, as purchases are sped up to turn a debasing currency into real value sooner, which increases another determinant of the price level: the velocity of money. With these factors feeding into each other, an exponential hyperinflationary process develops. The major trigger of this process is a government that uses money creation to meet current spending, which is a key difference with monetarily responsive sovereign entities that aim monetary policy at promoting (or sustaining) matching economic growth that would offset increases in the price level.

To incorporate this aspect in our analysis, we need to assume and test a hyperinflation process over time. There is a significant number of recorded bouts of hyperinflation throughout the twentieth century, and they can help us gain insights into the parameters of the hyperinflation process. Such examples are Germany in the 1920s, Brazil and Argentina in the late 1980s and early 1990s, and Zimbabwe in the early to mid-2000s (Federal Reserve Bank of Dallas 2011).

Zimbabwe, being the most recent example, is used as an illustration. Using U.S. dollar exchange rate data to the Zimbabwean dollar as proxy for inflation during hyperinflationary period of the 2000s, a process of the form

$H(t) = Value(t)/Value(0) = e^{1.94\times t-4.5}$

produces a fit with the observed data of $R^2 = 99\%$. The deterministic connection between the rate of inflation and passage of time is unquestionable. Observing the same type of time process in all other documented cases of hyperinflation will help us extend our sample and explore its dependence with variables like the initial deficit and debt level vis-à-vis the size of the economy, as initial state variables of the hyperinflation process. For example, we would be able to attribute a time process for a country provided that it was estimated on the most proximate historical hyperinflation case based on the size of its initial deficit and debt and initial national product growth rate.

With a projection for $H(t)$ at hand, our expression becomes

$P_{Sov\_rogue} = [D/H(t)] + [(D/MS_o)\exp(-rt)\int_{-\infty}^{D}(D-A)p(A)dA]$,

where $H(t)$ is the projected level of hyperinflation process.

In a sense our expression is trivial because the term $\{D - [D/H(t)]\}$ is very large and dwarfs the second term in the expression. In fact, in any of the historical cases of hyperinflation with inflation rates of a minimum 50 percent, it is so large that it ensures the erosion of all invested capital in a matter of one or two years. In that, it is equivalent to repudiation of all debt by the government.

We are then drawn to answer the more interesting question what is the value of default for an entity that we suspect might become, but is not yet, a rogue one. Using a binomial lattice similar to the one used to solve (8), we are able to unravel the scenarios under which hyperinflationary money creation are imminent. Note that under expression (9), technically we have made the assumption that the central bank will be the lender of last resort to the government, and the monetization of debt is just buying time for the government before it actually repays the particular tranche. Running an independent target debt level over the long run would mean that these tranches of monetization will have increased the government debt burden level over the long run. In most cases a government cannot avoid the need to maintain the historical debt level, if it is to keep the economy and social system operational. Under such a scenario, if, over time, monetization of prior debt raises the debt burden sufficiently, the government will be in default to its own central bank. At this point forward, irresponsible monetization will be a government reflex for short-term survival, and the rational expectations of the public will possess all ingredients for looming hyperinflation. Under such conditions the economic loss of explicit repudiation will be equivalent to repudiation through hyperinflation, and this deep state of “complete default” is where the three sovereign categories— fiscal, responsible monetary, and rogue monetary—converge.

9. Case Study

To test how the proposed sovereign debt risk methodology works in practice and form a judgment on its success, we have developed a case study on the sovereign debt of seven countries. Using national account data from the World Bank Data Bank, we have estimated traditional risk metrics and juxtaposed them with comparable market data. The purpose was not necessarily to produce a close match to market metrics, as we have stated our indifferent view of the correctness of existing sovereign credit models. Our model, building on theoretical premises rather than empirical relationships, is motivated by a need for change and is a prescription for change, and, as such, a perfect fit with existing data would invalidate its potential to add value. At the same time, if the results were far removed from the existing risk ranking and market metrics, which embed the collective market judgment and sense of those country's debt riskiness, it would cast doubt on the reasonableness of our approach. We are content to find that the results fall comfortably in between, simultaneously in proximity to what the market quotes as reasonable, and leaving room for the methodology to add value when applied in practice.

The procedure applied in the case study is as follows. We collected data on the economies and capital markets of the United States, United Kingdom, France, Germany, Italy, Spain, and Greece and divided them in two groups: the United States, United Kingdom, France, Germany, and separately Italy, Spain, and Greece. The grouping was thought pertinent for three reasons related to the revenue fiscal asset component of each country. First, the first four countries are large, very diversified, and capital-intensive economies, in distinction with the smaller, relatively more labor-intensive and less diversified economies of the other three countries—a distinction that we wanted to control for, in order to discover a better tailored relationship between the stock market and government revenue provided a certain type of economy. For example, we might suspect we would find a somewhat higher dependency on the country's stock market of an economy with higher capital intensity. We also might expect to find that a larger economy will have a higher correlation with a global stock index (also a component of the procedure) due to larger inclusion of its companies in that index. Another reason for the grouping was the ability to perform a more reliable revenue beta estimation through a pooled regression. The third reason has to do with the observation of closely clustered historical GDP volatilities within the two groups, with standard deviation of national income being an essential parameter in the estimation of the net fiscal asset value, as we shall see in Appendix A. The following steps were then performed:

1. Historical revenue and expenditure time series were used to calculate proportional changes in each of the two variables over time for each country.
2. In each of two groups the revenue proportional change series were regressed against the local market's S&P index, with a lag of one period on the dependent variable side. The observations were standardized to capture the correlation aspect properly, but not to allow generally larger observations from one country to influence the pooled regression with a higher weight. Separate test regressions for each country were performed and were found significant, and stepwise addition of countries from each group was done, to demonstrate that the significance of the regression was increasingly improved with addition of countries from the group, corroborating the appropriateness of grouping.
3. The coefficients from regression (2) were rescaled to reflect the original (destandardized) revenue and local market index series.

4. A regression of each local index against the S&P500 was performed. The coefficients of (3) were scaled by the regression coefficients from (4) to derive betas of the proportional change in revenues against a single reference index as described in the methodology section.[2]
5. The expenditure series of each country were regressed against the global S&P index. These regressions turned out to be significant, albeit with low regression coefficients. Our conjecture about the significance of this relationship was stated in the methodological section.
6. In our procedure, we differentiate between betas to the stock market for the purpose of valuation and betas of correlation. The reason behind this dual treatment is given in Appendix A. Beta of the revenue to the S&P 500 for the purpose of valuation for each country was calculated as the ratio of average revenue percent change standard deviation for the particular group to the S&P500 standard deviation.
7. Betas for the purpose of valuation of the expenditure stream was done similarly, but by using the country's own, not the group, expenditure proportional change standard deviation. The grouping was appropriate for revenue side analysis due to similarities in economic structure, while expenditure is predominantly influenced by a country's government decision-making "discretionary" idiosyncrasies.
8. A special type of average of (6) and (7) was calculated. This would be the actual valuation beta used to determine the discount rate for the combined stream of revenues minus expenditures. The exact formula and its derivation are provided in Appendix A.
9. The studies for the equity premium vary between 2 and 7 percent with more results gravitating to the upper bound (Song 2007). We assumed 5 percent in the analysis. One argument for our choice is practical: it is an upward-biased midpoint between two extremes. The other argument has to do with quantitative easing. Risk-averse monetary authorities will tend to intervene in pronounced economic downturns, to pull down the long-term risk-free rate, shifting left the maximum Sharpe ratio portfolio on the efficient frontier, which reflects generally lower return levels, and consequently lower required rates of return. They will tend to overshoot the length of market downturns to ensure the economy is out of recession, but reducing the long-run attainable rate of return. For this reason the maximum of 7 percent based on historical estimates when quantitative easing was not utilized would be upward biased.
10. Due to reasoning explained in the methodology section, the growth rate for the revenue stream is the forecast GDP growth rate. The forecast was the average of
    a. GDP growth average forecast of two models provided by Arnott and Chaves (2012).
    b. A weighted historical N observation average where the current observation ($t = 0$) has a weight that is $(N - 0)/(N - 1)$ times higher than the weight of the prior observation ($t = -1$). This gives exponentially higher weight to the most recent observations, making, for example, 50-year-old GDP growth to be of very little influence in the average and relatively more recent observations of significantly larger influence. These estimates were consistent with the estimates produced by the Arnott model, which supported their combined merit.[3]

11. Growth rate of expenditure was assumed to be historical average population growth calculated similarly to (10.b).

[2] Another possibility will be to derive a relationship of sovereign revenues against an index of financial companies and banks, which due to the connection between sovereign and bank should prove significant as well. The authors intend to pursue this in future extensions of the methodology.
[3] Inclusion of additional forecasts from GDP growth models would potentially improve overall accuracy. Our purpose of using two accessible models was to demonstrate the performance of the method using reasonable growth assumptions.

12. Expected bank loss was estimated using V-Lab SRISK—a free service provided by NYU, pioneered and administered by Robert Engle, Viral Acharya, and others.[4] SRISK, standing for Systemic Risk, is an estimate of the recapitalization need of each institution in the V-Lab database, under a significant decline in the stock market. SRISK is generally synonymous with the bailout put option that a government is short during a market crisis and in that sense a suitable proxy for the sovereign drain on assets under the correlated sovereign-bank-asset process described in the methodology section. We additionally calculated empirically the sensitivity of the SRISK-based proxy bailout option to the stock market.
13. Debt maturity was chosen as eight years—the prevailing approximate borrowing term of several of these governments.
14. Using the net fiscal asset values, volatilities, and Black-Scholes option pricing, we calculate the values of the default put options and their deltas.
15. Monetization of debt was assumed for the United States, United Kingdom, France, and Germany. Technically, in the Euro zone, the ECB is not allowed to finance directly government debt of more than three years' maturity, which leaves European banks as the main channel of monetization of long-term government debt. Banks are not likely to buy troubled government bonds at the premium prices as a central bank would, or extendedly rollover government debt unlike a central bank. The direct financing rule was not violated for Spain, Italy, or Greece, but we have confidence that if the largest economies Germany or France approach default, this "direct financing law" will be overridden under the ECB statute, which had allowed it to make limited purchases in 2012 of short-term debt of Italy, Greece, and Spain on the grounds that this prevents yields widening to levels that jeopardize the uniform conduct of monetary policy across the Euro zone. The larger the afflicted economies, the higher the chance that these limited exception becomes the norm.
16. Using the output for the default option values and deltas (14), the assumptions (15), and the expected bank losses (12) we additionally calculated the yield to maturity, risk-neutral probability of default, and credit-induced debt volatility.
17. The interim results of the analysis are presented in Table 2, and the output metrics (14) and (16) are presented in Table 3.

Table 2

Country	Net Fiscal Asset (billion LCU)	Debt (billion LCU)	Assets/ Debt	GDP Growth	*t*-statistic of Revenue Asset to Stock Market	Net Fiscal Asset Volatility	Valuation β	Correlation β	SRISK (billion LCU)
United States	108,435	11,100	9.7	2.15%	**6.3**	29%	0.19	0.95	581
United Kingdom	14,106	1,212	9.9	1.65		18	0.18	0.59	209

[4] The authors represent Northfield Information Service Inc., a company providing risk models for more than 30 years to more than 300 global financial institutions. Their conviction is that in an actual operational setting, the particular procedure laid out in the methodology section should be used to capture the dynamics between the sovereign and bank assets. Admittedly, using SRISK in our test framework assumes that a severe crisis in the stock market equates to a severe crisis with government finances, which is clearly a simplifying assumption. It could have been avoided if a particularly tailored risk model were used. However, it was the authors' intention not to use a risk model provided by a commercial vendor like Northfield in this exposition, in order to avoid marketing connotation, and to demonstrate that the methodology is attainable, in broad terms, using publicly available inputs. We would like to express our gratitude to the V-Lab creators and contributors for providing this valuable service to the public, and we hope to have contributed for its further popularization by the use in our analysis.

France	20,948	1,712	12.2	1.57		18	0.16	0.3	330
Germany	11,758	1,388	8.5	1.21		21	0.22	0.37	171
Italy	3,610	1,921	1.8	1.0		33	0.22	1.1	100
Spain	9,921	736	13.5	2.13	**3.9**	18	0.25	0.32	136
Greece	284	355	0.8	1.0		101	0.26	−0.31[a]	19

a. The source for negative correlation of the Greek net fiscal asset and the stock market was a particular feature of its expenditure side. Out of all countries in the analysis, Greece was the only one with a positive relationship between its expenditure and the stock market. A possible explanation might have been behavioral: the better market performed, the more government authorities might have been lax in regard to spending. With a very thin positive net asset balance between revenue and expenditure, the expenditure beta was the more pronounced component in the combined net fiscal asset beta.

Table 3

Country	Default Put Value (billion LCU)	Default Put Delta	Risk Neutral Probability of Default[a]	Forecast 8-Year Yield[b]	Market 10-Year Yield, Dec. 2011[c]	Market 10-Year Yield, Dec. 2012	Credit-Induced Debt Volatility
United States	283	≈ 0	0.03	2.4%	1.9%	1.8%	$\approx 0\%$
United Kingdom	47	≈ 0	0.04	2.9	2.0	1.8	≈ 0
France	3.72	≈ 0	0.002	2.0	3.0	2.0	≈ 0
Germany	1.57	≈ 0	0.001	2.0	1.9	1.4	≈ 0
Italy	343	0.095	0.25	5.0	7[d]	4.5	6.6
Spain	170	≈ 0	0.27	6.1	5.1	5.0	5.5
Greece	257	0.08	0.9	29	35	12[e]	23

a. Risk-neutral probability will necessarily be higher than the real world risk probability. *PD* is the variable that, in a pricing distribution, will bias the outcomes to more negative, reflecting the marginal market's investors risk aversion. Removing the risk aversion out of the distribution will push back to more positive outcomes, reducing risk-neutral to lower real world default probabilities. Furthermore, in regard to the case like Spain where the expected bank shortfall is significantly material in respect to the stand-alone sovereign put option value (with effect of bank defaults removed), and the fact that SRISK figures were used instead of the multinomial lattice procedure outlined in the methodology section, the SRISK component of the default put option delta was practically difficult to compute with a lack of a tailored factor risk model, so the particular *PD* value in the table shows the combined effect of the probability of default and the residual value as proportion of bond value. Also, for monetarily responsible entities the notion of probability of default should be more appropriately thought of as theprobability of a "commensurate monetary response," rather than explicit fiscal shortfall, as pointed out in the main discussion.

b. The comparison between the forecast yield with the 2011 market yield for each country produces 0.99 correlation across the analyzed countries.

c. As of the time the analysis was concluded, the latest debt data available from the World Bank were end of 2011. The latest revenue and expenditure data available are 2011 for some countries and 2010 for others. Given the slowly shifting country fundamentals, our conviction is that they are appropriate for proof-of-concept analysis. Depending on the accuracy and currency desired, one can obtain data for each country from a variety of sources, like the European Commission Directorate of Financial Affairs, U.S. Congressional Budget Office, and others where the latest quarter and projection data are often provided. The user friendliness and breadth of the dataset from the World Bank made it a natural base input point in our research.

d. The put option value for Greece indicates a discount of 72 percent to a unit of notional. A recent deal for repurchase of Greek national debt included a write-off between 67 and 70 percent (Landon 2012), which is very close to the discount estimated by the model. With the reduction of debt, and lower leverage, yields of Greek debt were reduced significantly.

e. In September 2012, the ECB announced a measured but decisive program for secondary market purchases of short-term debt of Euro zone governments, which had a dampening impact on the yields of Italian bonds (Warnock 2012).

10. Sovereign Risk and the Enterprise Portfolio

While producing reliable risk metrics of sovereign risk is a key analytical milestone, there is another equally important beneficial aspect of the taken approach. The risk factor framework is not only one of the building blocks in the described methodology, but also the natural environment where the extended output of the sovereign risk approach can add value as an enterprise risk management tool.

The view of a sovereign position as a portfolio of sovereign and bank assets has immediate implications. Given that both banks and sovereigns are naturally exposed to the same risk factors—clearly connected to the broad economy—we can use the factor exposures and build factor risk representations of each of these entities. Provided that the risk factor exposures combine linearly to produce an aggregate set of exposures, one can identify factor bets and correlations with seemingly unrelated investment positions and asset classes, transcending portfolios and company divisions as long as the same factor risk model is used in all of these subdomains.

The higher-moment behavior introduced by the default put options along each of the sovereign and bank asset dimensions can be handled in two ways. We have alluded to both in our exposition. First, we can take the delta approximation approach, calculate the parameters of the standalone sovereign default, the recapitalization (bailout put) options of banks, and the default put options of banks net of the recapitalization option. The last component is important to remember in light of the fact that the government is not committed to repaying all creditors of a troubled bank, but committed to keeping it in an operational state (i.e., be a conduit for *future* lending); the two objectives have different outcomes to the trouble bank's creditors. By overlaying delta-equivalent position of each entity in the portfolio we can achieve a portfolio of all or any combination of entities. The main benefit of the linear combination approach is that the output of the procedure will be directly usable in any type of mean-variance optimization.

Another way to incorporate the credit options will be to use directly the multinomial lattice described in the methodology section. Observing and numerically integrating all of the outcomes from identity (8), we would be able to compute probabilities, Value-at-Risk, expected tail loss, skew, kurtosis, etc. Due to today's technology, a multinomial lattice of this sort will able to handle billions of scenarios and hundreds of entities, in a matter of minutes. The benefit of this approach will be better accuracy of producing portfolio-level risk metrics reflective of higher moments. Our opinion is that an integrated approach that exploits the benefits of both methods should be used as a matter of robust risk management practice.

11. Conclusion

The purpose of our research and modeling work was to provide a framework for bottom-up, accurate measurement of sovereign credit risk, and by extension, bank risk, that is compatible with enterprise-wide risk management. Our belief is that the captures the dynamics of sovereign credit risk in an economically justified way. The approach is based on contingent claims analysis arguments and uses explicit measures of the sovereign net fiscal assets and their volatility—inputs that have been a key hurdle to applying the default option analysis to sovereign credit in the past. The methodology is comprehensive regarding the customary types of government responses to a credit crisis. The model limits the use of implied inputs to reduce model risk from feeding back erroneous market views. There is evidence for this pitfall provided the swift market "adjustments" during the Euro zone sovereign credit crisis. No assumptions for explicit modeling of debt renegotiation were deemed necessary; apart from that rational renegotiation is based on future capacity of government finances to bear the new debt burden, which this model is designed to capture. As an outcome of these features, the model offers results that are consistent with observed variables in the sovereign debt market. It also supports our initial goal to make it an integrated component of an enterprise-level risk model.

References

Arnott, R. D., and D. B. Chaves. 2012. Demographic Changes, Financial Markets, and the Economy. *Financial Analysts Journal* 68(1). Jan., 2012, pp. 36-37.

Blodget, H. 2012. Sorry, Folks, We Don't Just Have 'A Spending Problem'. *Business Insider,* December. http://www.businessinsider.com/government-spending-and-taxes-2012-12?utm_source=Pulse&utm_medium=App&utm_campaign=partner&utm_medium=referral&utm_source=pulsenews.

DiBartolomeo, D. 2007. Portfolio Analysis of Investment Funds with Undisclosed Holdings. Northfield Research. January. http://www.northinfo.com/Documents/508.pdf.

European Central Bank. 2012. Financial Integration of Europe. April, pp. 20–24. http://www.ecb.int/pub/pdf/other/financialintegrationineurope201204en.pdf.

Federal Reserve Bank of Dallas. 2011. Hyperinflation in Zimbabwe. Globalization and Monetary Policy Institute Annual Report. http://www.dallasfed.org/assets/documents/institute/annual/2011/annual11b.pdf.

François , P., G. Hübner, and J.-R. Sibille. 2011. A Structural Balance Sheet Model of Sovereign Credit Risk. December. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1975353.

Gray, D., R. Merton, and Z. Bodie. 2007. Contingent Claims Approach to Measuring and Managing Sovereign Credit Risk. *Journal of Investment Management* 5(4): 5–28.

Jeanneret, A. 2008. A Structural Model for Sovereign Credit Risk. Harvard University, Swiss Finance Institute, and University of Lausanne, April.

Landon, T., Jr. 2012. Terms of Greek Bond Buyback Top Expectations. *New York Times,* December 3. http://www.nytimes.com/2012/12/04/business/global/greece-announces-terms-of-13-billion-bond-buyback-to-slash-debt.html?_r=0.

Leland, H., and K. B. Toft. 1996. Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. *Journal of Finance* 51: 987–1019.

Merton, R. C. 1974. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance* 29: 449–470.

Sharpe, W. F. 1992. Asset Allocation: Management Style and Performance Measurement. *Journal of Portfolio Management* 18(2): 7–19.

Song, Z. 2007. The Equity Risk Premium: An Annotated Bibliography. *Research Foundation of CFA Institute Literature Review* 2(1). Jan., 2007, pp. 5-19.

Warnock, E. 2012. ECB's Noyer Says Bond Program Is 'Bearing Fruit'. *Wall Street Journal,* Dec. 2. http://online.wsj.com/news/articles/SB10001424127887323401904578154852649691068

Williams, R. 2011. The Numbers: What Are the Federal Government's Sources of Revenue? Tax Policy Center, Urban Institute and Brooking Institution. September. http://www.taxpolicycenter.org/briefing-book/background/numbers/revenue.cfm.

Appendix A

A major assertion in the model framework is that the beta of the combined revenue-expenditure position for risk-calculation purposes is different from that used for valuation purposes. To see the rationale for this, we should inspect it with two types of considerations.

First, the assumption poses no contradiction to the real world. Government revenues and expenditures are enormous and their diversification beyond the capability of any market player, or even beyond the unlikely concerted effort of a number of players. Both the revenue and expenditure sides are largely exogenous to any diversification-related decision making of even the largest player around—the government. The government in that sense is not in a role to optimize the portfolio of revenues and expenditures. If put in an asset management perspective, the government's bona fide priorities are in this order: (1) to optimize the social/economic utility of expenditure allocation itself and (2) the cash flow matching of the revenue and expenditure sides (i.e., similar to an asset-liability management objective). This is distinct from optimizing the one-period mean-variance return of a portfolio long the revenue stream and short the expenditure and debt service stream. Therefore, we deem reasonable that, in valuation of revenue/expenditure cash flows, one should use a beta that is the ratio of the volatility of the cash flow stream to the volatility of the broad market. In other words, the investor is unable to diversify, and the type of disutility he or she will require a premium for, apart from time value of money, will be the disutility from volatility relative to the market. For this reason, the size of the market premium would be an adequate measure of the premium investors will require to bear other types of undiversifiable risk of the same or proportional magnitude, in this case the volatility of the large government finance figures.

Second, the assumption does not pose any major difficulties in the calculation process. As we have seen previously, the revenue and expenditure side should both have estimable values (using valuation beta) and separately correlations with the broad market (using correlation-based beta). With these inputs at hand, we can readily pursue the calculation of the standard deviation of either side in isolation, and both sides in combination.

At first glance, we might think that the previous logic might pose a challenge when we deal with the combined discount rate of a portfolio of two cash flows like revenue and expenditure streams—each of undiversifiable nature, and each of different volatility relative to the market. As a start, we will support our decision to use a combined discount rate, and then we will address the computation of the combined discount rate.

The rationale to use one discount rate for both the revenue and the expenditure sides rests on two arguments. First, we are eventually modeling the combined position of the two sides, so combining the calculation of an expected discount rate, and using it over the two sides' separate marginal distributions, is mathematically legitimate. Second, the average of two estimates is significantly superior from an estimation efficiency point of view, rather than using two estimates for the expenditure and revenue sides separately (each with less number of observations). The accuracy of the discount rate estimate is important, as any estimation error will be magnified through the perpetuity cash flows stream discounting, which is sensitive to the discount rate input.

Using the utility argument mentioned earlier, we can proceed with estimating a valuation beta that will be reflective of the discount rate for the combinations of several undiversifiable positions. First, we shall postulate that the baseline utility for assuming undiversifiable risk represented by the broad market is

$$U_{market} = M_{market} - \left(\frac{1}{RAP}\right) \times \sigma^2_{market}, \tag{10}$$

where M_{market} is the expected market return, σ^2_{market} is its variance, and RAP is a parameter related to an investor's risk acceptance/aversion.

We have already made the conclusion that any investment that carries β times more undiversifiable risk than the market has utility

$$U_{investment\_A} = M_{investment\_A} - \left(\frac{1}{RAP}\right) \times \beta^2_{investment\_A} * \sigma^2_{market}. \tag{11}$$

Consequently our revenue and expenditure positions have utility respectively of

$$U_{expenditure} = M_{expenditure} - \left(\frac{1}{RAP}\right) \times \beta^2_{expenditure} \times \sigma^2_{market}, \tag{12}$$

$$U_{revenue} = M_{revenue} - \left(\frac{1}{RAP}\right) \times \beta^2_{revenue} \times \sigma^2_{market}. \tag{13}$$

Based on our premises that we penalize returns not for their directionality vis-à-vis the market and vis-à-vis each other, but based on their volatility as an idiosyncratic phenomenon, the only reasonable ensuing assumption is that they should be considered uncorrelated for the purpose of utility calculation. Hence the risky (unexpected) component of return stemming from the two positions *cannot* directly add or subtract as a unified argument of the combined to utility function, as we do not admit the diversification of systematic risk of the combined portfolio due to positions of potentially opposite sign. We also impose the natural constraint that a variance unit of any stand-alone position should impact the overall utility to the same investor in the same amount per dollar invested. In other words we, as a net effect, would feel inconvenienced the same if revenue has a volatility of *X* per dollar, or expenditure has a volatility of *X* per dollar. Then we can write our utility function as

$$U^*_{Combined} = M_{Portfolio} - \left[\left(\frac{1}{RAP}\right) * (|Value_{revenue}| \times \sigma^2_{revenue} + |Value_{expenditure}| \times \sigma^2_{expenditure}\right],$$

where *Value* is the signed market value of each component of the combined cash flow. To make this utility function positive homogeneous, a common desirable property for utility functions to ensure that they behave the same as we scale the portfolio up and down, we can multiply the variance terms on the right-hand side by

$$\frac{1}{|\mathrm{Value}_{\mathrm{revenue}}|+|\mathrm{Value}_{\mathrm{expenditure}}|},$$

and the expected return term by

$$\frac{1}{\mathrm{Value}_{\mathrm{revenue}}+\mathrm{Value}_{\mathrm{expenditure}}}.$$

The utility function then becomes

$$U_{Combined} = \frac{1}{Value_{revenue} + Value_{expenditure}} \times \quad M_{Combined} - \frac{1}{|Value_{revenue}|+|Value_{expenditure}|} [\left(\frac{1}{RAP}\right) \times (|Value_{revenue}| \times \sigma^2_{revenue} + |Value_{expenditure}| \times \sigma^2_{expenditure}].$$

Dropping the expected value terms for brevity we get to

$$-\frac{1}{|Value_{revenue}|+|Value_{expenditure}|} [\left(\frac{1}{RAP}\right) \times (|Value_{revenue}| \times \sigma^2_{revenue} + |Value_{expenditure}| \times \sigma^2_{expenditure}],$$

which relating to equations (12) and (13), we can write as

$$- [\left(\frac{1}{RAP}\right) \times \left(Absolute\_Weight_{revenue} \times \beta^2_{revenue} + Absolute\_Weight_{expenditure} \times \beta^2_{expenditure}\right)] \times \sigma^2_{market}.$$

We can then put in the convention that

$$\beta^2_{portfolio} = Absolute\_Weight_{revenue} \times \beta^2_{revenue} + Absolute\_Weight_{expenditure} \times \beta^2_{expenditure}.$$

Consequently, we can write the utility function of our portfolio as

$$U_{Combined} = M_{portfolio} - \left(\frac{1}{RAP}\right) * \beta^2_{rportfolio} \times \sigma^2_{market}. \quad (14)$$

We note that this utility function is not concave like the usual utility functions applied in portfolio management (i.e., it does not promote diversification), but this is in line with our assumption of nondiversifiable investment positions. In all other aspects it is like a customary utility function: It is positive homogenous, translation invariant, supports stochastic dominance, etc. Therefore, we accept it as our utility function.

The result then is that the combined portfolio has the same utility as a portfolio with relative volatility (beta) to the broad market equal to

$$\beta_{portfolio} = \sqrt{Absolute\_Weight_{revenue} \times \beta_{revenue}^2 + Absolute\_Weight_{expenditure} \times \beta_{expenditure}^2},$$

which would entail using this beta in the calculation of the discount rate for the net fiscal asset valuation.

We need to point out that that the linear weighted combination of expected returns of the two separate positions will be different than the expected return of the combined portfolio, but this is just a reincarnation of the inability to diversify (i.e., arbitrage away), due to the scale of positions. The world of the utility function (14) consists only of the government portfolio of the revenue net expenditure streams, which, even if a world of epic magnitude, is a lonely place where this theoretical anomaly remains contained.

As a final note we refer to a practical feature of the combined value calculation. Given that the combined beta for the purpose of valuation uses the cash flow present values, which in turn use the beta for the purpose of valuation, we had to take a circular approach. We sought a solution that will make the system of variables intrinsically compatible. The values displayed on Tables 2 and 3 are the output of this procedure.