TRANSACTIONS OF SOCIETY OF ACTUARIES
1950 VOL. 2 NO. 2

ACTUARIAL NOTE: ON AVERAGE AGE AT DEATH PROBLEMS

WALTER L. GRACE AND CECIL J. NESBITT

THE purpose of this note is to demonstrate a general method for the solution of average age at death problems. Although such problems are not of practical importance, the method and notations used here may, nevertheless, be of interest and value beyond the immediate problems.

A prerequisite for the understanding of this work is familiarity with the concepts of the survivorship group (cohort) and stationary population interpretations of a mortality table. It is assumed that the reader is familiar with the basic formulas in Chapters I and XII of reference 1 at the end of this note. For additional discussion of the cohort and stationary population ideas, the reader may refer to pp. 21–22 of reference 2 or pp. 1–2 of reference 3.

We shall use the following notation:

$$\begin{aligned} \eta &= \text{the number of deaths} \\ \tau &= \text{total lifetime of the } \eta \text{ persons who die} \\ a &= \text{average age at death of those } \eta \text{ persons} \\ &= \tau/\eta . \end{aligned}$$

Our method for the solution of average age at death problems depends upon the use of two special functions. The first, F_x, represents the total lifetime of the l_x persons who attain exact age x in the survivorship group represented by the mortality table, and thus

$$F_x = xl_x + T_x . \qquad \text{(see ref. 1, p. 207)} \quad (1)$$

We shall let G_x represent the total lifetime of the closed group formed from the T_x persons who are now aged x or more in the stationary population represented by the mortality table, that is,

$$G_x = xT_x + 2Y_x . \qquad \text{(see ref. 1, p. 211)} \quad (2)$$

As immediate properties of these functions, we have

$$dF_x = -xl_x\mu_x dx \quad (3)$$

$$dG_x = -F_x dx . \quad (4)$$

From these it follows that

$$F_x = \int_0^\infty (x+t)\, l_{x+t}\mu_{x+t}\,dt \tag{5}$$

$$G_x = \int_0^\infty F_{x+t}\,dt\,. \tag{6}$$

Our method applies to the wide variety of average age at death problems wherein the number of deaths η is expressible as a linear combination of certain values of l_x and T_x, say

$$\eta = a_1 l_{x_1} + a_2 l_{x_2} + \ldots + a_r l_{x_r} + b_1 T_{y_1} + b_2 T_{y_2} + \ldots + b_s T_{y_s} \tag{7}$$

or, on introduction of $\varphi(l_{x_i}, T_{y_j})$ to denote the right-hand member of (7),

$$\eta = \varphi\,(l_{x_i}, T_{y_j})\,. \tag{8}$$

In our method the total lifetime τ of the η persons who die is determined by simply replacing in $\varphi(l_{x_i}, T_{y_j})$ each l_{x_i} by the corresponding F_{x_i} and each T_{y_j} by the corresponding G_{y_j}, that is, we take

$$\tau = \varphi\,(F_{x_i}, G_{y_j})\,. \tag{9}$$

Thus, in such cases, the problem of determining the average age at death is reduced to the simpler subproblem of determining $\eta = \varphi(l_{x_i}, T_{y_j})$, whereupon

$$a = \frac{\varphi\,(F_{x_i}, G_{y_j})}{\varphi\,(l_{x_i}, T_{y_j})}\,. \tag{10}$$

It is difficult to present a short, yet rigorous proof of formula (10) which will cover the variety of cases to which it will apply. In place of such a general proof we shall show how the formula applies in simple cases and shall also illustrate its application in some more complicated problems. For a particular case the correctness of formula (10) is either immediately obvious or may be verified by examining the calculations for η and visualizing the parallel calculations for τ. In each of the illustrations the parallelism between the calculations for η and τ may be observed.

As standard problems we consider:

a. Determine the average age at death of those persons who will die between ages $x + m$ and $x + m + n$ from among l_x persons now of exact age x.

Here

$$\eta = \int_m^{m+n} l_{x+t}\mu_{x+t}dt = \int_m^{m+n} - dl_{x+t} = l_{x+m} - l_{x+m+n}$$

$$\tau = \int_m^{m+n} (x+t)\, l_{x+t}\mu_{x+t}dt = \int_m^{m+n} - dF_{x+t} = F_{x+m} - F_{x+m+n}.$$

Then

$$a = \frac{F_{x+m} - F_{x+m+n}}{l_{x+m} - l_{x+m+n}}$$

which is of the form (10).

b. Determine the average age at death of those persons who die between ages x and $x + n$ in any period of c years in the stationary population represented by the mortality table.

Since at each moment during the c years there are $l_{x+t}dt$ persons subject to the annual mortality rate μ_{x+t}

$$\eta = \int_0^n c\,\mu_{x+t}l_{x+t}dt = c\,(l_x - l_{x+n})$$

$$\tau = \int_0^n (x+t)\; c\,\mu_{x+t}l_{x+t}dt = c\,(F_x - F_{x+n})$$

and

$$a = \frac{F_x - F_{x+n}}{l_x - l_{x+n}}.$$

c. Find the average age at death of those persons who are now between ages x and $x + n$ in the stationary population represented by the mortality table.

In this problem we trace the closed group of persons consisting of the survivors from the $T_x - T_{x+n}$ persons who are now between ages x and $x + n$. Then

$$\eta = \int_0^n l_{x+t}dt = T_x - T_{x+n}$$

$$\tau = \int_0^n F_{x+t}dt = G_x - G_{x+n}$$

$$a = \frac{G_x - G_{x+n}}{T_x - T_{x+n}}.$$

As a first illustration of the use of the method in a more complicated situation, we take the problem:

a. In the stationary population represented by the mortality table, what is the average age at death of the persons who at the present moment are between ages 20 and 40 and who will die between ages 30 and 50?

From the $l_{20+t}dt$ persons now at age $20 + t$, $(t < 10)$ there will be $(l_{30} - l_{50})dt$ persons who die between ages 30 and 50. From the $l_{30+t}dt$ persons now at age $30 + t$, $(t < 10)$ there will be $(l_{30+t} - l_{50})dt$ persons who die between ages 30 and 50. Hence

$$\eta = \int_0^{10} (l_{30} - l_{50})\, dt + \int_0^{10} (l_{30+t} - l_{50})\, dt$$

$$= 10l_{30} - 20l_{50} + T_{30} - T_{40}\,.$$

Then, directly, or by use of example "a" of the preceding set, we have

$$\tau = \int_0^{10} (F_{30} - F_{50})\, dt + \int_0^{10} (F_{30+t} - F_{50})\, dt$$

$$= 10F_{30} - 20F_{50} + G_{30} - G_{40}$$

$$a = \frac{10F_{30} - 20F_{50} + G_{30} - G_{40}}{10l_{30} - 20l_{50} + T_{30} - T_{40}}\,.$$

Our second illustration is problem 5 of the 1942 Part 5 examination.

b. A nation with a stationary male civilian population of T_0 on January 1, 1941, conscripts by lot for its army 10 percent of all male civilians then between ages 21 and 28. Ten percent of those attaining age 21 after January 1, 1941, are immediately inducted. If no men are released from the army prior to January 1, 1942, determine the average age at death of those male civilians who die during 1941 (assuming equal distribution of deaths throughout each year of age).

The problem can be solved without use of the assumption in the parentheses. We shall proceed by finding η and τ for the army personnel and by subtracting from l_0 and T_0 obtain η and τ for the civilians.

For $t < 7$ there are at each moment during the year $\frac{1}{10}l_{21+t}dt$ persons at age $21 + t$ in the army who are subject to the annual mortality rate μ_{21+t}. This leads to $\frac{1}{10}\,\mu_{21+t}l_{21+t}dt$ deaths at age $21 + t$ during the year. For deaths between ages 28 and 29 we must take into account the fact that there will be conscripts at age $28 + x$ only during the last $(1 - x)$ part of the year. Hence, the number of deaths at age $28 + x$ will be $(1 - x)\mu_{28+x} \cdot \frac{1}{10}l_{28+x}dx$. Then,

$$\eta = \frac{1}{10}\left[\int_0^{7} \mu_{21+t}l_{21+t}dt + \int_0^{1} (1 - x)\, \mu_{28+x}l_{28+x}dx\right]$$

$$= \frac{1}{10}\left\{l_{21} - l_{28} - \left[\, (1 - x)\, l_{28+x}\right]_0^1 - \int_0^{1} l_{28+x}dx\right\}$$

$$= \tfrac{1}{10}\{ l_{21} - T_{28} + T_{29}\}\,.$$

We could now predict that

$$\tau = \tfrac{1}{10}\{F_{21} - G_{28} + G_{29}\}.$$

The verification is that

$$\tau = \frac{1}{10}\left[\int_0^7 (21+t)\,\mu_{21+t}l_{21+t}dt + \int_0^1 (1-x)(28+x)\,\mu_{28+x}l_{28+x}dx\right]$$

$$= \frac{1}{10}\left\{F_{21} - F_{28} - \left[(1-x)F_{28+x}\right]_0^1 - \int_0^1 F_{28+x}dx\right\}$$

which reduces to the above.

For the civilian deaths, it follows that

$$a = \frac{T_0 - \frac{1}{10}(F_{21} - G_{28} + G_{29})}{l_0 - \frac{1}{10}(l_{21} - T_{28} + T_{29})}$$

$$= \frac{F_0 - \frac{1}{10}(F_{21} - G_{28} + G_{29})}{l_0 - \frac{1}{10}(l_{21} - T_{28} + T_{29})}.$$

REFERENCES

1. Spurgeon, E. F. *Life Contingencies*, Cambridge: At the University Press, 1932.
2. Greville, T. N. E. *U.S. Life Tables*, 1939–41.
3. Bailey, W. G., and Haycocks, H. W. *Some Theoretical Aspects of Multiple Decrement Tables*, Edinburgh: Constable, 1946.