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CALCULATION OF APPROXIMATE ANNUITY VALUES ON A MORTALITY BASIS THAT PROVIDES FOR FUTURE IMPROVEMENTS IN MORTALITY

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I. INTRODUCTION

This paper presents a relatively simple method for calculating approximate annuity values on a mortality basis that provides for future improvements in mortality. This method involves the use of a special set of supplementary commutation columns in addition to the standard commutation columns that are generally used. Appropriate formulae are developed for calculating the approximate annuity values directly from these commutation columns. The approximate annuity values so calculated agree closely with the exact values calculated from a mortality table (without projection) and a projection scale for future improvements in mortality, such as were presented by Messrs. W. A. Jenkins and E. A. Lew in their paper, "A New Mortality Basis for Annuities."[1]

While this method is a general one in the sense that it may be used to calculate approximate annuity values on the basis of any mortality table (without projection), any reasonable projection scale of future improvements in mortality, and any interest rate, the particular supplementary

[1] *TSA* I, 369.

commutation columns presented in this paper are based on the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest. Annuity values calculated from these supplementary commutation columns reproduce very closely the values obtained by applying the Scale B projection factors from the Jenkins-Lew paper to annuity values calculated on the basis of the Annuity Table for 1949 (ultimate), without projection, and $2\frac{1}{2}\%$ interest. These supplementary commutation columns provide, therefore, a practical method for calculating approximate annuity values on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest.

The method presented here is applicable to all types of annuity contracts, immediate or deferred. It is applicable to nonrefund annuities and to all of the various types of refund annuities, including both cash and installment refund annuities and life annuities guaranteeing payments for a certain period. In fact, the approximate value of any type of benefit involving life contingencies may be calculated by this method, as each of the standard commutation symbols may be represented in terms of the supplementary commutation symbols. Approximate values of joint life annuities may also be calculated by using these supplementary commutation columns. While the method is particularly useful for valuation purposes, it may be used for any other purposes where exact annuity values are not required, as the errors involved are relatively small.

The particular advantage of using the proposed method for valuation purposes is that the same valuation factors may be used year after year even though the annuity contracts are valued on a mortality basis that provides for future improvements in mortality. If the exact annuity values that are produced from a mortality table (without projection) and a projection scale were used for valuation purposes, the valuation factors would have to be changed each year. This annual change in valuation factors is avoided under the proposed method by using two valuation factors for each attained age. One valuation factor represents the approximate value of the annuity in 1950, with provision for future improvements in mortality, and the other valuation factor represents the approximately constant annual increment in the annuity value that results from improving mortality. The same valuation factors could be used each year and the adjustment for improving mortality could be made in the aggregate, by multiplying the total annual increment for all ages combined by the number of years that elapse between 1950 and the year in which the valuation takes place.

Even though the annuity values produced by this method are approximate, they are always consistent, as all of the formulae in this paper are developed from the same basic assumptions. Comparisons between the approximate annuity values produced by the supplementary commutation columns and the corresponding exact annuity values indicate that the approximate values generally exceed the exact values. The maximum errors on immediate annuities are about $\frac{1}{2}\%$ of the annuity value at the younger ages and practically zero at the older ages. The errors on deferred annuities are only slightly larger. Test valuations based on a recent age distribution of immediate life annuity contracts in force in the Metropolitan Life Insurance Company indicate that in 1950 the aggregate reserve based on the approximate method would exceed the aggregate reserve based on exact annuity values by less than .01% for males and less than .02% for females. It is estimated that the corresponding errors in 1960 would be less than .1% for both males and females.

II. IMMEDIATE NONREFUND LIFE ANNUITIES

The basic principles underlying the proposed method may be most easily grasped by analyzing in detail the simplest problem, namely, that of calculating the approximate value of an immediate nonrefund life annuity. The application of this method to other types of annuities follows along the same general lines and will be discussed in subsequent sections.

The standard notation that will be used, such as i, d, v, q_x, p_x, N_x, D_x, a_x, may be considered as defined in terms of the Annuity Table for 1949 (ultimate), without projection, and $2\frac{1}{2}\%$ interest.[2] The new notation that will be used may be considered as defined in terms of the Annuity Table for 1949 (ultimate) with Projection Scale B[3] and $2\frac{1}{2}\%$ interest. This new notation is defined with reference to the calendar year 1950 in order to reflect the Jenkins-Lew assumption that the Annuity Table for 1949 (ultimate), without projection, represents the level of mortality in the base calendar year 1950. The basic new symbols that will be used are:

s_x = the annual rate of decrease in the mortality rate at attained age x (= $\frac{1}{100}$ times the s_x referred to on page 424 of the Jenkins-Lew paper)

$$^{1950+k}q_x = \text{the mortality rate at attained age } x \text{ in the year } 1950+k$$
$$= q_x(1-s_x)^k \quad (1)$$

$$^{1950+k}p_x = \text{the probability of surviving one year at attained age } x \text{ in the year } 1950+k$$
$$= 1 - {}^{1950+k}q_x \quad (2)$$

[2] See Table 9, *TSA* I, 386.

[3] See Table 19, *TSA* I, 417—the values at intervening ages were obtained by interpolation.

${}^{1950+k}_{n}p_x$ = the probability that a life aged x in the year $1950 + k$ will survive n years to attain age $x + n$ in the year $1950 + k + n$

$$= ({}^{1950+k}p_x)\,({}^{1950+k+1}p_{x+1})\;\ldots\;({}^{1950+k+n-1}p_{x+n-1}) \tag{3}$$

${}^{1950+k}a_x$ = the value of an immediate nonrefund life annuity issued to a life aged x in the year $1950 + k$

$$= v({}^{1950+k}p_x) + v^2({}^{1950+k}_{2}p_x) + \ldots + v^n({}^{1950+k}_{n}p_x) + \ldots . \tag{4}$$

The other new symbols will be defined when we need them but they will all be summarized in Appendix I.

As a matter of convenience, all of the above symbols were defined in terms of a specific stationary mortality table, a specific projection scale for future improvements in mortality, and a specific interest rate. It should be understood, however, that all of the formulae that are derived below may also be interpreted in terms of other mortality and interest bases. The accuracy of the proposed method will, of course, depend on the particular mortality and interest bases used, but the results should generally be satisfactory provided that the annual rates of decrease in mortality are not considerably higher than those defined in Projection Scale B.

It should also be noted that while all of the new symbols are defined on the assumption that 1950 is the base calendar year, the method described in this paper is perfectly general and may be used even though some calendar year other than 1950 is assumed to be the base calendar year. Thus, all of the formulae in this paper would still be applicable if the superscript "1950 + k" would be replaced by the superscript "k" and the new superscript "k" would be interpreted to indicate that the values are to be taken as of the calendar year which occurs "k" years after the base calendar year.

Our basic objective is to find a relatively simple formula that will produce approximate values of ${}^{1950+k}a_x$ on the basis of the Annuity Table for 1949 with Projection Scale B. The first step is to show that ${}^{1950+k}a_x$ may be considered equivalent to an annuity with variable payments calculated on the basis of the Annuity Table for 1949 without projection. If appropriate values of ${}_np_x$ are inserted in the numerator and denominator of each term of formula (4), that formula may be stated as follows:

$$\left.\begin{aligned} {}^{1950+k}a_x = {} & v\,p_x\left(\frac{{}^{1950+k}p_x}{p_x}\right) + v^2\,{}_2p_x\left(\frac{{}^{1950+k}_{2}p_x}{{}_2p_x}\right) + \ldots \\ & + v^n\,{}_np_x\left(\frac{{}^{1950+k}_{n}p_x}{{}_np_x}\right) + \ldots . \end{aligned}\right\} \tag{5}$$

The corresponding value of an annuity paying a level amount of $1.00 per year on the basis of the Annuity Table for 1949 without projection is

$$a_x = vp_x + v^2{}_2p_x + \ldots + v^n{}_np_x + \ldots . \qquad (6)$$

By comparing formula (5) with formula (6), it is readily apparent that we may consider ${}^{1950+k}a_x$ as representing the value of an annuity calculated on the basis of the Annuity Table for 1949 (without projection) but with an increased amount payable each year. The increased amount payable at the end of the nth year would be ${}^{1950+k}_{\;\;n}p_x/{}_np_x$ and, as might be expected, it merely represents the ratio by which the probability of surviving n years is increased because of the improvements in mortality that are assumed in Projection Scale B. Specimen values of the increased amounts payable each year for an immediate nonrefund life annuity issued in 1950 to a male life aged 65 are shown in column (1) of Table 1. These increased amounts are the exact values of ${}^{1950}_{\;\;n}p_{65}/{}_np_{65}$ calculated for a male life on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B.

The second step is to find some simple formula that would produce approximate values of the increased amounts that are payable each year. A simple method that might be considered at first would be to assume that the amounts payable increase each year by a constant amount. Preliminary trials quickly indicated that this assumption did not produce sufficiently accurate results. A glance at column (1) of Table 1 clearly indicates that the increased amounts do not progress by approximately constant steps.

In order to find a formula that would produce more accurate approximations to the increased amounts payable each year, let us consider the exact formula for the increased amount payable at the end of the nth year, namely ${}^{1950+k}_{\;\;n}p_x/{}_np_x$. The exact formula is:

$$\frac{{}^{1950+k}_{\;\;n}p_x}{{}_np_x} = \left(\frac{{}^{1950+k}p_x}{p_x}\right)\left(\frac{{}^{1950+k+1}p_{x+1}}{p_{x+1}}\right)\ldots\left(\frac{{}^{1950+k+n-1}p_{x+n-1}}{p_{x+n-1}}\right) \qquad (7)$$

from (3) above. The exact values of the numerators in formula (7) may be obtained from formulae (2) and (1) above.

Formula (1) may be expanded as follows:

$${}^{1950+k}q_x = q_x(1-s_x)^k = q_x\left[1 - k\,s_x + \frac{k\,(k-1)}{2}\,s_x^2 - \ldots\right]. \qquad (8)$$

As s_x is generally small, not greater than .0125 on Projection Scale B, the first approximation is to ignore the second and higher powers of s_x in formula (8). This produces the approximate formula

$${}^{1950+k}q_x \doteq q_x - k\,s_xq_x\,. \qquad (9)$$

From (2) and (9), we obtain

$$^{1950+k}p_x \doteq p_x + k\, s_x q_x \,. \tag{10}$$

Dividing both sides of (10) by p_x, we get

$$\frac{^{1950+k}p_x}{p_x} \doteq 1 + k\,\frac{s_x q_x}{p_x}. \tag{11}$$

The effect of this first approximation is to overstate slightly the true values of $^{1950+k}p_x/p_x$, as the next term in formula (11) would be $-(k[k-1]/2)(s_x^2 q_x/p_x)$. If we let $f_x = s_x q_x/p_x$, we may rewrite formula (11) and obtain similar expressions for the other terms in formula (7), so that

$$\left.\begin{aligned} \frac{^{1950+k}p_x}{p_x} &\doteq 1 + k f_x \\ \frac{^{1950+k+1}p_{x+1}}{p_{x+1}} &\doteq 1 + (k+1)\, f_{x+1} \\ &\cdots\cdots\cdots\cdots \\ \frac{^{1950+k+n-1}p_{x+n-1}}{p_{x+n-1}} &\doteq 1 + (k+n-1)\, f_{x+n-1}\,. \end{aligned}\right\} \tag{12}$$

The increased amount payable at the end of the nth year is defined by formula (7) as the product of all of the left-hand terms of (12), so that

$$\left.\begin{aligned} \frac{^{1950+k}_{\quad n}p_x}{_np_x} \doteq [1 + k f_x][1 + (k+1)\, f_{x+1}]\cdots \\ \times [1 + (k+n-1)\, f_{x+n-1}]\,. \end{aligned}\right\} \tag{13}$$

As each f_x term includes a corresponding s_x term, we may introduce another approximation in expanding the right-hand side of (13) and again take advantage of the small values of s_x by ignoring all terms involving the second and higher powers of f_x. This produces

$$\frac{^{1950+k}_{\quad n}p_x}{_np_x} \doteq 1 + k f_x + (k+1)\, f_{x+1} + \cdots + (k+n-1)\, f_{x+n-1}\,. \tag{14}$$

It should be noted that while the first approximation tends to overstate the true values of $^{1950+k}_{\quad n}p_x/_np_x$, the second tends to understate the true values of $^{1950+k}_{\quad n}p_x/_np_x$, as all of the terms that are ignored are positive. The fact that the two approximations tend to balance each other explains the very small differences between the approximate and exact annuity values.

We have now attained our second objective, as formula (14) produces approximate values of the increased amount that is payable at the end of the year. In order to illustrate the accuracy of formula (14) we may apply it to the specific example that was referred to earlier, namely, an immediate nonrefund life annuity issued in 1950 to a male life aged 65. The increased amounts for this example would be defined as

$$\frac{{}^{1950}_{\ n}p_{65}}{{}_np_{65}} \doteq 1 + f_{66} + 2f_{67} + \ldots + (n-1)\,f_{65+n-1}\,. \qquad (15)$$

The approximate values produced by this formula are shown in column (2) of Table 1 and may be compared with the exact amounts shown in col-

TABLE 1

COMPARISON OF EXACT AND APPROXIMATE VALUES OF INCREASED AMOUNTS PER $1 OF ANNUAL INCOME THAT REFLECT THE EFFECT OF IMPROVING MORTALITY

For Immediate Nonrefund Life Annuity Issued in 1950 to a Male Aged 65 on the Annuity Table for 1949 (Ultimate) with Projection Scale B

End of nth Year n	Exact Value $= \frac{{}^{1950}_{\ n}p_{65}}{{}_np_{65}}$ (1)	Approximate Value* $= 1+\sum_{t=0}^{n-1} t f_{65+t}$ (2)	Excess of Approximate Value over Exact Value $=(2)-(1)$ (3)
1	1.00000	1.00000	.00000
2	1.00027	1.00028	.00001
3	1.00085	1.00086	.00001
4	1.00177	1.00178	.00001
5	1.00306	1.00308	.00002
6	1.00476	1.00481	.00005
7	1.00689	1.00699	.00010
8	1.00949	1.00963	.00014
9	1.01261	1.01282	.00021
10	1.01627	1.01654	.00027
11	1.02051	1.02086	.00035
12	1.02532	1.02576	.00044
13	1.03071	1.03118	.00047
14	1.03667	1.03719	.00052
15	1.04320	1.04371	.00051
16	1.05026	1.05072	.00046
17	1.05781	1.05816	.00035
18	1.06576	1.06591	.00015
19	1.07400	1.07387	−.00013
20	1.08233	1.08185	−.00048
21	1.09052	1.08961	−.00091
22	1.09824	1.09681	−.00143
23	1.10505	1.10310	−.00195
24	1.11039	1.10800	−.00239
25 and over	1.11352	1.11083	−.00269
${}^{1950}a_{65}$†	11.74417	11.74445	.00028

* Produced by formula (15).

† ${}^{1950}a_{65}$ equals the present value of the increased amounts on the basis of the Annuity Table for 1949 (ultimate), without projection, and 2½% interest.

umn (1) of Table 1. Table 1 indicates that the approximate increased amounts produced by formula (15) are very close to the exact increased amounts and the value of ${}^{1950}a_{65}$ on the assumption that the approximate increased amounts are payable each year is 11.74445 as compared to the exact value of 11.74417.

Formula (14) is the basic formula in this paper. The only approximation introduced in deriving all of the subsequent formulae is the assumption that formula (14) is exact, *i.e.*, that the increased amounts payable each year may be represented by formula (14).

The third step is to show how supplementary commutation columns may be used to facilitate the calculation of annuity values in which the increased amount payable at the end of the nth year is represented by formula (14). If we substitute the values from formula (14) for each of the terms in parentheses in formula (5), we obtain

$$\left.\begin{aligned}{}^{1950+k}a_x \doteq\ & vp_x[1+kf_x]\\ &+v^2{}_2p_x[1+kf_x+(k+1)f_{x+1}]\\ &+\cdots\cdots\cdots\\ &+v^n{}_np_x[1+kf_x+(k+1)f_{x+1}+\ldots\\ &\qquad\qquad+(k+n-1)f_{x+n-1}]\\ &+\cdots\cdots\cdots\end{aligned}\right\}\quad(16)$$

where each of the terms in brackets represents the approximate increased amount payable at the end of that year due to improved mortality. By rearranging the terms in formula (16), we may rewrite it as follows:

$$\left.\begin{aligned}{}^{1950+k}a_x \doteq\ & vp_x+v^2{}_2p_x+\ldots+v^n{}_np_x+\ldots\\ &+kf_x[vp_x+v^2{}_2p_x+\ldots+v^n{}_np_x+\ldots]\\ &+(k+1)f_{x+1}[v^2{}_2p_x+\ldots+v^n{}_np_x+\ldots]\\ &+\cdots\cdots\cdots\\ &+(k+n-1)f_{x+n-1}[v^n{}_np_x+\ldots]\\ &+\cdots\cdots\cdots\end{aligned}\right\}\quad(17)$$

Substituting D_{x+n}/D_x for $v^n {}_np_x$ for all values of n in (17), we get

$$\left.\begin{aligned}{}^{1950+k}a_x \doteq{}& \frac{1}{D_x}[D_{x+1}+D_{x+2}+\ldots+D_{x+n}+\ldots]\\ &+\frac{k f_x}{D_x}[D_{x+1}+D_{x+2}+\ldots+D_{x+n}+\ldots]\\ &+\frac{(k+1)\,f_{x+1}}{D_x}[D_{x+2}+\ldots+D_{x+n}+\ldots]\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\\ &+\frac{(k+n-1)\,f_{x+n-1}}{D_x}[D_{x+n}+\ldots]\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad(18)$$

Substituting N_{x+n} for $D_{x+n}+\ldots$ for all values of n in (18), we get

$$\left.\begin{aligned}&{}^{1950+k}a_x\\ &\doteq\frac{N_{x+1}+k f_x N_{x+1}+(k+1)\,f_{x+1}N_{x+2}+\ldots+(k+n-1)\,f_{x+n-1}N_{x+n}+\ldots}{D_x}.\end{aligned}\right\}\quad(19)$$

If we let $h_x = f_x N_{x+1}$, we may rewrite formula (19) as follows:

$$\left.\begin{aligned}{}^{1950+k}a_x \doteq{}& \frac{N_{x+1}}{D_x}+\frac{k}{D_x}[h_x+h_{x+1}+\ldots+h_{x+n-1}+\ldots]\\ &+\frac{1}{D_x}[h_{x+1}+2h_{x+2}+\ldots+(n-1)\,h_{x+n-1}+\ldots].\end{aligned}\right\}\quad(20)$$

This formula suggests the particular supplementary commutation columns that would be useful in the calculation of ${}^{1950+k}a_x$. It might first be noted that, as $f_x = s_x q_x/p_x$ and as $s_x = 0$ for ages 90 and over on Projection Scale B, f_x and h_x are also equal to 0 at ages 90 and over. We may, therefore, define the supplementary commutation columns as follows:

$$H_x=\sum_{t=0}^{89-x} h_{x+t}=h_x+h_{x+1}+\ldots+h_{89}\quad(21)$$

and

$$\left.\begin{aligned}J_x={}&\sum_{t=0}^{89-x} t h_{x+t}=h_{x+1}+2h_{x+2}+\ldots+(89-x)\,h_{89}\\ ={}&\sum_{t=1}^{89-x} H_{x+t}=H_{x+1}+H_{x+2}+\ldots+H_{89}.\end{aligned}\right\}\quad(22)$$

Values of H_x and J_x on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest are shown separately for males and females in Appendix II. The other supplementary commutation columns in Appendix II are required for calculating the approximate values of other types of annuity contracts and will be discussed later.

Substituting H_x and J_x for the two series in formula (20), we obtain

$$ {}_{1950+k}a_x \doteq \frac{N_{x+1}}{D_x} + \frac{J_x}{D_x} + k\,\frac{H_x}{D_x}. \tag{23} $$

TABLE 2

COMPARISON OF EXACT AND APPROXIMATE VALUES OF IMMEDIATE NON-REFUND LIFE ANNUITIES ISSUED IN 1950 AND 1960

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—$2\frac{1}{2}\%$ Interest

Age x	Annuities Issued at Age x in 1950				Annuities Issued at Age x in 1960			
	Exact Value of ${}_{1950}a_x$ (1)	Approximate Value of ${}_{1950}a_x$* (2)	Error (2)−(1) (3)	Error (3)÷(1) (4)	Exact Value of ${}_{1960}a_x$ (5)	Approximate Value of ${}_{1960}a_x$† (6)	Error (6)−(5) (7)	Error (7)÷(5) (8)
	Male							
15	30.917	31.018	.101	.33%	31.134	31.298	.164	.53%
25	28.296	28.370	.074	.26	28.574	28.704	.130	.45
35	24.962	25.005	.043	.17	25.307	25.401	.094	.37
45	20.849	20.867	.018	.09	21.263	21.319	.056	.26
55	16.330	16.336	.006	.04	16.759	16.785	.026	.16
65	11.744	11.744	.000	.00	12.092	12.100	.008	.07
75	7.396	7.395	−.001	−.01	7.588	7.590	.002	.03
85	3.927	3.927	.000	.00	3.965	3.965	.000	.00
	Female							
15	31.935	32.032	.097	.30%	32.078	32.229	.151	.47%
25	29.611	29.685	.074	.25	29.797	29.922	.125	.42
35	26.672	26.719	.047	.18	26.906	26.999	.093	.35
45	23.018	23.043	.025	.11	23.299	23.360	.061	.26
55	18.640	18.649	.009	.05	18.943	18.978	.035	.18
65	13.686	13.687	.001	.01	13.963	13.976	.013	.09
75	8.714	8.713	−.001	−.01	8.883	8.886	.003	.03
85	4.564	4.564	.000	.00	4.599	4.599	.000	.00

* Obtained by formula (24).

† Obtained by formula (23), $k = 10$.

Formula (23) represents our basic objective, a relatively simple formula that will produce approximate values of immediate nonrefund life annuities issued at any age x in any year $1950 + k$. By dividing both sides of (23) by a_x we obtain formula (112), which may be used to calculate approximate values of the projection factors presented in the Jenkins-Lew paper.

The formula for a corresponding annuity issued in 1950 may be obtained by letting $k = 0$ in formula (23), so that

$$^{1950}a_x \doteq \frac{N_{x+1}}{D_x} + \frac{J_x}{D_x}. \tag{24}$$

The accuracy of formulae (23) and (24) is demonstrated in Table 2, where the approximate values of $^{1950}a_x$ and $^{1960}a_x$ produced by these formulae are compared for male and female lives with the corresponding exact annuity values. These exact values are the same ones that were used to produce the projection factors in the Jenkins-Lew paper. The maximum error is about $\frac{1}{2}\%$ of the exact annuity value at the very young ages and the errors at the older ages, where most of the annuity business is concentrated, are less than $\frac{1}{10}\%$. The errors will generally increase as the value of k increases, as is indicated by Table 3, which compares the approximate values of an immediate nonrefund life annuity issued at age 65 in decennial years from 1950 to 2000 with the corresponding exact values.

It is apparent from Table 3 that the proposed method will provide a high degree of accuracy for a long period of time. It is likely that by the time the degree of accuracy of this method becomes questionable, new basic tables and new projection scales will have come into use. Corresponding new supplementary commutation columns may then be used to again reduce the errors to negligible proportions.

Up to this point we have been dealing only with annuity values computed on the basis of the ultimate part of the Annuity Table for 1949. It might be considered desirable to adjust these annuity values for the effect of select mortality. For annuities issued in the year $1950 + k$, this adjustment could be made by multiplying the ultimate annuity values by the following approximate factor:

$$\frac{^{1950+k}p_{[x]}}{^{1950+k}p_x} \doteq \frac{p_{[x]} + k\, s_x q_{[x]}}{p_x + k\, s_x q_x}. \tag{25}$$

Before taking up the application of this method for valuation purposes, it seems worth while to consider the meaning of the three separate terms in formula (23) as all of the subsequent formulae may be analyzed in the same manner. The first term N_{x+1}/D_x is equal to a_x and represents the exact value of the annuity we are considering on the basis of the stationary mortality table, *i.e.*, the Annuity Table for 1949 (ultimate) without projection. The second term J_x/D_x, which we may designate by ${}^{i}a_x$, represents the approximate increase in the value of an annuity issued in the

TABLE 3

COMPARISON OF EXACT AND APPROXIMATE VALUES OF IMMEDIATE NONREFUND LIFE ANNUITIES ISSUED AT AGE 65 IN DECENNIAL YEARS FROM 1950 TO 2000

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—2½% Interest

Issue Year 1950+k	Value of k	Annuities Issued at Age 65 in 1950+k			
		Exact Value of ${}^{1950+k}a_{65}$ (1)	Approximate Value of ${}^{1950+k}a_{65}$* (2)	Error: (2)−(1) (3)	Error: (3)÷(1) (4)
Males:					
1950......	0	11.744	11.744	.000	.00%
1960......	10	12.092	12.100	.008	.07
1970......	20	12.425	12.456	.031	.25
1980......	30	12.745	12.812	.067	.53
1990......	40	13.051	13.168	.117	.90
2000......	50	13.343	13.524	.181	1.36
Females:					
1950......	0	13.686	13.687	.001	.01%
1960......	10	13.963	13.976	.013	.09
1970......	20	14.227	14.264	.037	.26
1980......	30	14.477	14.552	.075	.52
1990......	40	14.715	14.840	.125	.85
2000......	50	14.939	15.128	.189	1.27

* Obtained by formula (23).

year represented by the stationary mortality table (1950) because of the future improvements in mortality assumed in the projection scale, so that

$$ {}^{1950}a_x \doteq a_x + {}^{i}a_x \,. \tag{26} $$

In the third term, the value of H_x/D_x, which we may designate by ${}^{\Delta}a_x$, represents the approximate annual increment in the annuity value that results from shifting all of the annuity payments forward by one year, so that

$$ {}^{1950+k}a_x \doteq a_x + {}^{i}a_x + k\,{}^{\Delta}a_x \tag{27} $$

and

$$ {}^{1950+k}a_x \doteq {}^{1950}a_x + k\,{}^{\Delta}a_x \,. \tag{28} $$

Formulae (26), (27), and (28) are perfectly general and will be shown later to apply to any type of annuity or insurance benefit. This means, for example, that we can replace a_x in these formulae by $_{n|}a_x$ or A_x or any similar symbol and the formulae will still hold.

Formula (28) may be used for valuing all annuities on which an immediate nonrefund life annuity is the only benefit payable at the time of valuation. All of these annuities would first have to be classified separately by sex and then by attained age. Two valuation factors would be required for each attained age, namely

$$(A): \quad {}^{1950}a_x \doteq \frac{N_{x+1}+J_x}{D_x} \tag{29}$$

and

$$(B): \quad {}^{\Delta}a_x = \frac{H_x}{D_x}. \tag{30}$$

For a valuation in the year $1950 + k$, we may let ${}^{1950+k}T_x$ represent the amount of annual income in force at attained age x, and then

$$\sum_x {}^{1950+k}T_x\,({}^{1950+k}a_x) \doteq \sum_x {}^{1950+k}T_x\,({}^{1950}a_x) + k\sum_x {}^{1950+k}T_x\,({}^{\Delta}a_x)\,. \tag{31}$$

Formula (31) indicates that the same valuation factors may be used each year even though the annuities are valued on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B. The adjustment for improving mortality would only have to be made in the aggregate, by using the appropriate value of k. For example, the value of k would be 5 for a valuation in 1955 or 10 for a valuation in 1960, but the same valuation factors ${}^{1950}a_x$ and ${}^{\Delta}a_x$ would be used in each of those years.

In actual practice, where the attained age x generally denotes the age attained on the contract anniversary and the valuation is performed at the end of the calendar year, the valuation factors could be adjusted to a mean reserve basis as follows:

$$\text{Valuation Factor (A)} = \tfrac{1}{2}\,({}^{1950}a_x) + \tfrac{1}{2}\,(1 + {}^{1950}a_{x+1}) \tag{32}$$

and

$$\text{Valuation Factor (B)} = \tfrac{1}{2}\,({}^{\Delta}a_x) + \tfrac{1}{2}\,({}^{\Delta}a_{x+1})\,. \tag{33}$$

If it were considered desirable to assume continuous improvement in mortality, $k + \frac{1}{2}$ could be substituted for k in formula (31) for a valuation at the end of the calendar year $1950 + k$. In that case the aggregate annual increment would be multiplied by $k = \frac{1}{2}$ for a valuation at the end of 1950 and by $k = 10\frac{1}{2}$ for a valuation at the end of 1960.

III. STANDARD AND SUPPLEMENTARY COMMUTATION COLUMNS

The formulae for other types of annuities may be derived from the basic formula (14) by following a similar procedure to that used in Section II for immediate nonrefund life annuities. It is more convenient, however, to obtain the formulae for other types of annuities by using some general rules for expressing the standard commutation columns in terms of the supplementary commutation columns. These general rules will be explained and illustrated in this section.

The standard commutation columns may be considered to represent the value of a particular benefit on the basis of the Annuity Table for 1949 (ultimate), without projection, and $2\frac{1}{2}\%$ interest. We may designate the value of this same benefit on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest by placing parentheses around the expression in standard commutation columns and indicating the calendar year in which the benefit is issued by a superscript in the upper left-hand corner. Thus, we may express formula (23) as follows:

$$ {}^{1950+k}a_x = {}^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1}}{D_x} + \frac{J_x}{D_x} + k\,\frac{H_x}{D_x}. \tag{34} $$

The superscript $1950 + k$ corresponds to the calendar year in which age x is attained.

Similarly, we may express the general formula (27) as follows:

$$ {}^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1}}{D_x} + {}^{i}\left(\frac{N_{x+1}}{D_x}\right) + k\,{}^{\Delta}\left(\frac{N_{x+1}}{D_x}\right). \tag{35} $$

The basic formula (14) may be expressed as follows:

$$ \left.\begin{aligned} &{}^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) \doteq \frac{D_{x+n}}{D_x} \\ &\quad \times [1 + k f_x + (k+1) f_{x+1} + \dots + (k+n-1) f_{x+n-1}]\,. \end{aligned}\right\} \tag{36} $$

This formula states that for a life aged x in the year $1950 + k$, the value of a pure endowment of \$1 at the end of n years computed on the Annuity Table for 1949 with Projection Scale B is approximately equal to the value of a pure endowment providing the increased amount in brackets at the end of n years, computed on the Annuity Table for 1949 without projection.

The value of the increased amount that is defined by the expression in brackets in formula (36) may be obtained more conveniently by using two additional supplementary commutation columns, F_x and G_x. These are defined as follows:

$$F_x = \sum_{t=0}^{89-x} f_{x+t} = f_x + f_{x+1} + \ldots + f_{89} \tag{37}$$

$$\left.\begin{aligned} G_x &= \sum_{t=0}^{89-x} t f_{x+t} = f_{x+1} + 2 f_{x+2} + \ldots + (89-x) f_{89} \\ &= \sum_{t=1}^{89-x} F_{x+t} = F_{x+1} + F_{x+2} + \ldots + F_{89}\,. \end{aligned}\right\} \tag{38}$$

Values of F_x and G_x on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B are shown separately for males and females in Appendix II.

Noting that

$$F_x - F_{x+n} = f_x + f_{x+1} + \ldots + f_{x+n-1} \tag{39}$$

and

$$G_x - G_{x+n} - nF_{x+n} = f_{x+1} + 2 f_{x+2} + \ldots + (n-1) f_{x+n-1} \tag{40}$$

we may write formula (36) as follows:

$$^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) \doteq \frac{D_{x+n}}{D_x} [1 + G_x - G_{x+n} - nF_{x+n} + k (F_x - F_{x+n})]\,. \tag{41}$$

The use of the supplementary commutation columns F_x and G_x may be illustrated by referring to formula (15), which was used to calculate the values shown in column (2) of Table 1. Any value in column (2) could be calculated more easily by expressing formula (15) as follows:

$$\frac{^{1950}{}_n p_{65}}{{}_n p_{65}} \doteq 1 + G_{65} - G_{65+n} - nF_{65+n}\,. \tag{42}$$

As the expression in brackets in formula (41) will be used quite often, it will be convenient to introduce the following symbol:

$$^{1950+k}{}_n I_x = G_x - G_{x+n} - nF_{x+n} + k (F_x - F_{x+n}) \tag{43}$$

so that formula (41) may be expressed as follows:

$$^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) \doteq \frac{D_{x+n}}{D_x} (1 + {}^{1950+k}{}_n I_x)\,. \tag{44}$$

$^{1950+k}{}_n I_x$ represents the approximate additional amount that should be paid in order to reflect the effect of taking account of improving mortality.

The general rules for manipulating these commutation columns may be illustrated by a specific example. Let us consider the problem of obtaining the approximate formula for

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right),$$

the value of a deferred nonrefund life annuity.

One way of obtaining the required formula is to take advantage of formulae (34) and (44) and split

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right)$$

into two factors as follows:

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right)= {}^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) {}^{1950+k+n}\left(\frac{N_{x+n+1}}{D_{x+n}}\right). \tag{45}$$

Note that the year of birth, *i.e.*, the difference between the superscript and the attained age in the denominator, should be the same for each factor.

The approximate formula for

$$^{1950+k}\left(\frac{D_{x+n}}{D_x}\right)$$

is given by (44).

By using formula (34), we may write

$$^{1950+k+n}\left(\frac{N_{x+n+1}}{D_{x+n}}\right) \doteq \frac{N_{x+n+1}+J_{x+n}+(k+n)H_{x+n}}{D_{x+n}}. \tag{46}$$

In multiplying formula (44) by formula (46), it should be noted that the product of any two supplementary commutation symbols should always be set equal to 0. The reason for this rule is that each term in each supplementary commutation symbol involves s_x and each term in the product of two such symbols would involve s_x^2. As the basic formula (14) is based on the assumption that the second and higher powers of s_x will be ignored, we should ignore any products of two supplementary commutation symbols in order to get consistent results. The product of formulae (44) and (46) may, therefore, be expressed as follows:

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right) \doteq \frac{N_{x+n+1}(1+{}^{1950+k}_{\ \ n}I_x)+J_{x+n}+(k+n)H_{x+n}}{D_x}. \tag{47}$$

A question might be raised as to what formula would have been produced if we had started with either of the following relationships.

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right)= {}^{1950+k}\left(\frac{D_{x+n+1}}{D_x}\right) {}^{1950+k+n+1}\left(\frac{N_{x+n+1}}{D_{x+n+1}}\right) \tag{48}$$

or

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right) = {}^{1950+k}\left(\frac{D_{x+n+1}}{D_x}\right) + {}^{1950+k}\left(\frac{D_{x+n+2}}{D_x}\right) + \dots \qquad (49)$$

The answer is that while the approximate formulae that would result from (48) or (49) might at first glance appear to differ from formula (47), they would produce exactly the same numerical results. By a little manipulation of the supplementary commutation symbols, it can be proved that the formulae resulting from (48) or (49) are identical with formula (47).

In order to show how two formulae that appear to differ from each other may be proved to be identical and at the same time illustrate the use of some of the more important relationships between the supplementary commutation columns, let us consider formula (34) again.

$$^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1}}{D_x} + \frac{J_x}{D_x} + k\,\frac{H_x}{D_x}. \qquad (34) \text{ repeated}$$

By adding 1 to each side of formula (34), we get

$$^{1950+k}\left(\frac{N_x}{D_x}\right) \doteq \frac{N_x}{D_x} + \frac{J_x}{D_x} + k\,\frac{H_x}{D_x}. \qquad (50)$$

Substituting $x + 1$ for x and $k + 1$ for k in formula (50), we get

$$^{1950+k+1}\left(\frac{N_{x+1}}{D_{x+1}}\right) \doteq \frac{N_{x+1}}{D_{x+1}} + \frac{J_{x+1}}{D_{x+1}} + (k+1)\,\frac{H_{x+1}}{D_{x+1}}. \qquad (51)$$

Letting n equal 1 in formula (41), we get

$$^{1950+k}\left(\frac{D_{x+1}}{D_x}\right) \doteq \frac{D_{x+1}}{D_x}\left[1 + G_x - G_{x+1} - F_{x+1} + k\,(F_x - F_{x+1})\right]. \qquad (52)$$

As

$$^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) = {}^{1950+k}\left(\frac{D_{x+1}}{D_x}\right)\,{}^{1950+k+1}\left(\frac{N_{x+1}}{D_{x+1}}\right)$$

we may multiply the right-hand sides of formulae (51) and (52), again ignoring any products of two supplementary commutation symbols, and write

$$\left.\begin{aligned} &^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \\ &\doteq \frac{N_{x+1}\left[1+G_x-G_{x+1}-F_{x+1}+k\,(F_x-F_{x+1})\right]+J_{x+1}+(k+1)\,H_{x+1}}{D_x}. \end{aligned}\right\} \qquad (53)$$

Thus we have formulae (53) and (34) both representing the approximate value of $^{1950+k}(N_{x+1}/D_x)$.

In order to show that they will both produce the same results, we may transform the right-hand side of (53) into the right-hand side of (34) by using the following relationships:

$$G_x = G_{x+1} + F_{x+1} \qquad \text{from (38)}$$

$$f_x = F_x - F_{x+1} \qquad \text{from (37)}$$

$$J_x = J_{x+1} + H_{x+1} \qquad \text{from (22)}$$

This permits us to write formula (53) as follows:

$$^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1}(1+kf_x) + J_x + kH_{x+1}}{D_x}. \tag{54}$$

As

$$h_x = f_x N_{x+1}, \qquad ^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1} + kh_x + J_x + kH_{x+1}}{D_x}. \tag{55}$$

As

$$H_x = h_x + H_{x+1}, \qquad ^{1950+k}\left(\frac{N_{x+1}}{D_x}\right) \doteq \frac{N_{x+1} + J_x + kH_x}{D_x}$$

and we have derived formula (34) from formula (53).

Up to this point, we have considered only the commutation columns D_x and N_x. The commutation columns C_x and M_x may also be expressed in terms of the same supplementary commutation columns F_x, G_x, H_x, J_x.

In order to get the approximate formula for $^{1950+k}(C_x/D_x)$, we may use the relationship:

$$C_x = vD_x - D_{x+1}$$

or

$$\frac{C_x}{D_x} = v - \frac{D_{x+1}}{D_x}$$

so that

$$^{1950+k}\left(\frac{C_x}{D_x}\right) = v - {}^{1950+k}\left(\frac{D_{x+1}}{D_x}\right). \tag{56}$$

From (52),

$$^{1950+k}\left(\frac{D_{x+1}}{D_x}\right) \doteq \frac{D_{x+1}}{D_x}[1 + k(F_x - F_{x+1})] \tag{57}$$

since

$$G_x - G_{x+1} - F_{x+1} = 0\,.$$

Consequently,

$$^{1950+k}\left(\frac{C_x}{D_x}\right) \doteq v - \frac{D_{x+1}}{D_x} - k\frac{D_{x+1}}{D_x}(F_x - F_{x+1}) \tag{58}$$

or

$$^{1950+k}\left(\frac{C_x}{D_x}\right) \doteq \frac{C_x}{D_x} - k\frac{D_{x+1}}{D_x}(F_x - F_{x+1})\,. \tag{59}$$

In order to get the formula for ${}^{1950+k}(C_{x+n}/D_x)$, we would use the following relationship:

$$ {}^{1950+k}\left(\frac{C_{x+n}}{D_x}\right)= {}^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) {}^{1950+k+n}\left(\frac{C_{x+n}}{D_{x+n}}\right). \qquad (60) $$

By using formula (44) and formula (59) and following the rules described in deriving (47), we get

$$ \left.\begin{aligned} &{}^{1950+k}\left(\frac{C_{x+n}}{D_x}\right) \\ &\doteq \frac{C_{x+n}\,(1+{}^{1950+k}_{\quad n}I_x) - (k+n)\,D_{x+n+1}\,(F_{x+n}-F_{x+n+1})}{D_x}. \end{aligned}\right\} \qquad (61) $$

This same procedure is applicable to all of the other commutation symbols that will be discussed below.

To get the approximate formula for ${}^{1950+k}(M_x/D_x)$, we may use the relationship

$$ \frac{M_x}{D_x}=1-d\,\frac{N_x}{D_x}. $$

We may, therefore, write

$$ {}^{1950+k}\left(\frac{M_x}{D_x}\right)=1-d\left[{}^{1950+k}\left(\frac{N_x}{D_x}\right)\right]. \qquad (62) $$

Using formula (50), we may write

$$ {}^{1950+k}\left(\frac{M_x}{D_x}\right)\doteq 1-d\,\frac{N_x}{D_x}-d\,\frac{J_x}{D_x}-dk\,\frac{H_x}{D_x} \qquad (63) $$

$$ \doteq \frac{M_x-d\,(J_x+kH_x)}{D_x}. \qquad (64) $$

The value of ${}^{1950+k}(M_{x+n}/D_x)$ may be obtained by using formulae (44) and (64) and following the same procedure indicated for obtaining ${}^{1950+k}(C_{x+n}/D_{x+n})$. The formula would be

$$ {}^{1950+k}\left(\frac{M_{x+n}}{D_x}\right)\doteq \frac{M_{x+n}\,(1+{}^{1950+k}_{\quad n}I_x)-d\,[J_{x+n}+(k+n)\,H_{x+n}]}{D_x}. \qquad (65) $$

We may break this formula into its component parts following the general formula (27) as follows

$$ {}^{1950+k}\left(\frac{M_{x+n}}{D_x}\right)\doteq \frac{M_{x+n}}{D_x}+{}^{i}\left(\frac{M_{x+n}}{D_x}\right)+k\,{}^{\Delta}\left(\frac{M_{x+n}}{D_x}\right). \qquad (66) $$

Using (39), (40) and (43), we get

$$\left.\begin{aligned}&{}^{i}\!\left(\frac{M_{x+n}}{D_x}\right)\\&=\frac{M_{x+n}[f_{x+1}+2f_{x+2}+\ldots+(n-1)f_{x+n-1}]-d(J_{x+n}+nH_{x+n})}{D_x}\end{aligned}\right\}\quad(67)$$

and

$${}^{\Delta}\!\left(\frac{M_{x+n}}{D_x}\right)=\frac{M_{x+n}[f_x+f_{x+1}+\ldots+f_{x+n-1}]-dH_{x+n}}{D_x}.\quad(68)$$

For cash refund annuities, we also need formulae that express R_x in terms of supplementary commutation columns. To derive these formulae, we start with

$${}^{1950+k}\!\left(\frac{R_x}{D_x}\right)={}^{1950+k}\!\left(\frac{M_x}{D_x}\right)+{}^{1950+k}\!\left(\frac{M_{x+1}}{D_x}\right)+\ldots+{}^{1950+k}\!\left(\frac{M_{x+n}}{D_x}\right)+\ldots\quad(69)$$

and the general formula

$${}^{1950+k}\!\left(\frac{R_x}{D_x}\right)\doteq\frac{R_x}{D_x}+{}^{i}\!\left(\frac{R_x}{D_x}\right)+k\,{}^{\Delta}\!\left(\frac{R_x}{D_x}\right)\quad(70)$$

where

$${}^{i}\!\left(\frac{R_x}{D_x}\right)={}^{i}\!\left(\frac{M_x}{D_x}\right)+{}^{i}\!\left(\frac{M_{x+1}}{D_x}\right)+\ldots+{}^{i}\!\left(\frac{M_{x+n}}{D_x}\right)+\ldots\quad(71)$$

and

$${}^{\Delta}\!\left(\frac{R_x}{D_x}\right)={}^{\Delta}\!\left(\frac{M_x}{D_x}\right)+{}^{\Delta}\!\left(\frac{M_{x+1}}{D_x}\right)+\ldots+{}^{\Delta}\!\left(\frac{M_{x+n}}{D_x}\right)+\ldots.\quad(72)$$

Using formula (67), we may write (71) as follows:

$${}^{i}\!\left(\frac{R_x}{D_x}\right)=\frac{1}{D_x}\left\{\begin{array}{ll}M_x[0] & -dJ_x\\ +M_{x+1}[0] & -dJ_{x+1}-dH_{x+1}\\ +M_{x+2}[f_{x+1}] & -dJ_{x+2}-2dH_{x+2}\\ +M_{x+3}[f_{x+1}+2f_{x+2}] & -dJ_{x+3}-3dH_{x+3}\\ +\ldots\ldots\ldots & \ldots\ldots\\ +M_{x+n}[f_{x+1}+2f_{x+2}+\ldots+(n-1)f_{x+n-1}] & \\ & -dJ_{x+n}-ndH_{x+n}\\ +\ldots\ldots\ldots & \ldots\ldots\end{array}\right\}\quad(73)$$

Formula (73) may be summarized as follows:

$$\left.\begin{aligned}{}^{i}\!\left(\frac{R_x}{D_x}\right)=&\frac{1}{D_x}[f_{x+1}R_{x+2}+2f_{x+2}R_{x+3}+\ldots\\&+(n-1)f_{x+n-1}R_{x+n}+\ldots]\\&-\frac{2d}{D_x}[J_x+J_{x+1}+\ldots+J_{x+n}+\ldots]\end{aligned}\right\}\quad(74)$$

since

$$H_{x+1}+2H_{x+2}+\ldots+nH_{x+n}+\ldots=J_x+J_{x+1}+\ldots+J_{x+n}+\ldots.$$

Using formula (68) we may write (72) as follows:

$$\Delta\left(\frac{R_x}{D_x}\right)=\frac{1}{D_x}\left\{\begin{array}{ll} M_x(0) & -dH_x \\ +M_{x+1}(f_x) & -dH_{x+1} \\ +M_{x+2}(f_x+f_{x+1}) & -dH_{x+2} \\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ +M_{x+n}(f_x+f_{x+1}+\ldots+f_{x+n-1}) & -dH_{x+n} \\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}\right\} \quad (75)$$

Formula (75) may be summarized as follows, using (22),

$$\left.\begin{array}{l} \Delta\left(\dfrac{R_x}{D_x}\right)=\dfrac{1}{D_x}[f_xR_{x+1}+f_{x+1}R_{x+2}+\ldots+f_{x+n-1}R_{x+n}+\ldots] \\ \qquad\qquad\qquad\qquad -\dfrac{d}{D_x}[H_x+J_x]. \end{array}\right\} \quad (76)$$

From (74) and (76), it is apparent that we need three additional supplementary commutation columns to evaluate ${}^{i}(R_x/D_x)$ and ${}^{\Delta}(R_x/D_x)$. These may be defined as follows, if we let $y_x = f_xR_{x+1}$.

$$\left.\begin{array}{l} K_x=\sum_{t=0}^{89-x}J_{x+t}=J_x+J_{x+1}+\ldots+J_{89} \\ \quad=\sum_{t=0}^{89-x}tH_{x+t}=H_{x+1}+2H_{x+2}+\ldots+(89-x)H_{89} \\ \quad=\sum_{t=0}^{89-x}\dfrac{t(t+1)}{2}h_{x+t}=h_{x+1}+3h_{x+2}+\ldots \\ \qquad\qquad\qquad +\dfrac{(89-x)(90-x)}{2}h_{89}. \end{array}\right\} \quad (77)$$

$$Y_x=\sum_{t=0}^{89-x}y_{x+t}=y_x+y_{x+1}+\ldots+y_{89}. \quad (78)$$

$$\left.\begin{array}{l} Z_x=\sum_{t=0}^{89-x}ty_{x+t}=y_{x+1}+2y_{x+2}+\ldots+(89-x)y_{89} \\ \quad=\sum_{t=1}^{89-x}Y_{x+t}=Y_{x+1}+Y_{x+2}+\ldots+Y_{89}. \end{array}\right\} \quad (79)$$

Values of K_x, Y_x, and Z_x on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest are shown in Appendix II, separately for males and females. Using these supplementary commutation columns, we may express formulae (74) and (76) as follows:

$$^{i}\left(\frac{R_x}{D_x}\right)=\frac{Z_x-2\,dK_x}{D_x} \tag{80}$$

$$^{\Delta}\left(\frac{R_x}{D_x}\right)=\frac{Y_x-d\,(H_x+J_x)}{D_x}. \tag{81}$$

Using formula (70), we may now write:

$$^{1950+k}\left(\frac{R_x}{D_x}\right)\doteq\frac{R_x+Z_x-2\,dK_x+k\,[Y_x-d\,(H_x+J_x)]}{D_x}. \tag{82}$$

The value of $^{1950+k}(R_{x+n}/D_x)$ may be obtained by using formulae (44) and (82) and following the same procedure indicated for obtaining formula (61).

The formula for $^{1950+k}(S_x/D_x)$ may be obtained by using the relationship:

$$R_x=vS_x-S_{x+1}=N_x-dS_x$$

so that

$$S_x=\frac{1}{d}\,(N_x-R_x)\,.$$

Consequently,

$$^{1950+k}\left(\frac{S_x}{D_x}\right)=\frac{1}{d}\left[{}^{1950+k}\left(\frac{N_x}{D_x}\right)-{}^{1950+k}\left(\frac{R_x}{D_x}\right)\right] \tag{83}$$

and from (82) and (50), we get

$$^{1950+k}\left(\frac{S_x}{D_x}\right)\doteq\frac{S_x+2K_x+\frac{1}{d}\,(J_x-Z_x)+k\left[H_x+J_x+\frac{1}{d}\,(H_x-Y_x)\right]}{D_x}. \tag{84}$$

A convenient summary of the principal formulae that were developed in this section is presented in Appendix III. By using these formulae, any benefit expressible in terms of standard commutation symbols may be expressed in terms of both standard and supplementary commutation symbols so that the approximate value of the benefit may be obtained on a mortality basis that provides for future improvements in mortality.

IV. DEFERRED NONREFUND LIFE ANNUITIES

The value of a nonrefund life annuity, deferred n years, issued at age x may be expressed in terms of standard commutation symbols as:

$$_{n|}a_x=\frac{N_{x+n+1}}{D_x}.$$

The exact value of this deferred annuity on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest may be designated as:

$$^{1950+k}{}_{n|}a_x = {}^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right)$$

where $1950 + k$ represents the year in which the deferred annuity is issued.

By using the general formula (27) and formula (47), the approximate value of this n-year deferred annuity issued at age x in the year $1950 + k$ may be expressed as follows:

$$^{1950+k}\left(\frac{N_{x+n+1}}{D_x}\right) \doteq \frac{N_{x+n+1}}{D_x} + {}^{i}\left(\frac{N_{x+n+1}}{D_x}\right) + k\,{}^{\Delta}\left(\frac{N_{x+n+1}}{D_x}\right) \qquad (85)$$

where

$$^{i}\left(\frac{N_{x+n+1}}{D_x}\right) = \frac{N_{x+n+1}(G_x - G_{x+n} - nF_{x+n}) + J_{x+n} + nH_{x+n}}{D_x} \qquad (86)$$

and

$$^{\Delta}\left(\frac{N_{x+n+1}}{D_x}\right) = \frac{N_{x+n+1}(F_x - F_{x+n}) + H_{x+n}}{D_x}. \qquad (87)$$

As indicated previously, (86) indicates the part of the annuity value that provides for future improvements in mortality on a contract issued in 1950, while (87) represents the approximate annual increment in the annuity value that takes account of the fact that all payments are shifted forward one year when the contract is issued in 1951 instead of 1950. Consequently, the formula for an n-year deferred annuity issued at age x in 1950 would simply be

$$^{1950}\left(\frac{N_{x+n+1}}{D_x}\right) \doteq \frac{N_{x+n+1}(1+G_x-G_{x+n}-nF_{x+n}) + J_{x+n} + nH_{x+n}}{D_x}. \qquad (88)$$

The accuracy of formulae (85) and (88) for a nonrefund deferred life annuity with the first payment due at age 66 is tested in Table 4 for annuities issued in 1950 and 1960. While the errors produced by these approximate formulae are a little larger than those produced in the case of immediate life annuities, they do not exceed 1% of the exact annuity value except at the very young ages where a long deferred period is involved.

After the deferred period, these annuities may be valued together with the immediate nonrefund life annuities as indicated in Section II. During the deferred period, we have a choice of two possible valuation procedures. The same valuation factors may be used each year under either of these two valuation procedures.

Under the first procedure, the deferred annuities for males and females would each have to be classified by attained age (x) and by the number of years before the annuity is entered upon (n). Two valuation factors would be required for each combination of x and n, namely:

$$(A): \quad {}^{1950}\left(\frac{N_{x+n+1}}{D_x}\right) \text{ from } (88)$$

and

$$(B): \quad {}^{\Delta}\left(\frac{N_{x+n+1}}{D_x}\right) \text{ from } (87).$$

TABLE 4

COMPARISON OF EXACT AND APPROXIMATE VALUES OF NONREFUND LIFE ANNUITIES DEFERRED TO AGE 65

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—$2\frac{1}{2}$% Interest

Issue Age x	Years Deferred $65-x=n$	Immediate Annuity Entered Upon in	Exact Value (1)	Approximate Value (2)	Error (2)−(1) (3)	Error (3)÷(1) (4)
	Deferred Annuities Issued in 1950—${}^{1950}_{\ \ n|}a_x$*					
Male:						
25	40	1990	4.078	4.117	.039	.96%
35	30	1980	5.036	5.061	.025	.50
45	20	1970	6.271	6.284	.013	.21
55	10	1960	8.160	8.164	.004	.05
Female:						
25	40	1990	5.013	5.068	.055	1.10%
35	30	1980	6.286	6.324	.038	.60
45	20	1970	7.925	7.947	.022	.28
55	10	1960	10.165	10.174	.009	.09
	Deferred Annuities Issued in 1960—${}^{1960}_{\ \ n|}a_x$†					
Male:						
25	40	2000	4.254	4.321	.067	1.57%
35	30	1990	5.269	5.320	.051	.97
45	20	1980	6.573	6.607	.034	.52
55	10	1970	8.524	8.544	.020	.23
Female:						
25	40	2000	5.142	5.229	.087	1.69%
35	30	1990	6.458	6.528	.070	1.08
45	20	1980	8.149	8.199	.050	.61
55	10	1970	10.439	10.469	.030	.29

* Approximate values obtained by formula (88).

† Approximate values obtained by formula (85), with $k = 10$.

The aggregate reserve in the year $1950 + k$ would be the aggregate of valuation factor (A) $+ k$ times the aggregate of valuation factor (B).

In actual practice, for a valuation at the end of calendar year $1950 + k$, further adjustments similar to those described for immediate life annuities in Section II [see (32) and (33)] might be made. This would amount to replacing x by $x + \frac{1}{2}$ and n by $n - \frac{1}{2}$ in formulae (87) and (88) and replacing k by $k + \frac{1}{2}$ as the multiple for the aggregate of valuation factor (B). Further approximations might, of course, be introduced in order to reduce the number of classifications, such as using central values of x or n for corresponding groups of values of x or n.

The second valuation procedure for deferred annuities might be more desirable in some cases as the deferred annuities for each sex would have to be classified only by attained age (x). Two valuation constants could be punched on the valuation card at the time of issue. Valuation constant (a) would be the amount of annual income multiplied by N_{x+n+1}, where $x + n$ is the age at which the immediate life annuity is entered upon. Valuation constant (b) would be the amount of annual income multiplied by

$$N_{x+n+1}\,[1 - G_{x+n} - (k+n)\,F_{x+n}] + J_{x+n} + (k+n)\,H_{x+n}$$

where $1950 + k + n$ is the calendar year in which age $x + n$ is attained. Note that while x, k, and n all vary with the duration of the contract, the values of $x + n$ and $k + n$ are fixed at the time of issue and remain constant until the immediate life annuity is entered upon.

In this case three valuation factors, based only on attained age x, would be required. These valuation factors, which could be used year after year, are

$$\frac{1}{D_x}, \quad \frac{G_x}{D_x}, \quad \text{and} \quad \frac{F_x}{D_x}.$$

The aggregate reserve in the year $1950 + k$ would be

$$\sum_x \text{(b)}\frac{1}{D_x} + \sum_x \text{(a)}\frac{G_x}{D_x} + k\sum_x \text{(a)}\frac{F_x}{D_x}. \qquad (89)$$

In actual practice, we may again replace x by $x + \frac{1}{2}$ in the three valuation factors and k by $k + \frac{1}{2}$ in formula (89) when the valuation is performed at the end of the year $1950 + k$.

Both valuation procedures will produce exactly the same reserves, as may be seen from the following equality:

$$\frac{N_{x+n+1}(1+G_x-G_{x+n}-nF_{x+n})+J_{x+n}+nH_{x+n}}{D_x}+k\,\frac{N_{x+n+1}(F_x-F_{x+n})+H_{x+n}}{D_x}$$

$$=\frac{N_{x+n+1}[1-G_{x+n}-(k+n)F_{x+n}]+J_{x+n}+(k+n)H_{x+n}}{D_x}+\frac{N_{x+n+1}G_x}{D_x}+k\,\frac{N_{x+n+1}F_x}{D_x}.$$

It might be noted at this point that the formulae for an annuity due, deferred n years, might be expressed as follows:

$$_{n|}\ddot{a}_x=\frac{N_{x+n}}{D_x}.$$

$$^{1950+k}\!\left(\frac{N_{x+n}}{D_x}\right)\doteq\frac{N_{x+n}}{D_x}+{}^{i}\!\left(\frac{N_{x+n}}{D_x}\right)+k\,{}^{\Delta}\!\left(\frac{N_{x+n}}{D_x}\right) \tag{90}$$

where

$$^{i}\!\left(\frac{N_{x+n}}{D_x}\right)=\frac{N_{x+n}(G_x-G_{x+n}-nF_{x+n})+J_{x+n}+nH_{x+n}}{D_x} \tag{91}$$

and

$$^{\Delta}\!\left(\frac{N_{x+n}}{D_x}\right)=\frac{N_{x+n}(F_x-F_{x+n})+H_{x+n}}{D_x}. \tag{92}$$

The above formulae may be obtained by using the following relationship

$$^{1950+k}\!\left(\frac{N_{x+n}}{D_x}\right)={}^{1950+k}\!\left(\frac{N_{x+n+1}}{D_x}\right)+{}^{1950+k}\!\left(\frac{D_{x+n}}{D_x}\right).$$

The apparent inconsistency between formulae (90), (91), and (92) and the ones that would be obtained if $n-1$ were substituted for n in formulae (85), (86), and (87) may be explained by verifying the following equalities:

$$N_{x+n}[G_x-G_{x+n-1}-(n-1)F_{x+n-1}]+J_{x+n-1}+(n-1)H_{x+n-1}$$
$$=N_{x+n}(G_x-G_{x+n}-nF_{x+n})+J_{x+n}+nH_{x+n}$$

and

$$N_{x+n}(F_x-F_{x+n-1})+H_{x+n-1}=N_{x+n}(F_x-F_{x+n})+H_{x+n}.$$

V. LIFE ANNUITIES GUARANTEEING PAYMENTS FOR A CERTAIN PERIOD

An immediate life annuity guaranteeing payments for an n-year certain period may be considered as made up of two parts, an n-year annuity certain and a nonrefund life annuity, deferred n years. The value of this benefit in terms of standard commutation symbols would be

$$a_{\overline{n}|} + {}_{n|}a_x = a_{\overline{n}|} + \frac{N_{x+n+1}}{D_x}.$$

The value of the annuity certain part of the benefit does not depend on mortality and would, therefore, not be influenced by any improvements in mortality. As we have already considered deferred nonrefund life an-

TABLE 5

COMPARISON OF EXACT AND APPROXIMATE VALUES OF IMMEDIATE ANNUITIES WITH 10-YEAR CERTAIN PERIOD ISSUED IN 1950 AND 1960

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—$2\frac{1}{2}$% Interest

Age x	Annuities Issued at Age x in 1950: Exact Value of $a_{\overline{10}|} + {}^{1950}_{10|}a_x$ (1)	Approximate Value of $a_{\overline{10}|} + {}^{1950}_{10|}a_x$* (2)	Error: (2)−(1) (3)	Error: (3)÷(1) (4)	Annuities Issued at Age x in 1960: Exact Value of $a_{\overline{10}|} + {}^{1960}_{10|}a_x$ (5)	Approximate Value of $a_{\overline{10}|} + {}^{1960}_{10|}a_x$† (6)	Error: (6)−(5) (7)	Error: (7)÷(5) (8)
	Male							
15	30.944	31.044	.100	.32%	31.157	31.320	.163	.52%
25	28.337	28.410	.073	.26	28.610	28.739	.129	.45
35	25.042	25.084	.042	.17	25.378	25.470	.092	.36
45	21.082	21.099	.017	.08	21.469	21.521	.052	.24
55	16.912	16.916	.004	.02	17.276	17.296	.020	.12
65	12.979	12.979	.000	.00	13.219	13.220	.001	.01
75	10.055	10.055	.000	.00	10.125	10.124	−.001	−.01
85	8.882	8.881	−.001	−.01	8.883	8.883	.000	.00
	Female							
15	31.949	32.047	.098	.31%	32.093	32.243	.150	.47%
25	29.639	29.712	.073	.25	29.822	29.946	.124	.42
35	26.725	26.772	.047	.18	26.952	27.045	.093	.35
45	23.133	23.159	.026	.11	23.401	23.461	.060	.26
55	18.917	18.926	.009	.05	19.191	19.221	.030	.16
65	14.427	14.428	.001	.01	14.636	14.645	.009	.06
75	10.691	10.691	.000	.00	10.765	10.765	.000	.00
85	8.961	8.961	.000	.00	8.963	8.963	.000	.00

* Obtained by formula (93), $k = 0$.

† Obtained by formula (93), $k = 10$.

nuities in Section IV, it is a simple matter to obtain the approximate value of a life annuity with an n-year certain period on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest.

Thus, the approximate value of an immediate life annuity with a 10-year certain period issued to a life aged x in the year $1950+k$ could be obtained from (85), (86), and (87) as follows:

$$a_{\overline{10}|}+\frac{N_{x+11}}{D_x}+{}^{i}\!\left(\frac{N_{x+11}}{D_x}\right)+k\,{}^{\Delta}\!\left(\frac{N_{x+11}}{D_x}\right) \tag{93}$$

where

$${}^{i}\!\left(\frac{N_{x+11}}{D_x}\right)=\frac{N_{x+11}\left(G_x-G_{x+10}-10F_{x+10}\right)+J_{x+10}+10H_{x+10}}{D_x} \tag{94}$$

and

$${}^{\Delta}\!\left(\frac{N_{x+11}}{D_x}\right)=\frac{N_{x+11}\left(F_x-F_{x+10}\right)+H_{x+10}}{D_x}. \tag{95}$$

The approximate formula for a similar life annuity with a 20-year certain period would be:

$$a_{\overline{20}|}+\frac{N_{x+21}}{D_x}+{}^{i}\!\left(\frac{N_{x+21}}{D_x}\right)+k\,{}^{\Delta}\!\left(\frac{N_{x+21}}{D_x}\right) \tag{96}$$

where

$${}^{i}\!\left(\frac{N_{x+21}}{D_x}\right)=\frac{N_{x+21}\left(G_x-G_{x+20}-20F_{x+20}\right)+J_{x+20}+20H_{x+20}}{D_x} \tag{97}$$

and

$${}^{\Delta}\!\left(\frac{N_{x+21}}{D_x}\right)=\frac{N_{x+21}\left(F_x-F_{x+20}\right)+H_{x+20}}{D_x}. \tag{98}$$

The accuracy of formulae (93) and (96) is tested in Tables 5 and 6 for annuities issued in 1950 and 1960. A comparison of these tables with Table 2 indicates that the errors produced by the approximate formulae for a life annuity with an n-year certain period are even less than those for immediate nonrefund life annuities. This result might be expected as there are no approximations involved in the annuity certain part of the contract.

Similar formulae may be used to obtain the value of an installment refund annuity, where n would represent the number of annual payments that would have to be made before the consideration paid for the annuity contract has been returned to the annuitant.

In the case of life income settlement options, the formulae would involve an annuity-due instead of an immediate life annuity. For example, if the benefit provided were an annuity-due with a 10-year certain period, with the first payment starting at age x in the year $1950+k$, the approximate value of the benefit could be obtained from formulae (90), (91), and (92) as follows:

$$\ddot{a}_{\overline{10}|}+\frac{N_{x+10}}{D_x}+{}^{i}\!\left(\frac{N_{x+10}}{D_x}\right)+k\,{}^{\Delta}\!\left(\frac{N_{x+10}}{D_x}\right) \tag{99}$$

where

$$ {}^{i}\left(\frac{N_{x+10}}{D_x}\right) = \frac{N_{x+10}\,(G_x - G_{x+10} - 10F_{x+10}) + J_{x+10} + 10H_{x+10}}{D_x} \quad (100)$$

and

$$ {}^{\Delta}\left(\frac{N_{x+10}}{D_x}\right) = \frac{N_{x+10}\,(F_x - F_{x+10}) + H_{x+10}}{D_x}. \quad (101)$$

The accuracy of formula (99) is tested in Table 7 for settlement options starting in the years 1970 and 1980. The errors in this table are a little

TABLE 6

COMPARISON OF EXACT AND APPROXIMATE VALUES OF IMMEDIATE ANNUITIES WITH 20-YEAR CERTAIN PERIOD ISSUED IN 1950 AND 1960

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—2½% Interest

Age x	Annuities Issued at Age x in 1950				Annuities Issued at Age x in 1960			
	Exact Value of $a_{\overline{20}\rvert}+{}^{1950}_{\ \ 20\rvert}a_x$ (1)	Approximate Value of $a_{\overline{20}\rvert}+{}^{1950}_{\ \ 20\rvert}a_x$* (2)	Error (2)−(1) (3)	Error (3)÷(1) (4)	Exact Value of $a_{\overline{20}\rvert}+{}^{1960}_{\ \ 20\rvert}a_x$ (5)	Approximate Value of $a_{\overline{20}\rvert}+{}^{1960}_{\ \ 20\rvert}a_x$† (6)	Error (6)−(5) (7)	Error (7)÷(5) (8)
	Male							
15	31.011	31.111	.100	.32%	31.216	31.378	.162	.52%
25	28.456	28.526	.070	.25	28.713	28.839	.126	.44
35	25.331	25.371	.040	.16	25.634	25.717	.083	.32
45	21.860	21.873	.013	.06	22.162	22.196	.034	.15
55	18.603	18.602	−.001	−.01	18.814	18.813	−.001	−.01
65	16.354	16.353	−.001	−.01	16.418	16.414	−.004	−.02
75	15.632	15.632	.000	.00	15.635	15.634	−.001	−.01
	Female							
15	31.991	32.088	.097	.30%	32.130	32.278	.148	.46%
25	29.717	29.790	.073	.25	29.891	30.013	.122	.41
35	26.889	26.935	.046	.17	27.097	27.185	.088	.32
45	23.514	23.536	.022	.09	23.737	23.788	.051	.21
55	19.872	19.878	.006	.03	20.056	20.070	.014	.07
65	16.875	16.874	−.001	−.01	16.946	16.944	−.002	−.01
75	15.678	15.678	.000	.00	15.682	15.682	.000	.00

* Obtained by formula (96), $k = 0$. † Obtained by formula (96), $k = 10$.

larger than in Table 5, as the annuity payments start at a later date in Table 7 than in Table 5.

Another type of annuity benefit which might be considered at this point is the one available at the maturity of a retirement income policy or an accumulative type of deferred annuity contract where mortality is not a factor during the deferred period. In these cases it is assumed that we are interested in the value of an annuity-due, with a 10-year certain period, at

TABLE 7

COMPARISON OF EXACT AND APPROXIMATE VALUES OF LIFE INCOME SETTLEMENT OPTIONS WITH 10-YEAR CERTAIN PERIOD

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—$2\frac{1}{2}\%$ Interest

AGE OF PAYEE WHEN INCOME COMMENCES x	EXACT VALUE (1)	APPROXIMATE VALUE (2)	ERROR (2)−(1) (3)	ERROR (3)÷(1) (4)
	Life Income Commences in 1970—$\ddot{a}_{\overline{10}\rceil}+{}^{1970}_{10|}\ddot{a}_x$*			
Male:				
35	26.679	26.844	.165	.62%
45	22.798	22.910	.112	.49
55	18.539	18.594	.055	.30
65	14.262	14.274	.012	.08
75	10.772	10.770	−.002	−.02
Female:				
35	28.155	28.311	.156	.55%
45	24.632	24.746	.114	.46
55	20.408	20.476	.068	.33
65	15.721	15.744	.023	.15
75	11.503	11.505	.002	.02
	Life Income Commences in 1980—$\ddot{a}_{\overline{10}\rceil}+{}^{1980}_{10|}\ddot{a}_x$*			
Male:				
35	26.972	27.232	.260	.96%
45	23.145	23.337	.192	.83
55	18.880	18.987	.107	.57
65	14.506	14.534	.028	.19
75	10.856	10.854	−.002	−.02
Female:				
35	28.351	28.585	.234	.83%
45	24.864	25.051	.187	.75
55	20.656	20.778	.122	.59
65	15.924	15.973	.049	.31
75	11.587	11.592	.005	.04

* Approximate values obtained by formula (99), with $k = 20$ for life incomes commencing in 1970 and $k = 30$ for life incomes commencing in 1980.

the time the contract matures and the annuity payments begin. The approximate value of this type of benefit may also be obtained by using formulae (90), (91), and (92). For example, if the annuity payments start at age 65 in the year 1950 + k, the approximate value of the benefit at the time payments start would be

$$\ddot{a}_{\overline{10}|} + \frac{N_{75}}{D_{65}} + {}^{i}\!\left(\frac{N_{75}}{D_{65}}\right) + k\,{}^{\Delta}\!\left(\frac{N_{75}}{D_{65}}\right) \tag{102}$$

where

$${}^{i}\!\left(\frac{N_{75}}{D_{65}}\right) = \frac{N_{75}\left(G_{65} - G_{75} - 10F_{75}\right) + J_{75} + 10H_{75}}{D_{65}} \tag{103}$$

and

$${}^{\Delta}\!\left(\frac{N_{75}}{D_{65}}\right) = \frac{N_{75}\left(F_{65} - F_{75}\right) + H_{75}}{D_{65}}. \tag{104}$$

The accuracy of formula (102) for a maturity benefit starting at age 65 and of a similar formula for a maturity benefit starting at age 55 is tested in Table 8 for the maturity values of original contracts issued in 1950.

The problem of valuing all of the annuity benefits considered in this section may be disposed of easily as they all consist of an annuity certain and a deferred nonrefund life annuity. The annuity certain part of the benefit may be valued separately in the usual manner, while the deferred annuity part of the benefit may be valued by either of the procedures presented in Section IV. After the annuity certain period of the contract has elapsed, the contracts may be valued together with the immediate nonrefund life annuity contracts as indicated in Section II.

VI. CASH REFUND LIFE ANNUITIES

The value of an immediate life annuity that provides for the payment, at the death of the annuitant, of that part of the annuity consideration that had not been returned in the form of annuity payments may be expressed in terms of standard commutation symbols as:

$$\frac{N_{x+1}}{D_x} + \frac{nM_x - R_{x+1} + R_{x+n+1}}{D_x}$$

where x represents the age of the annuitant and n represents the number of annual payments that have to be made until the total annuity consideration has been returned to the annuitant. For convenience, it is assumed that n is an integer and that the death benefit will be paid on the contract anniversary.

The value of this cash refund annuity may be broken up into two parts, which may be designated as follows:

$$a_x = \frac{N_{x+1}}{D_x} = \text{the value of an immediate nonrefund life annuity}$$

$$_nB_x = \frac{n\,M_x - R_{x+1} + R_{x+n+1}}{D_x} = \text{the value of the decreasing death benefit.}$$

TABLE 8

COMPARISON OF EXACT AND APPROXIMATE VALUES OF AN ANNUITY DUE WITH A 10-YEAR CERTAIN PERIOD AT THE MATURITY OF A RETIREMENT INCOME POLICY OR A DEFERRED REFUND ANNUITY ISSUED IN 1950

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—2½% Interest

Age at Issue of Original Contract	Maturity Year of Original Contract 1950+k	Value of k	Exact Value (1)	Approximate Value (2)	Error: (2)−(1) (3)	Error: (3)÷(1) (4)
	Maturity Benefit at Age 55—$\ddot{a}_{\overline{10}\rvert} + {}^{1950+k}_{\quad 10\rvert}\ddot{a}_{55}$*					
Male:						
15........	1990	40	19.203	19.379	.176	.92%
25........	1980	30	18.880	18.987	.107	.57
35........	1970	20	18.539	18.594	.055	.30
45........	1960	10	18.181	18.202	.021	.12
Female:						
15........	1990	40	20.888	21.080	.192	.92%
25........	1980	30	20.656	20.778	.122	.59
35........	1970	20	20.408	20.476	.068	.33
45........	1960	10	20.144	20.174	.030	.15
	Maturity Benefit at Age 65—$\ddot{a}_{\overline{10}\rvert} + {}^{1950+k}_{\quad 10\rvert}\ddot{a}_{65}$†					
Male:						
15........	2000	50	14.971	15.053	.082	.55%
25........	1990	40	14.742	14.794	.052	.35
35........	1980	30	14.506	14.534	.028	.19
45........	1970	20	14.262	14.274	.012	.08
Female:						
15........	2000	50	16.309	16.431	.122	.75%
25........	1990	40	16.121	16.202	.081	.50
.35........	1980	30	15.924	15.973	.049	.31
45........	1970	20	15.721	15.744	.023	.15

* Approximate values obtained by formula (99), with $x = 55$.

† Approximate values obtained by formula (102).

In calculating the approximate value of this cash refund annuity on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest, the two parts may be considered separately. The approximate value of the immediate nonrefund life annuity part of the contract has already been treated in Section II. The value of the decreasing death benefit part of the contract may be expressed as follows:

$$ {}^{1950+k}_{n}B_x = n\left[{}^{1950+k}\left(\frac{M_x}{D_x}\right)\right] - {}^{1950+k}\left(\frac{R_{x+1}}{D_x}\right) + {}^{1950+k}\left(\frac{R_{x+n+1}}{D_x}\right) \quad (105) $$

where $1950 + k$ represents the calendar year in which age x is attained. From (64), we get

$$ n\left[{}^{1950+k}\left(\frac{M_x}{D_x}\right)\right] \doteq \frac{nM_x - nd\,(J_x + kH_x)}{D_x}. \quad (106) $$

Subtracting (64) from (82), we get

$$ {}^{1950+k}\left(\frac{R_{x+1}}{D_x}\right) \doteq \frac{R_{x+1} + Z_x - d\,(2K_x - J_x) + k\,(Y_x - dJ_x)}{D_x}. \quad (107) $$

From (44) and (107), using the procedure followed in deriving (47), we get

$$ \left.\begin{aligned} &{}^{1950+k}\left(\frac{R_{x+n+1}}{D_x}\right) \\ &\doteq \frac{R_{x+n+1}(1 + {}^{1950+k}_{n}I_x) + Z_{x+n} - d\,(2K_{x+n} - J_{x+n}) + (k+n)\,(Y_{x+n} - d\,J_{x+n})}{D_x}. \end{aligned}\right\} \quad (108) $$

Combining these three formulae and using the general formula (27) we may write

$$ {}^{1950+k}_{n}B_x \doteq \frac{nM_x - R_{x+1} + R_{x+n+1}}{D_x} + {}^{i}_{n}B_x + k\,{}^{\Delta}_{n}B_x \quad (109) $$

where

$$ \left.\begin{aligned} &{}^{i}_{n}B_x \\ &= \frac{R_{x+n+1}(G_x - G_{x+n} - nF_{x+n}) + Z_{x+n} - Z_x + nY_{x+n} - d\,[2K_{x+n} - 2K_x + (n-1)\,(J_{x+n} + J_x)]}{D_x} \end{aligned}\right\} \quad (11 $$

and

$$ {}^{\Delta}_{n}B_x = \frac{R_{x+n+1}\,(F_x - F_{x+n}) + Y_{x+n} - Y_x - d\,(J_{x+n} - J_x + nH_x)}{D_x}. \quad (111) $$

The value of ${}^{i}_{n}B_x$ represents the approximate decrease due to improving mortality in the value of a death benefit of this type issued in 1950, while ${}^{\Delta}_{n}B_x$ represents the approximate annual decrement that results when the benefit is issued in some subsequent year.

The accuracy of formula (109) is tested in Table 9 where the approximate values of ${}^{1950}_{10}B_x$ and ${}^{1960}_{10}B_x$ that are produced by this formula are compared with the corresponding exact values. Table 10 shows a similar comparison for the combination of the immediate life annuity and the decreasing death benefit. A comparison of this table with Table 2 indicates that the errors produced by the approximate formula for cash refund annuities are even less than those arising on immediate nonrefund life annuities.

TABLE 9

COMPARISON OF EXACT AND APPROXIMATE VALUES OF A 10-YEAR DECREASING DEATH BENEFIT* OF TYPE USED IN A CASH REFUND ANNUITY

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—2½% Interest

Issue Age x	Exact Value (1)	Approximate Value (2)	Error: (2)−(1) (3)	Error: (3)÷(1) (4)
	10-Year Decreasing Death Benefit Issued in 1950—${}^{1950}_{10}B_x$†			
Male:				
25	.043	.043	.000	.00%
45	.249	.248	−.001	−.40
65	1.325	1.325	.000	.00
85	5.371	5.371	.000	.00
Female:				
25	.029	.029	.000	.00%
45	.125	.125	.000	.00
65	.793	.793	.000	.00
85	4.752	4.752	.000	.00
	10-Year Decreasing Death Benefit Issued in 1960—${}^{1960}_{10}B_x$‡			
Male:				
25	.038	.038	.000	.00%
45	.220	.216	−.004	−1.82
65	1.209	1.201	−.008	−.66
85	5.330	5.330	.000	.00
Female:				
25	.025	.025	.000	.00%
45	.110	.109	−.001	−.91
65	.721	.716	−.005	−.69
85	4.715	4.715	.000	.00

* ${}_{10}B_x = \dfrac{10M_x - R_{x+1} + R_{x+11}}{D_x}$.

† Approximate values obtained by formula (109), with $k = 0$, $n = 10$.

‡ Approximate values obtained by formula (109), with $k = 10$, $n = 10$.

The two parts of a cash refund annuity may also be considered separately for valuation purposes. The annuity part of the contract may be valued together with the immediate nonrefund life annuities in the manner indicated in Section II. In valuing the decreasing death benefit, it is

TABLE 10

COMPARISON OF EXACT AND APPROXIMATE VALUES OF THE COMBINATION* OF AN IMMEDIATE ANNUITY AND A 10-YEAR DECREASING DEATH BENEFIT CORRESPONDING TO THE BENEFITS PROVIDED BY A CASH REFUND ANNUITY

Based on the Annuity Table for 1949 (Ultimate) with Projection Scale B—2½% Interest

Issue Age x	Exact Value (1)	Approximate Value (2)	Error: (2)−(1) (3)	Error: (3)÷(1) (4)
	Combination of Benefits Issued in 1950—${}^{1960}a_x + {}^{1950}_{10}B_x$†			
Male:				
25	28.339	28.413	.074	.26%
45	21.098	21.115	.017	.08
65	13.069	13.069	.000	.00
85	9.298	9.298	.000	.00
Female:				
25	29.640	29.714	.074	.25%
45	23.143	23.168	.025	.11
65	14.479	14.480	.001	.01
85	9.316	9.316	.000	.00
	Combination of Benefits Issued in 1960—${}^{1960}a_x + {}^{1960}_{10}B_x$†			
Male:				
25	28.612	28.742	.130	.45%
45	21.483	21.535	.052	.24
65	13.301	13.301	.000	.00
85	9.295	9.295	.000	.00
Female:				
25	29.822	29.948	.126	.42%
45	23.409	23.469	.060	.26
65	14.684	14.692	.008	.05
85	9.314	9.314	.000	.00

* $a_x + {}_{10}B_x = \dfrac{N_{x+1}}{D_x} + \dfrac{10M_x - R_{x+1} + R_{x+11}}{D_x}$.

† Approximate values obtained by adding appropriate values from Table 2 and Table 9.

again possible to use two valuation factors that will remain the same from year to year. These valuation factors would be

$$(A):\ {}^{1950}_{\ \ n}B_x \doteq \frac{n\,M_x - R_{x+1} + R_{x+n+1}}{D_x} + {}^{i}_{n}B_x$$

where ${}^{i}_{n}B_x$ is defined by (110), and

$$(B):\ {}^{\Delta}_{n}B_x \text{ as defined by (111)}.$$

The aggregate reserve in the year 1950 + k would be the aggregate of valuation factor (A) plus k times the aggregate of valuation factor (B).

The method involves a classification of the cash refund annuities for each sex by attained age (x) and by the number of years before the decreasing death benefit is exhausted (n). In order to cut down the number of classifications, it might be desirable to use age-groups and central ages for the classification by attained age x or some other approximation.

In actual practice, $x + \frac{1}{2}$ could be substituted for x and $n - \frac{1}{2}$ for n in the two valuation factors and $k + \frac{1}{2}$ could be used instead of k when the valuation takes place at the end of the calendar year 1950 + k.

VII. JOINT LIFE ANNUITIES

The Jenkins-Lew paper presented two approximate methods[4] for taking account of the effect of future improvements in mortality on joint life annuities. As both of these methods depend on determining the effect of future improvements in mortality on a single life annuity, either method may be adapted for use with the supplementary commutation columns presented in this paper.

In general, the procedure designated as Method A in the paper referred to above seems preferable for use with the supplementary commutation columns. This is particularly true for valuation purposes, as Method A permits the same valuation factors to be used year after year. Method A involves multiplying the value of the joint life annuity on the Annuity Table for 1949 (without projection) by the projection factor for a single life of the same sex at the equivalent equal age (with the male factor used for a joint life annuity on one male and one female). The projection factor for an immediate nonrefund life annuity issued in the year 1950 + k on a single life aged x may be designated as

$$\frac{{}^{1950+k}a_x}{a} \doteq 1 + \frac{1}{a_x}\left(\frac{J_x}{D_x}\right) + \frac{k}{a_x}\left(\frac{H_x}{D_x}\right). \qquad (112)$$

[4] *TSA* I, 459.

The value of a joint life annuity issued to two lives aged x in the year $1950 + k$ on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest may be designated as ${}^{1950+k}a_{xx}$. According to Method A, the approximate value of ${}^{1950+k}a_{xx}$ would be given by the following formula:

$$ {}^{1950+k}a_{xx} \doteq a_{xx}\left(\frac{{}^{1950+k}a_x}{a_x}\right). \tag{113} $$

From (112) and (113), we obtain

$$ {}^{1950+k}a_{xx} \doteq a_{xx} + \frac{a_{xx}}{a_x}\left(\frac{\mathrm{J}_x}{\mathrm{D}_x}\right) + k\,\frac{a_{xx}}{a_x}\left(\frac{\mathrm{H}_x}{\mathrm{D}_x}\right) \tag{114} $$

where we may designate the second term as ${}^{i}a_{xx}$ and the third term as $k^{\Delta}a_{xx}$.

The two constant valuation factors would therefore be

$$ (\mathrm{A}):\quad {}^{1950}a_{xx} \doteq a_{xx} + {}^{i}a_{xx} \doteq a_{xx} + \frac{a_{xx}}{a_x}\left(\frac{\mathrm{J}_x}{\mathrm{D}_x}\right) $$

and

$$ (\mathrm{B}):\quad {}^{\Delta}a_{xx} \doteq \frac{a_{xx}}{a_x}\left(\frac{\mathrm{H}_x}{\mathrm{D}_x}\right). $$

The aggregate reserve for a valuation in the year $1950 + k$ would be the aggregate of valuation factor (A) plus k times the aggregate of valuation factor (B). In actual practice, valuation factors (A) and (B) may be adjusted to a mean reserve basis in a similar manner to that indicated for single life annuities in Section II, (32) and (33), and $k + \frac{1}{2}$ may be used instead of k for a valuation at the end of the year $1950 + k$.

The test of the accuracy of Method A in the Jenkins-Lew paper[5] should suffice for the method described above, as Table 2 clearly indicates that the projection factors for single life immediate nonrefund annuities may be closely reproduced by the supplementary commutation columns.

The procedure designated as Method B in the Jenkins-Lew paper requires no special comment, as the supplementary commutation columns may easily be used to determine the age setback that would make a single life annuity on the basis of the Annuity Table for 1949 without projection equal to a corresponding single life annuity on the basis of the Annuity Table for 1949 with Projection Scale B.

[5] See Table 37, *TSA* I, 460.

If greater accuracy is desired, supplementary commutation columns may be constructed for joint lives. These would be defined as follows:

$$f_x = \frac{s_x q_x}{p_x}, \qquad h_{xx} = f_x N_{x+1:x+1}$$

$$H_{xx} = \sum_{t=0}^{89-x} h_{x+t:x+t} = h_{xx} + h_{x+1:x+1} + \ldots + h_{89:89} \tag{115}$$

and

$$J_{xx} = \sum_{t=1}^{89-x} H_{x+t:x+t} = H_{x+1:x+1} + H_{x+2:x+2} + \ldots + H_{89:89} . \tag{116}$$

The approximate formula for a joint life annuity would be

$$^{1950+k}a_{xx} \doteq a_{xx} + 2\,\frac{J_{xx}}{D_{xx}} + 2k\,\frac{H_{xx}}{D_{xx}}. \tag{117}$$

These supplementary commutation columns for joint lives have not been constructed for this paper as Method A, discussed above, seems to provide sufficient accuracy for joint life annuities.

APPENDIX I

SUMMARY OF NEW NOTATION

The following summary of the more important new notation used in this paper is intended to supplement the basic symbols defined at the beginning of Section II. The numbers on the left indicate the Section and formula where this new notation is first used.

A. *Auxiliary Symbols*

II, (12) $f_x = \dfrac{s_x q_x}{p_x}$

II, (20) $h_x = f_x N_{x+1}$

III, (78) $y_x = f_x R_{x+1}$.

B. *Supplementary Commutation Columns*

III, (37) $F_x = \sum_{t=0}^{89-x} f_{x+t} = F_{x+1} + f_x$

III, (38) $G_x = \sum_{t=1}^{89-x} F_{x+t} = G_{x+1} + F_{x+1}$

$$\text{II, (21)}\quad H_x = \sum_{t=0}^{89-x} h_{x+t} = H_{x+1} + h_x$$

$$\text{II, (22)}\quad J_x = \sum_{t=1}^{89-x} H_{x+t} = J_{x+1} + H_{x+1}$$

$$\text{III, (77)}\quad K_x = \sum_{t=0}^{89-x} J_{x+t} = K_{x+1} + J_x$$

$$\text{III, (78)}\quad Y_x = \sum_{t=0}^{89-x} y_{x+t} = Y_{x+1} + y_x$$

$$\text{III, (79)}\quad Z_x = \sum_{t=1}^{89-x} Y_{x+t} = Z_{x+1} + Y_{x+1}\,.$$

The limiting age on the summations shown above is 89 because Projection Scale B does not involve any improvements in mortality at ages 90 and over. If these supplementary commutation columns are constructed for a projection scale with a different terminal age, the summations should, of course, be adjusted to cover all ages which involve any improvements in mortality.

C. *Modification of Standard Commutation Column Symbols*

The following notation will, as a matter of convenience, be defined with reference to a specific expression in standard commutation column symbols and a specific mortality basis. This notation may, however, be interpreted in a similar manner with reference to any other expression or any other mortality basis.

III, (34) N_x/D_x: This symbol designates the exact value of N_x/D_x on the Annuity Table for 1949 (ultimate), without projection, and $2\frac{1}{2}\%$ interest.

III, (34) ${}^{1950+k}(N_x/D_x)$: This symbol designates the exact value of N_x/D_x on the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest; the superscript $1950+k$ designates that the value is to be calculated for a life aged x in the calendar year $1950+k$. Note that ${}^{1950+k}(N_x/D_x)$ is equal to N_x/D_x multiplied by the appropriate projection factor from the Jenkins-Lew paper.

II, (26) ${}^{i}(\mathrm{N}_x/\mathrm{D}_x)$: This symbol designates the exact value that must be added to $\mathrm{N}_x/\mathrm{D}_x$ in order to produce the particular approximate value of ${}^{1950}(\mathrm{N}_x/\mathrm{D}_x)$ that results from the assumption that the basic formula (14) is exact. This definition implies the following general formula:

$$ {}^{1950}\left(\frac{\mathrm{N}_x}{\mathrm{D}_x}\right) \doteq \frac{\mathrm{N}_x}{\mathrm{D}_x} + {}^{i}\left(\frac{\mathrm{N}_x}{\mathrm{D}_x}\right). $$

II, (27) ${}^{\Delta}(\mathrm{N}_x/\mathrm{D}_x)$: This symbol designates the exact value that must be added to the approximate value of ${}^{1950+k}(\mathrm{N}_x/\mathrm{D}_x)$ in order to produce the approximate value of ${}^{1950+k+1}(\mathrm{N}_x/\mathrm{D}_x)$, where both of these approximate values are the particular ones that result from the assumption that the basic formula (14) is exact. This assumption implies that the value of ${}^{\Delta}(\mathrm{N}_x/\mathrm{D}_x)$ is independent of k so that the following general formula holds:

$$ {}^{1950+k}\left(\frac{\mathrm{N}_x}{\mathrm{D}_x}\right) \doteq \frac{\mathrm{N}_x}{\mathrm{D}_x} + {}^{i}\left(\frac{\mathrm{N}_x}{\mathrm{D}_x}\right) + k\,{}^{\Delta}\left(\frac{\mathrm{N}_x}{\mathrm{D}_x}\right) $$

D. *Other Notation*

III, (43) $$ {}^{1950+k}_{\quad n}I_x = \mathrm{G}_x - \mathrm{G}_{x+n} - n\mathrm{F}_{x+n} + k(\mathrm{F}_x - \mathrm{F}_{x+n}). $$

The assumption that the basic formula (14) is exact implies that for a life aged x in the calendar year $1950 + k$, the probability of surviving n years on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B is equal to $(1 + {}^{1950+k}_{\quad n}I_x)$ times the probability of surviving n years on the basis of the Annuity Table for 1949 (ultimate) without projection.

VI, (105) $$ {}_nB_x = \frac{n\mathrm{M}_x - \mathrm{R}_{x+1} + \mathrm{R}_{x+n+1}}{\mathrm{D}_x}. $$

The symbol ${}_nB_x$ is introduced for convenience to denote the value of a decreasing death benefit which provides n in the first year, $n - 1$ in the second year, etc.

APPENDIX II

SUPPLEMENTARY COMMUTATION COLUMNS FOR APPROXIMATE ANNUITY VALUES ON THE ANNUITY TABLE FOR 1949 (ULTIMATE) WITH PROJECTION SCALE B AND 2½% INTEREST

AGE x	MALES						
	F_x	G_x	H_x	J_x	K_x	Y_x	Z_x
10	.012354	.753464	19.9808	845.7395	21530.1417	13.81336	629.48084
11	.012348	.741116	19.8343	825.9052	20684.4022	13.75425	615.72659
12	.012342	.728774	19.6900	806.2152	19858.4970	13.69508	602.03151
13	.012336	.716438	19.5472	786.6680	19052.2818	13.63579	588.39572
14	.012329	.704109	19.4062	767.2618	18265.6138	13.57639	574.81933
15	.012323	.691786	19.2666	747.9952	17498.3520	13.51673	561.30260
16	.012316	.679470	19.1283	728.8669	16750.3568	13.45670	547.84590
17	.012309	.667161	18.9907	709.8762	16021.4899	13.39628	534.44962
18	.012302	.654859	18.8539	691.0223	15311.6137	13.33530	521.11432
19	.012295	.642564	18.7178	672.3045	14620.5914	13.27372	507.84060
20	.012287	.630277	18.5818	653.7227	13948.2869	13.21139	494.62921
21	.012279	.617998	18.4457	635.2770	13294.5642	13.14820	481.48101
22	.012271	.605727	18.3094	616.9676	12659.2872	13.08393	468.39708
23	.012263	.593464	18.1724	598.7952	12042.3196	13.01848	455.37860
24	.012254	.581210	18.0346	580.7606	11443.5244	12.95178	442.42682
25	.012245	.568965	17.8958	562.8648	10862.7638	12.88364	429.54318
26	.012235	.556730	17.7554	545.1094	10299.8990	12.81384	416.72934
27	.012226	.544504	17.6135	527.4959	9754.7896	12.74224	403.98710
28	.012214	.532290	17.4691	510.0268	9227.2937	12.66858	391.31852
29	.012203	.520087	17.3227	492.7041	8717.2669	12.59266	378.72586
30	.012192	.507895	17.1732	475.5309	8224.5628	12.51437	366.21149
31	.012179	.495716	17.0209	458.5100	7749.0319	12.43341	353.77808
32	.012165	.483551	16.8652	441.6448	7290.5219	12.34953	341.42855
33	.012152	.471399	16.7055	424.9393	6848.8771	12.26254	329.16601
34	.012136	.459263	16.5419	408.3974	6423.9378	12.17210	316.99391
35	.012120	.447143	16.3736	392.0238	6015.5404	12.07801	304.91590
36	.012103	.435040	16.2003	375.8235	5623.5166	11.97988	292.93602
37	.012083	.422957	16.0219	359.8016	5247.6931	11.87748	281.05854
38	.012064	.410893	15.8376	343.9640	4887.8915	11.77053	269.28801
39	.012042	.398851	15.6474	328.3166	4543.9275	11.65864	257.62937
40	.012019	.386832	15.4503	312.8663	4215.6109	11.54146	246.08791
41	.011993	.374839	15.2467	297.6196	3902.7446	11.41869	234.66922
42	.011965	.362874	15.0328	282.5868	3605.1250	11.28843	223.38079
43	.011935	.350939	14.8048	267.7820	3322.5382	11.14770	212.23309
44	.011899	.339040	14.5584	253.2236	3054.7562	10.99403	201.23906
45	.011859	.327181	14.2916	238.9320	2801.5326	10.82548	190.41358
46	.011814	.315367	14.0020	224.9300	2562.6006	10.64068	179.77290
47	.011762	.303605	13.6893	211.2407	2337.6706	10.43861	169.33429
48	.011703	.291902	13.3525	197.8882	2126.4299	10.21869	159.11560
49	.011638	.280264	12.9922	184.8960	1928.5417	9.98067	149.13493

Age x	Males						
	F_x	G_x	H_x	J_x	K_x	Y_x	Z_x
50	.011564	.268700	12.6088	172.2872	1743.6457	9.72470	139.41023
51	.011481	.257219	12.2034	160.0838	1571.3585	9.45116	129.95907
52	.011390	.245829	11.7811	148.3027	1411.2747	9.16295	120.79612
53	.011290	.234539	11.3398	136.9629	1262.9720	8.85874	111.93738
54	.011180	.223359	10.8854	126.0775	1126.0091	8.54208	103.39530
55	.011060	.212299	10.4160	115.6615	999.9316	8.21167	95.18363
56	.010930	.201369	9.9377	105.7238	884.2701	7.87152	87.31211
57	.010787	.190582	9.4488	96.2750	778.5463	7.52025	79.79186
58	.010635	.179947	8.9555	87.3195	682.2713	7.16224	72.62962
59	.010470	.169477	8.4557	78.8638	594.9518	6.79595	65.83367
60	.010293	.159184	7.9557	70.9081	516.0880	6.42592	59.40775
61	.010102	.149082	7.4532	63.4549	445.1799	6.05053	53.35722
62	.009899	.139183	6.9580	56.4969	381.7250	5.67686	47.68036
63	.009685	.129498	6.4700	50.0269	325.2281	5.30531	42.37505
64	.009455	.120043	5.9899	44.0370	275.2012	4.93630	37.43875
65	.009212	.110831	5.5181	38.5189	231.1642	4.57038	32.86837
66	.008953	.101878	5.0553	33.4636	192.6453	4.20816	28.66021
67	.008677	.093201	4.6063	28.8573	159.1817	3.85370	24.80651
68	.008387	.084814	4.1720	24.6853	130.3244	3.50784	21.29867
69	.008079	.076735	3.7532	20.9321	105.6391	3.17145	18.12722
70	.007753	.068982	3.3509	17.5812	84.7070	2.84549	15.28173
71	.007408	.061574	2.9656	14.6156	67.1258	2.53091	12.75082
72	.007045	.054529	2.6025	12.0131	52.5102	2.23197	10.51885
73	.006667	.047862	2.2622	9.7509	40.4971	1.94959	8.56926
74	.006269	.041593	1.9452	7.8057	30.7462	1.68454	6.88472
75	.005855	.035738	1.6520	6.1537	22.9405	1.43756	5.44716
76	.005423	.030315	1.3832	4.7705	16.7868	1.20924	4.23792
77	.004978	.025337	1.1416	3.6289	12.0163	1.00295	3.23497
78	.004526	.020811	.9278	2.7011	8.3874	.81871	2.41626
79	.004064	.016747	.7403	1.9608	5.6863	.65631	1.75995
80	.003598	.013149	.5786	1.3822	3.7255	.51524	1.24471
81	.003131	.010018	.4415	.9407	2.3433	.39477	.84994
82	.002666	.007352	.3271	.6136	1.4026	.29386	.55608
83	.002210	.005142	.2342	.3794	.7890	.21121	.34487
84	.001768	.003374	.1606	.2188	.4096	.14532	.19955
85	.001348	.002026	.1039	.1149	.1908	.09451	.10504
86	.000960	.001066	.0624	.0525	.0759	.05691	.04813
87	.000617	.000449	.0334	.0191	.0234	.03057	.01756
88	.000331	.000118	.0148	.0043	.0043	.01358	.00398
89	.000118	.000000	.0043	.0000	.0000	.00398	.00000

NOTE.—All of the supplementary commutation columns are equal to 0 at ages 90 and over.

APPENDIX II—*Continued*

SUPPLEMENTARY COMMUTATION COLUMNS FOR APPROXIMATE ANNUITY VALUES ON THE ANNUITY TABLE FOR 1949 (ULTIMATE) WITH PROJECTION SCALE B AND 2½% INTEREST

AGE x	FEMALES						
	F_x	G_x	H_x	J_x	K_x	Y_x	Z_x
10	.007721	.483011	13.9405	627.0073	16777.7810	9.51917	462.74000
11	.007719	.475292	13.8805	613.1268	16150.7737	9.49674	453.24326
12	.007716	.467576	13.8169	599.3099	15537.6469	9.47269	443.77057
13	.007713	.459863	13.7505	585.5594	14938.3370	9.44710	434.32347
14	.007710	.452153	13.6811	571.8783	14352.7776	9.42002	424.90345
15	.007707	.444446	13.6087	558.2696	13780.8993	9.39140	415.51205
16	.007704	.436742	13.5341	544.7355	13222.6297	9.36133	406.15072
17	.007699	.429043	13.4567	531.2788	12677.8942	9.32984	396.82088
18	.007696	.421347	13.3773	517.9015	12146.6154	9.29692	387.52396
19	.007692	.413655	13.2954	504.6061	11628.7139	9.26262	378.26134
20	.007687	.405968	13.2116	491.3945	11124.1078	9.22691	369.03443
21	.007682	.398286	13.1255	478.2690	10632.7133	9.18969	359.84474
22	.007678	.390608	13.0371	465.2319	10154.4443	9.15100	350.69374
23	.007672	.382936	12.9469	452.2850	9689.2124	9.11085	341.58289
24	.007667	.375269	12.8542	439.4308	9236.9274	9.06913	332.51376
25	.007660	.367609	12.7593	426.6715	8797.4966	9.02575	323.48801
26	.007655	.359954	12.6622	414.0093	8370.8251	8.98070	314.50731
27	.007648	.352306	12.5627	401.4466	7956.8158	8.93393	305.57338
28	.007641	.344665	12.4607	388.9859	7555.3692	8.88534	296.68804
29	.007633	.337032	12.3562	376.6297	7166.3833	8.83483	287.85321
30	.007625	.329407	12.2489	364.3808	6789.7536	8.78225	279.07096
31	.007617	.321790	12.1387	352.2421	6425.3728	8.72749	270.34347
32	.007608	.314182	12.0253	340.2168	6073.1307	8.67047	261.67300
33	.007598	.306584	11.9088	328.3080	5732.9139	8.61096	253.06204
34	.007588	.298996	11.7887	316.5193	5404.6059	8.54889	244.51315
35	.007577	.291419	11.6652	304.8541	5088.0866	8.48409	236.02906
36	.007565	.283854	11.5376	293.3165	4783.2325	8.41631	227.61275
37	.007553	.276301	11.4059	281.9106	4489.9160	8.34543	219.26732
38	.007539	.268762	11.2697	270.6409	4208.0054	8.27121	210.99611
39	.007524	.261238	11.1291	259.5118	3937.3645	8.19342	202.80269
40	.007509	.253729	10.9834	248.5284	3677.8527	8.11192	194.69077
41	.007491	.246238	10.8327	237.6957	3429.3243	8.02636	186.66441
42	.007474	.238764	10.6760	227.0197	3191.6286	7.93646	178.72795
43	.007453	.231311	10.5137	216.5060	2964.6089	7.84201	170.88594
44	.007432	.223879	10.3452	206.1608	2748.1029	7.74265	163.14329
45	.007409	.216470	10.1699	195.9909	2541.9421	7.63815	155.50514
46	.007384	.209086	9.9878	186.0031	2345.9512	7.52810	147.97704
47	.007355	.201731	9.7983	176.2048	2159.9481	7.41217	140.56487
48	.007327	.194404	9.6010	166.6038	1983.7433	7.29000	133.27487
49	.007293	.187111	9.3956	157.2082	1817.1395	7.16122	126.11365

Age x	Females						
	F_x	G_x	H_x	J_x	K_x	Y_x	Z_x
50	.007258	.179853	9.1816	148.0266	1659.9313	7.02543	119.08822
51	.007218	.172635	8.9586	139.0680	1511.9047	6.88222	112.20600
52	.007177	.165458	8.7307	130.3373	1372.8367	6.73416	105.47184
53	.007132	.158326	8.4959	121.8414	1242.4994	6.57967	98.89217
54	.007082	.151244	8.2552	113.5862	1120.6580	6.41954	92.47263
55	.007030	.144214	8.0063	105.5799	1007.0718	6.25201	86.22062
56	.006972	.137242	7.7507	97.8292	901.4919	6.07798	80.14264
57	.006908	.130334	7.4861	90.3431	803.6627	5.89566	74.24698
58	.006840	.123494	7.2137	83.1294	713.3196	5.70607	68.54091
59	.006765	.116729	6.9317	76.1977	630.1902	5.50727	63.03364
60	.006682	.110047	6.6412	69.5565	553.9925	5.30049	57.73315
61	.006592	.103455	6.3402	63.2163	484.4360	5.08372	52.64943
62	.006493	.096962	6.0331	57.1832	421.2197	4.86033	47.78910
63	.006386	.090576	5.7206	51.4626	364.0365	4.63037	43.15873
64	.006270	.084306	5.4024	46.0602	312.5739	4.39396	38.76477
65	.006142	.078164	5.0793	40.9809	266.5137	4.15125	34.61352
66	.006005	.072159	4.7515	36.2294	225.5328	3.90254	30.71098
67	.005855	.066304	4.4227	31.8067	189.3034	3.65059	27.06039
68	.005694	.060610	4.0939	27.7128	157.4967	3.39606	23.66433
69	.005520	.055090	3.7658	23.9470	129.7839	3.13972	20.52461
70	.005332	.049758	3.4399	20.5071	105.8369	2.88247	17.64214
71	.005128	.044630	3.1170	17.3901	85.3298	2.62531	15.01683
72	.004910	.039720	2.8020	14.5881	67.9397	2.37213	12.64470
73	.004679	.035041	2.4965	12.0916	53.3516	2.12424	10.52046
74	.004432	.030609	2.2016	9.8900	41.2600	1.88303	8.63743
75	.004170	.026439	1.9194	7.9706	31.3700	1.64990	6.98753
76	.003891	.022548	1.6506	6.3200	23.3994	1.42625	5.56128
77	.003599	.018949	1.4007	4.9193	17.0794	1.21644	4.34484
78	.003298	.015651	1.1708	3.7485	12.1601	1.02180	3.32304
79	.002984	.012667	.9615	2.7870	8.4116	.84342	2.47962
80	.002664	.010003	.7740	2.0130	5.6246	.68226	1.79736
81	.002337	.007666	.6086	1.4044	3.6116	.53896	1.25840
82	.002006	.005660	.4652	.9392	2.2072	.41391	.84449
83	.001677	.003983	.3436	.5956	1.2680	.30711	.53738
84	.001353	.002630	.2430	.3526	.6724	.21828	.31910
85	.001040	.001590	.1628	.1898	.3198	.14669	.17241
86	.000749	.000841	.1007	.0891	.1300	.09131	.08110
87	.000484	.000357	.0559	.0332	.0409	.05074	.03036
88	.000262	.000095	.0255	.0077	.0077	.02329	.00707
89	.000095	.000000	.0077	.0000	.0000	.00707	.00000

NOTE.—All of the supplementary commutation columns are equal to 0 at ages 90 and over.

APPENDIX III

BASIC FORMULAE

The following summary of basic formulae will explain how to obtain the approximate value of any expression in standard commutation column symbols on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest by using the supplementary commutation columns in Appendix II. The numbers on the left indicate where these formulae are derived in the text.

III, (50)
$$^{1950+k}\left(\frac{N_x}{D_x}\right) \doteq \frac{N_x}{D_x} + {}^{i}\left(\frac{N_x}{D_x}\right) + k\,{}^{\Delta}\left(\frac{N_x}{D_x}\right)$$

$$^{i}\left(\frac{N_x}{D_x}\right) = \frac{J_x}{D_x}$$

$$^{\Delta}\left(\frac{N_x}{D_x}\right) = \frac{H_x}{D_x}.$$

III, (59)
$$^{1950+k}\left(\frac{C_x}{D_x}\right) \doteq \frac{C_x}{D_x} + {}^{i}\left(\frac{C_x}{D_x}\right) + k\,{}^{\Delta}\left(\frac{C_x}{D_x}\right)$$

$$^{i}\left(\frac{C_x}{D_x}\right) = 0$$

$$^{\Delta}\left(\frac{C_x}{D_x}\right) = -\frac{D_{x+1}}{D_x}(F - F_{x+1}).$$

III, (64)
$$^{1950+k}\left(\frac{M_x}{D_x}\right) \doteq \frac{M_x}{D_x} + {}^{i}\left(\frac{M_x}{D_x}\right) + k\,{}^{\Delta}\left(\frac{M_x}{D_x}\right)$$

$$^{i}\left(\frac{M_x}{D_x}\right) = -d\,\frac{J_x}{D_x}$$

$$^{\Delta}\left(\frac{M_x}{D_x}\right) = -d\,\frac{H_x}{D_x}.$$

III, (82)
$$^{1950+k}\left(\frac{R_x}{D_x}\right) \doteq \frac{R_x}{D_x} + {}^{i}\left(\frac{R_x}{D_x}\right) + k\,{}^{\Delta}\left(\frac{R_x}{D_x}\right)$$

$$^{i}\left(\frac{R_x}{D_x}\right) = -\frac{2d\,K_x - Z_x}{D_x}$$

$$^{\Delta}\left(\frac{R_x}{D_x}\right) = -\frac{d\,(H_x + J_x) - Y_x}{D_x}.$$

III, (84)
$$^{1950+k}\left(\frac{S_x}{D_x}\right) \doteq \frac{S_x}{D_x} + {}^{i}\left(\frac{S_x}{D_x}\right) + k\,{}^{\Delta}\left(\frac{S_x}{D_x}\right)$$

$$^{i}\left(\frac{S_x}{D_x}\right) = \frac{2\,K_x + \frac{1}{d}(J_x - Z_x)}{D_x}$$

$$^{\Delta}\left(\frac{S_x}{D_x}\right) = \frac{H_x + J_x + \frac{1}{d}(H_x - Y_x)}{D_x}.$$

III, (41)
$$^{1950+k}\left(\frac{D_{x+n}}{D_x}\right) \doteq \frac{D_{x+n}}{D_x} + {}^{i}\left(\frac{D_{x+n}}{D_x}\right) + k\,{}^{\Delta}\left(\frac{D_{x+n}}{D_x}\right)$$

$$^{i}\left(\frac{D_{x+n}}{D_x}\right) = \frac{D_{x+n}}{D_x}(G_x - G_{x+n} - nF_{x+n})$$

$$^{\Delta}\left(\frac{D_{x+n}}{D_x}\right) = \frac{D_{x+n}}{D_x}(F_x - F_{x+n}).$$

III, (47) If we let Q_x denote any one of the standard commutation column symbols (*i.e.*, C_x, D_x, M_x, N_x, R_x, or S_x), then the formula for the approximate value of $^{1950+k}(Q_{x+n}/D_x)$ may be obtained from the preceding formulae by multiplying the approximate formula for $^{1950+k}(D_{x+n}/D_x)$ by the approximate formula for $^{1950+k+n}(Q_{x+n}/D_{x+n})$ and eliminating any terms involving products of two supplementary commutation columns (*i.e.*, any products of two expressions with superscripts i or Δ). This produces the following general formula:

$$^{1950+k}\left(\frac{Q_{x+n}}{D_x}\right) \doteq \frac{Q_{x+n}}{D_x} + {}^{i}\left(\frac{Q_{x+n}}{D_x}\right) + k\,{}^{\Delta}\left(\frac{Q_{x+n}}{D_x}\right)$$

where

$$^{i}\left(\frac{Q_{x+n}}{D_x}\right) = {}^{i}\left(\frac{Q_{x+n}}{D_{x+n}}\right)\frac{D_{x+n}}{D_x} + n\,{}^{\Delta}\left(\frac{Q_{x+n}}{D_{x+n}}\right)\frac{D_{x+n}}{D_x} + {}^{i}\left(\frac{D_{x+n}}{D_x}\right)\frac{Q_{x+n}}{D_{x+n}}$$

and

$$^{\Delta}\left(\frac{Q_{x+n}}{D_x}\right) = {}^{\Delta}\left(\frac{Q_{x+n}}{D_{x+n}}\right)\frac{D_{x+n}}{D_x} + {}^{\Delta}\left(\frac{D_{x+n}}{D_x}\right)\frac{Q_{x+n}}{D_{x+n}}.$$

Any one of the standard commutation column symbols (C_x, D_x, M_x, N_x, R_x, S_x) may be substituted for Q_x in this general formula. The formulae for $^{i}(Q_{x+n}/D_{x+n})$ and $^{\Delta}(Q_{x+n}/D_{x+n})$, with a standard commutation column symbol substituted for Q_{x+n}, and the formulae for $^{i}(D_{x+n}/D_x)$ and $^{\Delta}(D_{x+n}/D_x)$ are shown at the beginning of this Appendix.