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# ACTUARIAL NOTE: TERMINAL RESERVES FROM MEAN RESERVES AND NET PREMIUMS

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FREQUENTLY calculations or tabulations must be made in terms of terminal reserves. Gross premium valuation and terminal reserves released on surrender, lapse or change are examples. It would sometimes be more convenient and efficient to be able to work from mean reserves and net premiums. This is especially true of a company which uses a seriatim valuation method and carries mean reserves and the net premium on the standard valuation card. This note suggests a very simple approximate formula for this purpose, an expression for the error in this approximate formula, and an exact formula for terminal reserves in terms of mean reserves, amounts of insurance, and a set of constants which vary by attained age only. As will be seen from the nature of the formulas they may be applied to individual policies or by mass methods to large groups.

Mean reserves are nearly always calculated by a formula which in effect assumes straight line interpolation between the initial and the terminal reserve. This formula may be stated for ordinary insurance, assuming for illustrative purposes an $n$ year endowment policy issued at age $x$,

$$ {}_{t}MR_{x:\overline{n}|} = \frac{{}_{t-1}V_{x:\overline{n}|} + {}_{t}V_{x:\overline{n}|} + {}_{t-1}P^{N}_{x:\overline{n}|}}{2} \tag{1} $$

where

${}_{t}MR_{x:\overline{n}|}$ = mean reserve during policy year $t$,

${}_{t}V_{x:\overline{n}|}$ = terminal reserve at the end of policy year $t$, and

${}_{t-1}P^{N}_{x:\overline{n}|}$ = net premium collected on the $t-1$ policy anniversary.

These same symbols are used to represent reserves and net premiums per dollar of insurance and to represent reserves and net premiums on a given policy or group of policies.

The quantity

$$ {}_{t}V^{\text{Approx}}_{x:\overline{n}|} = \frac{{}_{t}MR_{x:\overline{n}|} + {}_{t+1}MR_{x:\overline{n}|} - {}_{t}P^{N}_{x:\overline{n}|}}{2} \tag{2} $$

differs from the true terminal reserve by as much as two percent at very short durations, extremely young and old ages, and on unusual plans.

But in the aggregate, for a representative distribution of plans, ages and years of issue, this formula will give a very close approximation. It was applied to a closed block of business which had a very heterogeneous distribution. The aggregate error was less than five hundredths of one percent, that is, less than fifty cents per thousand dollars of reserve.

The formula

$$E_Y = \frac{[{}_tV^{\text{Approx}}_{x:\overline{n}|}] \cdot K^{(1)}_Y + [{}_tP^{\text{N}}_{x:\overline{n}|}] \cdot K^{(2)}_Y + [\text{Amount}] \cdot K^{(3)}_Y}{1 + K^{(1)}_Y} \tag{3}$$

is an exact expression for the difference between the true terminal reserve and the approximation of formula (2) at attained age $x + t = Y$. $K^{(1)}_Y$, $K^{(2)}_Y$, and $K^{(3)}_Y$ are independent of the plan of insurance, age at issue and year of issue, depending only on attained age and the valuation mortality and interest rate.

The quantities in the square brackets may be the terminal reserve approximation, net premiums, and amounts of insurance on a single policy or the sums of these quantities for any group of policies at attained age $Y$. The quantities $K^{(1)}_Y$, $K^{(2)}_Y$, and $K^{(3)}_Y$ may be expressed in a number of ways:

$$K^{(1)}_Y = \frac{1}{4}\left[\frac{vl_Y}{l_{Y-1}} + \frac{l_Y}{vl_{Y+1}} - 2\right] = \frac{1}{4}\left[\frac{D_Y}{D_{Y-1}} + \frac{D_Y}{D_{Y+1}} - 2\right] = \frac{1}{4}\left[\frac{1}{u_{Y-1}} + u_Y - 2\right]$$

$$K^{(2)}_Y = \frac{1}{4}\left[\frac{l_Y}{vl_{Y+1}} - 1\right] = \frac{1}{4}\left[\frac{D_Y}{D_{Y+1}} - 1\right] = \tfrac{1}{4}\,[u_Y - 1]$$

$$K^{(3)}_Y = \frac{1}{4}\left[\frac{vd_{Y-1}}{l_{Y-1}} - \frac{d_Y}{l_{Y+1}}\right] = \frac{1}{4}\left[\frac{C_{Y-1}}{D_{Y-1}} - \frac{C_Y}{D_{Y+1}}\right] = \tfrac{1}{4}\,[c_{Y-1} - c_Y u_Y]\ .$$

The remainder of this note is devoted to the derivation of formula (3) and an exact attained age formula for the terminal reserves in terms of mean reserves and amounts of insurance.

$${}_tV^{\text{Approx}}_{x:\overline{n}|} = {}_tV_{x:\overline{n}|} + E_{x+t} = \frac{{}_tMR_{x:\overline{n}|} + {}_{t+1}MR_{x:\overline{n}|} - {}_tP^{\text{N}}_{x:\overline{n}|}}{2} \tag{4}$$

By definition,

$${}_tMR_{x:\overline{n}|} = \frac{{}_{t-1}V_{x:\overline{n}|} + {}_tV_{x:\overline{n}|} + {}_{t-1}P^{\text{N}}_{x:\overline{n}|}}{2} \tag{5}$$

and

$${}_{t+1}MR_{x:\overline{n}|} = \frac{{}_tV_{x:\overline{n}|} + {}_{t+1}V_{x:\overline{n}|} + {}_tP^{\text{N}}_{x:\overline{n}|}}{2}\,. \tag{6}$$

Substituting (5) and (6) in (4),

$$ {}_tV_{x:\overline{n}|} + E_{x+t} = \frac{{}_{t-1}V_{x:\overline{n}|} + 2\,{}_tV_{x:\overline{n}|} + {}_{t+1}V_{x:\overline{n}|} + {}_{t-1}P^{N}_{x:\overline{n}|} - {}_tP^{N}_{x:\overline{n}|}}{4}. \quad (7) $$

From the fundamental relation

$$ [{}_tV_{x:\overline{n}|} + {}_tP^{N}_{x:\overline{n}|}]\,(1+i)\,l_{x+t} - d_{x+t} = l_{x+t+1} \cdot {}_{t+1}V_{x:\overline{n}|}, \quad (8) $$

$$ {}_{t-1}V_{x:\overline{n}|} = \frac{vl_{x+t}}{l_{x+t-1}} \cdot {}_tV_{x:\overline{n}|} + \frac{vd_{x+t-1}}{l_{x+t-1}} - {}_{t-1}P^{N}_{x:\overline{n}|} \quad (9) $$

and

$$ {}_{t+1}V_{x:\overline{n}|} = [{}_tV_{x:\overline{n}|} + {}_tP^{N}_{x:\overline{n}|}]\,(1+i)\,\frac{l_{x+t}}{l_{x+t+1}} - \frac{d_{x+t}}{l_{x+t+1}}. \quad (10) $$

Substituting (9) and (10) in (7) and combining,

$$ \begin{aligned} {}_tV_{x:\overline{n}|} + E_{x+t} = \frac{1}{4}\Bigg[&{}_tV_{x:\overline{n}|}\left(\frac{vl_{x+t}}{l_{x+t-1}} + \frac{l_{x+t}}{vl_{x+t+1}} + 2\right) \\ &+ {}_tP^{N}_{x:\overline{n}|}\left(\frac{l_{x+t}}{vl_{x+t+1}} - 1\right) + \left(\frac{vd_{x+t-1}}{l_{x+t-1}} - \frac{d_{x+t}}{l_{x+t+1}}\right)\Bigg]. \end{aligned} \quad (11) $$

Therefore,

$$ E_Y = {}_tV_{x:\overline{n}|} \cdot K^{(1)}_Y + {}_tP^{N}_{x:\overline{n}|} \cdot K^{(2)}_Y + K^{(3)}_Y \quad (12) $$

where $Y$, $K^{(1)}_Y$, $K^{(2)}_Y$, and $K^{(3)}_Y$ are as stated before.

However, under practical circumstances ${}_tV^{\text{Approx}}_{x:\overline{n}|}$ will be known and ${}_tV_{x:\overline{n}|}$ will not. Substituting

$$ {}_tV_{x:\overline{n}|} = {}_tV^{\text{Approx}}_{x:\overline{n}|} - E_Y \quad (13) $$

in (12) and solving for $E_Y$ gives

$$ E_Y = \frac{{}_tV^{\text{Approx}}_{x:\overline{n}|} \cdot K^{(1)}_Y + {}_tP^{N}_{x:\overline{n}|} \cdot K^{(2)}_Y + K^{(3)}_Y}{1 + K^{(1)}_Y} \quad (14) $$

which is per dollar of insurance. Formula (3) is the same relation for a general amount of insurance.

A very simple expression for terminal reserves in terms of mean reserves and amounts of insurance may be obtained from relations (5) and (9). (5) may be rewritten

$$ {}_{t-1}V_{x:\overline{n}|} + {}_{t-1}P^{N}_{x:\overline{n}|} = 2\,{}_tMR_{x:\overline{n}|} - {}_tV_{x:\overline{n}|}; \quad (15) $$

(9) may be rewritten

$$ {}_tV_{x:\overline{n}|} = [{}_{t-1}V_{x:\overline{n}|} + {}_{t-1}P^{N}_{x:\overline{n}|}]\,\frac{D_{x+t-1}}{D_{x+t}} - \frac{C_{x+t-1}}{D_{x+t}}. \quad (16) $$

Substituting (15) in (16) and solving for ${}_tV_{x:\overline{n}|}$,

$$ {}_tV_{x:\overline{n}|} = \frac{2\,{}_tMR_{x:\overline{n}|} \cdot D_{x+t-1} - C_{x+t-1}}{D_{x+t} + D_{x+t-1}}\,; \qquad (17) $$

or at attained age $x + t = Y$

$$ [{}_tV_{x:\overline{n}|}] = [{}_tMR_{x:\overline{n}|}] \cdot K_Y^{(4)} - [\text{Amount}] \cdot K_Y^{(5)}\,. \qquad (18) $$

The quantities in square brackets are again attained age summations and

$$ K_Y^{(4)} = \frac{2\,D_{Y-1}}{D_Y + D_{Y-1}} $$

$$ K_Y^{(5)} = \frac{C_{Y-1}}{D_Y + D_{Y-1}}\,. $$

It should be noted that for the calculation of terminal reserves for a given interval within a single calendar year we need for formula (18) an attained age sorting of the mean reserves that are usually referred to as the "previous" year's, that is, the reserves that would have been used for the statement of December 31 of the previous year, and corresponding amounts of insurance. For each attained age within a group we must carry out two multiplications, and then a summation for the group.

To apply formula (2) which is approximate, we need the "previous" and "current" year's mean reserves and the net premiums due on the policy anniversary for which the terminal reserve is being calculated. We do not need to sort by attained age; only a simple crossfooting and division by two is required.

Where terminal reserves are needed for a large number of groups and precision is not essential, formula (2) may be more advantageous than formula (18), even though it is not exact.

Terminal reserves calculated by either of these formulas will be subject to error due to the rounding off of the mean reserves when they were originally calculated. This error will be quite small in the aggregate and may be determined within close limits by the methods described by Chalmers L. Weaver, "Allowance for Rounding Errors in the Summation Check," *TASA* XLVIII, 267.