Estimation of Probability of Defaults (PD) for Low-Default Portfolios: An Actuarial Approach

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Abstract:

Global financial crises like the one recently experienced affect both large and small institutions. Today, when there is heightened need for enhanced risk management tools, some entities are unable to employ sophisticated mechanisms due to limited data availability. Moreover, from the Basel Committee on Banking Supervision's point of view via Basel II and Basel III, the internal ratings-based (IRB) approach requires institutions have some reliable estimates of default probabilities for each rating grade. Taking the work of previous researches a step further, this paper intends to propose a new dynamic mechanism for the risk management industry to calculate probabilities of default (PD). Through this, we calculate the realized probability of defaults and Bayesian estimates in the initial phase and then, using these estimates as inputs for the core model, we generate implied probability of default through actuarial estimation tools and different probability distributions. This mechanism is specialized to work best for low-default portfolios (LDPs). Furthermore, scenario testing is adopted to validate the model against any model-specific bias.

Key Words: *Probability of Defaults (PDs), Realized PDs, Bayesian Estimates, Probability Distributions*

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1. Introduction

Today, in this gigantic pasture of risk management, improved credit risk management has become the need of financial institutions around the world. Specifically, many financial institutions have either moved or are about to move toward the internal ratings-based (IRB) approach proposed by Basel Committee on Banking Supervision's Basel II. The most important step in switching to the IRB approach—whether foundation or advanced—is to determine probability of default (PD) for each risk grade. Probability of default has much significance as it is one of the core parts for improved allocation of capital, pricing, client judgment, regulatory compliance and, finally, monitoring of high-risk customers. Due to these significant reasons, a financial institution should be assured that the PD determination is sophisticated and, more importantly, shows the true picture of the portfolio in present, as well as future, scenarios.

Many financial institutions use long-term realized probability of default for calculating capital charges but this methodology has its limitations. Another issue raised in last few years is the estimation of probability of default for low-default portfolios (LDPs). For LDPs, realized PDs cannot show the true behavior of defaults. Less defaults or data always creates a hurdle in determining the true probability of default. Despite that, realized probability of defaults cannot be ignored and should be used as an input in determining the final results.

Another important property to take into account is the posterior probability of default of each grade. Knowledge of how specific grades perform within the default portfolio or, alternatively, the weight of default of each grade within the portfolio should also be used as input to evaluate this behavior. Bayes' theorem is a widely used criteria to obtain each grade's weight of default within the total number of the portfolio's defaults. This paper also uses the Bayesian estimates inputs for the model.

The subprime crisis taught financial institutions several lessons in enhanced risk management. For this practical reason, we believe every low-grade portfolio should take into account the behavior of a higher-grade portfolio. Big organizations having better credit ratings start to default, and, simultaneously, organizations with lower ratings follow suit. This paper captures this relationship between the grades through specific models and brief cases.

Taking into account all of the above features, we propose a new mechanism to obtain the probability of default for every grade. This model is very dynamic; it incorporates all the necessary aspects and returns an implied probability of default for each grade. The theme of the model is mainly based on a mechanism called convolution. Being over 100 years old with applications in signal processing, optics, engineering, statistics and actuarial sciences, practitioners must be aware of this mechanism. Also, this mechanism had been used in one of the approaches to develop operational value at risk (VaR) models through loss distribution. Convolution actually combines two probability distributions to produce a new and modified distribution. We will further explain the mechanism in the following section.

Revisiting LDPs, few practitioners have in last few years developed sophisticated mechanisms for probability of default estimations. Pluto and Tasche (2005), Kiefer (2008) and a few more practitioners proposed some refined tools and methodologies for the same purpose. Now, stepping forward, through this paper we propose another advanced mechanism that takes into account the inputs in very different manner. The model incorporates simple inputs from different angles but returns a single result in the form of implied PD, eliminating the problem of limited data.

In short, implied probability of default will be the terminology of our desired results. One of the probabilities used will be Bayesian estimates and the other will be the realized probability of default of each grade (number of defaults divided by number of customers).

2. The Model

This paper presents a new methodology for obtaining rating grades' probability of default that can be further used in the IRB approach to credit risk. This model specifically caters to the issue of LDPs for obtaining probability of defaults.[1] Another specialty of the model is to incorporate the relationship between the grades. For instance, a major change in speculative grades will result in a change in the investment grades and vice versa. This model is suitable during times of financial crises where highly rated institutions default.

The main idea behind this paper is to propose a new dynamic model that can be widely used in credit risk management to obtain probability of default. We are using an actuarial methodology of convolution, which will be the base of our model. Mathematically speaking, convolution is an operation on two functions, f(x) and g(x), that returns a third function which is actually the modified version of one of the original functions. Here, we are convoluting two probability distributions that return a modified new distribution which forms the cross of those distributions. Convolution has also been used in developing an operational VaR model but this is the first time it is being applied for credit risk management. Up till now, many practitioners have used different distributions for obtaining probability of default of each grade, but here, we are combining two probability distributions to get a new modified probability distribution. The results will definitely provide better estimates and the model can be widely used in every kind of portfolio, especially in LDPs.

Our model will utilize simple information from the portfolio. As discussed in the preceding section, the model only uses the total number of customers and the total number of defaults in each grade. One of our main concerns is to utilize the weight of default of each grade within the defaulted portfolio, which will be obtained simply by applying Bayes' theorem. It will produce the probability of default in each grade of the next customer who will be part of the portfolio. From the following portfolio, we can discover the Bayesian estimate.

[1] The use of this model is not restricted and can be applied to a variety of portfolios.

Grade	Number of Obligors	Number of Defaults
AAA	34	1
AA	56	1
A	119	3
BBB	257	2
BB	191	2
B	102	6
CCC	50	3
CC	34	1
C	12	2
Sum of Defaults		**21**

Table 1.1

Now, as Bayes' theorem says,

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}.$$

A is a percentage of obligors in a grade and B is an event of default.

Our table will provide results for each grade in this way.

Grade	Number of Obligors	Number of Defaults	Bayesian Estimates
AAA	34	1	4.76%
AA	56	1	4.76%
A	119	3	14.29%
BBB	257	2	9.52%
BB	191	2	9.52%
B	102	6	28.57%
CCC	50	3	14.29%
CC	34	1	4.76%
C	12	2	9.52%
Sum of Defaults		**21**	

Table 1.2

The above derives the Bayesian estimate, which provides the weights of default in each grade given the total number of defaults of the whole portfolio or, simply, the probabilities of each grade given the total number of defaults in that grade. This estimate can only answer the question that, given a default, what is the probability the obligor has a particular grade.

Therefore, to make this estimate useful, we will develop a probability distribution function that will enable us to calculate the probabilities of grades with multiple defaults, given the total number of defaults in that grade. For example, in the above table, for grade BBB, Bayesian estimate generates 0.0952, which shows the probability of the grade BBB if the default occurs in the portfolio or we can state that given a default in the portfolio, there is a 9.52 percent chance the default belongs to the grade BBB.

Going forward, one of our objectives is to determine the probability of the number of defaults differing from the number of default in grade BBB. For example, in our portfolio, the number of defaults in grade BBB is 2 but we want to know the probability if the number of defaults is other than 2. For this purpose, the binomial distribution is the most suited distribution, which will provide the desired probability at a varying number of defaults in a particular grade.

Considering the above example, we have 21 defaults in our portfolio and we want to know the probability of every possible occurrence of default in grade BBB.

As we know, the binomial distribution has the probability mass function (pmf),

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where the parameters are defined as,

n = total number of defaults in the portfolio,
k = number of defaults in particular grade, and
p = probability as estimated by Bayes' theorem.

By doing so, we are able to get the results for each grade (e.g., BBB in the following table) in the form shown in Table 1.3.

Grade	**BBB**
Total Defaults	2
Bayesian estimate	9.52%

Table 1.3

Hence, the estimated probabilities of default of different occurrences are generated through binomial distribution as

x	**P (X=x)**
0	0.358942365
1	0.376889483
2	0.188444741
3	0.059674168
4	0.013426688

5	0.002282537
6	0.000304338
7	3.26077E-05
8	2.85317E-06
9	2.06062E-07
10	1.23637E-08
11	6.18187E-10
12	2.57578E-11
13	8.91616E-13
14	2.54747E-14
15	5.94411E-16
16	1.11452E-17
17	1.639E-19
18	1.82111E-21
19	1.43772E-23
20	7.1886E-26
.	.
.	.

Table 1.4

Similar tables for remaining grades will be illustrated later in the paper.

Up till now, we have generated probabilities of default by just using the actual and total number of defaults in the portfolio. We have not taken into account the number of customers in each grade (or the default frequencies). Next we take into account the above as well and generate a frequency distribution, with Poisson distribution being the most suitable.

Refer to Table 1.1; we first calculate the parameter of the distribution, lambda (λ), which will take the impact of number of obligors and defaults against them in each grade. Results are shown in Table 1.5.

Grade	Number of Obligors	Number of Defaults	Lambda (λ)
AAA	34	1	2.9%
AA	56	1	1.8%
A	119	3	2.5%
BBB	257	2	0.8%
BB	191	2	1.0%
B	102	6	5.9%
CCC	50	3	6.0%
CC	34	1	2.9%
C	12	2	16.7%

Table 1.5

Once the lambda for each grade has been estimated, we can fit the Poisson distribution, results of which will be further included in our next step, convolution.

As we know, the Poisson distribution's pmf is:

$$p(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x = 0, 1, 2, \cdots$$

where,

λ = frequency of default in each grade, and
x = number of incremental defaults in the specific grade.

Poisson distribution will generate the probabilities of incremental default in every grade and these results will then be injected into our foundation model, convolution. In our example of grade BBB, the results follow.

λ	**0.0077821**
n	**P (N=n)**
0	0.9922481
1	0.0077218
2	0.0000300
3	0.0000001
4	0.0000000
5	0.0000000
6	0.0000000
7	0.0000000
.	.
.	.

Table 1.6

The results after running the convolution model provide a matrix for every grade. The resultant matrix for our example of BBB grade is provided here.

	f(x)	f1(x)	f2(x)	f3(x)	f4(x)	f5(x)	f6(x)	f7(x)	
0	1.00								0.96935110
1	0.00	0.00186750							0.00005635
2	0.00	0.00828704	0.00000349						0.00025006
3	0.00	0.02417052	0.00003095	0.00000001					0.00072934
4	0.00	0.05211769	0.00015895	0.00000009	0.00000000				0.00157269
5	0.00	0.08860008	0.00059526	0.00000064	0.00000000	0.00000000			0.00267373
6	0.00	0.12367094	0.00177894	0.00000336	0.00000000	0.00000000	0.00000000		0.00373253
7	0.00	0.14575503	0.00444980	0.00001402	0.00000001	0.00000000	0.00000000	0.00000000	0.00440015
8	0.00	0.14803246	0.00959340	0.00004890	0.00000007	0.00000000	0.00000000	0.00000000	0.00447129
9	0.00	0.13158441	0.01818230	0.00014723	0.00000033	0.00000000	0.00000000	0.00000000	0.00397901
10	0.00	0.10362272	0.03073178	0.00039115	0.00000126	0.00000000	0.00000000	0.00000000	0.00314118
11	0.00	0.07300692	0.04683131	0.00093130	0.00000425	0.00000001	0.00000000	0.00000000	0.00222493
12	0.00	0.04638981	0.06490359	0.00201058	0.00001285	0.00000003	0.00000000	0.00000000	0.00143027
13	0.00	0.02676335	0.08239097	0.00397180	0.00003526	0.00000012	0.00000000	0.00000000	0.00084628
14	0.00	0.01409855	0.09637515	0.00723185	0.00008870	0.00000038	0.00000000	0.00000000	0.00047071
15	0.00	0.00681430	0.10440992	0.01220961	0.00020614	0.00000116	0.00000000	0.00000000	0.00025471
16	0.00	0.00303449	0.10522721	0.01920946	0.00044535	0.00000324	0.00000001	0.00000000	0.00014108
17	0.00	0.00124950	0.09903804	0.02828389	0.00089914	0.00000843	0.00000004	0.00000000	0.00008435
18	0.00	0.00047724	0.08734631	0.03911759	0.00170386	0.00002055	0.00000011	0.00000000	0.00005561
19	0.00	0.00016955	0.07240559	0.05098156	0.00304198	0.00004707	0.00000030	0.00000000	0.00003937
20	0.00	0.00005616	0.05115924	0.06279133	0.00513337	0.00010174	0.00000080	0.00000000	0.00002603
21	0.00	0.00001738	0.04175177	0.07326154	0.00821128	0.00020818	0.00000144	0.00000001	0.00002049
22	0.00	0.00000504	0.02917880	0.08114747	0.01248175	0.00040447	0.00000483	0.00000003	0.00001425
23	0.00	0.00000137	0.01934681	0.08548131	0.01807050	0.00074806	0.00001096	0.00000007	0.00000954
24	0.00	0.00000035	0.01219252	0.08578059	0.02496707	0.00132003	0.00002467	0.00000018	0.00000616
25	0.00	0.00000008	0.01387388	0.08214354	0.03298016	0.00222698	0.00004895	0.00000046	0.00000692
26	0.00	0.00000002	0.00418532	0.07521490	0.04158442	0.00359855	0.00009674	0.00000111	0.00000233
27	0.00	0.00000000	0.00228642	0.06600578	0.05061230	0.00557856	0.00018333	0.00000254	0.00000140
28	0.00	0.00000000	0.00119425	0.05567002	0.05896651	0.00830911	0.00033366	0.00000556	0.00000083
29	0.00	0.00000000	0.00059712	0.04526262	0.06605829	0.01190756	0.00058429	0.00001164	0.00000050
30	0.00	0.00000000	0.00028612	0.03557606	0.07124197	0.01643955	0.00098555	0.00002333	0.00000031
31	0.00	0.00000000	0.00013152	0.02708284	0.07405044	0.02189192	0.00160344	0.00004501	0.00000020
32	0.00	0.00000000	0.00005806	0.01997583	0.07998445	0.02815056	0.00251916	0.00008376	0.00000013
33	0.00	0.00000000	0.00002017	0.01425728	0.07194993	0.03500002	0.00382616	0.00015067	0.00000008
34	0.00	0.00000000	0.00001005	0.00982314	0.06741280	0.04212260	0.00562359	0.00026229	0.00000006
35	0.00	0.00000000	0.00000395	0.00646835	0.05242444	0.04913030	0.00800613	0.00044228	0.00000004
36	0.00	0.00000000	0.00000149	0.00415373	0.05378246	0.05556377	0.01013803	0.00072307	0.00000002
37	0.00	0.00000000	0.00000055	0.00254205	0.04590126	0.06093202	0.01480040	0.00114534	0.00000001
38	0.00	0.00000000	0.00000036	0.00149432	0.03805710	0.06475348	0.01925087	0.00175947	0.00000001
39	0.00	0.00000000	0.00000007	0.00084508	0.03067917	0.06667035	0.02433588	0.00262346	0.00000001
40	0.00	0.00000000	0.00000002	0.00056379	0.02406176	0.06655026	0.02991997	0.00380444	0.00000000
41	0.00	0.00000000	0.00000001	0.00024296	0.01836844	0.06451775	0.03579687	0.00537768	0.00000000
42	0.00	0.00000000	0.00000000	0.00012416	0.01365145	0.06089151	0.04169667	0.00741047	0.00000000
43	0.00	0.00000000	0.00000000	0.00006164	0.00987867	0.05607366	0.04730336	0.00999722	0.00000000
44	0.00	0.00000000	0.00000000	0.00002978	0.00696097	0.05045905	0.05228268	0.01048966	0.00000000
45	0.00	0.00000000	0.00000000	0.00001403	0.01105983	0.04439653	0.05631805	0.01477617	0.00000000

Similarly, matrices for each grade have been generated that provided us with the final results of the model. The values in the last column give the convoluted probabilities for BBB grade. The number of defaults was 2 in BBB in our example; hence we are interested to pick the value calculated in front of number 2, i.e., 0.00025006. Be sure, this is not the probability of default for grade BBB. To obtain the final probability of default, we must calculate the convoluted probability against the original number of defaults in a specific grade and then the resulting cumulated probabilities will be the desired probabilities of default for that grade. Results are given in the table below.

Grade	PDs
AAA	1.08%
AA	1.74%
A	2.33%
BBB	2.55%
BB	2.85%
B	3.90%
CCC	5.28%
CC	6.36%
C	10.46%

Table 1.7

3. Scenarios

In this section, we intend to develop various scenarios and evaluate the model. We appraise the behavior of the model in different circumstances and the PD behavior in a specific grade along with its impact on the whole portfolio. For instance, increasing the number of customers, making the first probability distribution active, changes the realized probability of default and then convolutes with the second probability distribution, providing modified probability distribution to produce the implied probability of default for each grade. On the other hand, when we change the number of defaults in any grade, the first and second probability distributions both become active, and the realized probability of default and the Bayesian estimates both change and then convolute with each other to produce modified probability distribution. Finally, the implied PD for each grade will be produced. Let's create different scenarios and see the results.

3.1. Actual Portfolio

First, we gathered all the inputs and results of the actual portfolio[2] as tabulated below.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio
AAA	34	1	2.94%	4.76%	1.08%
AA	56	1	1.79%	4.76%	1.74%
A	119	3	2.52%	14.29%	2.33%
BBB	257	2	0.78%	9.52%	2.55%
BB	191	2	1.05%	9.52%	2.85%
B	102	6	5.88%	28.57%	3.90%
CCC	50	3	6.00%	14.29%	5.28%
CC	34	1	2.94%	4.76%	6.36%
C	12	2	16.67%	9.52%	10.46%
Total	**855**	**21**			

Table 2.1

3.2. Scenario 1

In our first scenario, we simply study the model behavior by increasing the number of customers in the portfolio. The details of the inputs, implied probabilities of default from the actual portfolio and the implied probability of default under the given scenario are as follows.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	68	1	1.47%	4.76%	1.08%	0.55%
AA	112	1	0.89%	4.76%	1.74%	0.88%
A	238	3	1.26%	14.29%	2.33%	1.18%
BBB	514	2	0.39%	9.52%	2.55%	1.29%
BB	382	2	0.52%	9.52%	2.85%	1.44%
B	204	6	2.94%	28.57%	3.90%	1.98%
CCC	100	3	3.00%	14.29%	5.28%	2.69%
CC	68	1	1.47%	4.76%	6.36%	3.24%
C	24	2	8.33%	9.52%	10.46%	5.44%
Total	**1,710**	**21**				

Table 2.2

[2] For the purpose of comparison, we show the final results from the actual portfolio under all scenarios.

We see the number of customers has doubled in each grade. Realized probabilities of default change and become less for each grade, Bayesian estimates are unchanged and, finally, the implied probabilities of default decrease with the realized probabilities of default.

3.3. Scenario 2

In the second scenario, we have increased the number of defaults, in fact, doubled the number in each grade. The table below shows the complete details.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	2	5.88%	4.76%	1.08%	1.55%
AA	56	2	3.57%	4.76%	1.74%	2.51%
A	119	6	5.04%	14.29%	2.33%	3.34%
BBB	257	4	1.56%	9.52%	2.55%	3.66%
BB	191	4	2.09%	9.52%	2.85%	4.08%
B	102	12	11.76%	28.57%	3.90%	5.50%
CCC	50	6	12.00%	14.29%	5.28%	7.36%
CC	34	2	5.88%	4.76%	6.36%	8.91%
C	12	4	33.33%	9.52%	10.46%	14.00%
Total	**855**	**42**				

Table 2.3

Table 2.3 shows that as the number of defaults increase, the probabilities of default also increase. However, it is the results of Bayesian estimates that are noteworthy. If we compare the Bayesian estimates of Table 2.2 with Table 2.3, we will find no change in any grade. This is because the defaults increase with the same weightage in all the grades. Therefore, in convoluted probabilities, the process only takes effect on the increment in defaults from the realized probabilities of default, while the Bayesian estimates show the same properties in both cases.

3.4. Scenario 3

Under this scenario, we try to find the relationship between implied probabilities of default and all the other inputs if both the number of defaults and number of customers increase. Applying this scenario to the realized probabilities of default would not change values since the number of customers and the number of defaults are both doubled. Here are the details after running the model.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	68	2	2.94%	4.76%	1.08%	0.80%
AA	112	2	1.79%	4.76%	1.74%	1.28%
A	238	6	2.52%	14.29%	2.33%	1.71%
BBB	514	4	0.78%	9.52%	2.55%	1.87%
BB	382	4	1.05%	9.52%	2.85%	2.08%
B	204	12	5.88%	28.57%	3.90%	2.83%
CCC	100	6	6.00%	14.29%	5.28%	3.81%
CC	68	2	2.94%	4.76%	6.36%	4.61%
C	24	4	16.67%	9.52%	10.46%	7.56%
Total	**1,710**	**42**				

Table 2.4

The above table illustrates that the implied probabilities of default under this scenario change compared to the implied probabilities of default of the actual portfolio. It is interesting to note that although inputs in both scenarios were the same, the probabilities of default have decreased. This proves the practicality and uniqueness of the model.

3.5. Scenario 4

In this scenario, we observe the behavior of the model if the defaults occur only in the higher-level grades. Let's see the results first.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	3	8.82%	8.57%	1.08%	1.92%
AA	56	7	10.71%	20.00%	1.74%	3.65%
A	119	9	5.88%	25.71%	2.33%	4.58%
BBB	257	2	0.78%	5.71%	2.55%	4.80%
BB	191	2	1.05%	5.71%	2.85%	5.09%
B	102	6	5.88%	17.14%	3.90%	6.08%
CCC	50	3	6.00%	8.57%	5.28%	7.42%
CC	34	1	2.94%	2.86%	6.36%	8.49%
C	12	2	16.67%	5.71%	10.46%	12.52%
Total	**855**	**35**				

Table 2.5

It's evident that as the number of defaults increase in the higher grades, the implied probabilities of default also increase and, as per our model, it creates an impact on the lower grades as well; hence, the implied probabilities for lower grades. This behavior happens due to higher realized probabilities of default as well as the higher Bayesian estimates for the upper grades.

3.6. Scenario 5

We want to see the behavior if the number of defaults increases only in the lower grades. By doing so, realized probabilities of default and the Bayesian estimates both will increase in the lower grades. Let's check the behavior of these changes on the whole portfolio.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	1	2.94%	3.70%	1.08%	1.07%
AA	56	1	1.79%	3.70%	1.74%	1.73%
A	119	3	2.52%	11.11%	2.33%	2.31%
BBB	257	2	0.78%	7.41%	2.55%	2.53%
BB	191	2	1.05%	7.41%	2.85%	2.82%
B	102	6	5.88%	22.22%	3.90%	3.84%
CCC	50	6	12.00%	22.22%	5.28%	5.78%
CC	34	2	5.88%	7.41%	6.36%	7.35%
C	12	4	33.33%	14.81%	10.46%	12.59%
Total	**855**	**27**				

Table 2.6

This scenario produces some interesting results. The implied probabilities of default of the lower grades increased as expected but, amazingly, the implied probabilities of default in the upper grades slightly decreased. This happens due the decreasing Bayesian estimates in the upper grades. If only realized probabilities of default are considered or only Bayesian estimates are considered, the dynamic nature of the model could not be observed.

3.7. Scenario 6

In this scenario, we will determine the impact on implied probabilities of default if only the customers in the middle grade, i.e., from BBB to B, default. The results follow.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	1	2.94%	3.03%	1.08%	1.07%
AA	56	1	1.79%	3.03%	1.74%	1.72%
A	119	3	2.52%	9.09%	2.33%	2.30%
BBB	257	4	1.56%	12.12%	2.55%	2.62%
BB	191	7	3.66%	21.21%	2.85%	3.21%
B	102	11	10.78%	33.33%	3.90%	4.63%
CCC	50	3	6.00%	9.09%	5.28%	5.97%
CC	34	1	2.94%	3.03%	6.36%	7.03%
C	12	2	16.67%	6.06%	10.46%	11.06%
Total	**855**	**33**				

Table 2.7

The realized probabilities of default as well as the Bayesian estimates of the middle grades increased. Due to this, the lower grades, i.e., from CCC to C, received a negative impact and slightly increased. Actually, the decreasing Bayesian estimates in the lower grades are netting off the implied probabilities of default in these grades, thus the implied probabilities of default increased but not as much as in the middle grades. Higher grades showed interesting behavior too as the implied probabilities of default decreased with a minimal margin. This is because the activeness of defaults in these grades decreased due to the decreasing Bayesian estimates.

3.8. Scenario 7

In this scenario, we ignore the increase or decrease in the number of defaults. However, we will see the behavior of the portfolio if the number of customers increases instead. Therefore, we doubled the number of customers in the upper grade. Here are the results.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	68	1	1.47%	4.76%	1.08%	0.55%
AA	112	1	0.89%	4.76%	1.74%	0.88%
A	238	3	1.26%	14.29%	2.33%	1.18%
BBB	257	2	0.78%	9.52%	2.55%	1.40%
BB	191	2	1.05%	9.52%	2.85%	1.70%
B	102	6	5.88%	28.57%	3.90%	2.75%
CCC	50	3	6.00%	14.29%	5.28%	4.13%
CC	34	1	2.94%	4.76%	6.36%	5.20%
C	12	2	16.67%	9.52%	10.46%	9.30%
Total	**1,064**	**21**				

Table 2.8

The results show that realized probabilities of default only decreased in the upper grades, while Bayesian estimates remained the same in the whole portfolio. In this case, the model is only taking the effect of decreasing realized probabilities of default in the upper grades while running the convolution mechanism. All other inputs are the same for the process. In the end, the implied probabilities of default show the decreasing behavior in the whole portfolio. It started with a major fall in the upper grades impacting middle and lower grades too.

3.9. Scenario 8

Similarly, in this scenario we will increase the number of customers in the middle grades, i.e., from BBB to B, given that the number of defaults remain same. Results are below.

Grades	No. of Customers	No. of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	1	2.94%	4.76%	1.08%	1.08%
AA	56	1	1.79%	4.76%	1.74%	1.74%
A	119	3	2.52%	14.29%	2.33%	2.33%
BBB	514	2	0.39%	9.52%	2.55%	2.44%
BB	382	2	0.52%	9.52%	2.85%	2.59%
B	204	6	2.94%	28.57%	3.90%	3.14%
CCC	50	3	6.00%	14.29%	5.28%	4.51%
CC	34	1	2.94%	4.76%	6.36%	5.59%
C	12	2	16.67%	9.52%	10.46%	9.69%
Total	**1,405**	**21**				

Table 2.9

Interestingly, implied probabilities of default of the middle grades decreased due to the decrement in the realized probabilities of default, while Bayesian estimates remained the same for the entire portfolio. This is the main reason the higher grades, i.e., from AAA to A, faced no impact in their implied probabilities of default. However, the implied probabilities of default of the lower grades decreased as per the mechanism of the model taking the decreasing effect from middle grades.

3.10. Scenario 9

Similarly as in scenarios 7 and 8, in the final scenario, we increase the number of customers. However, this time we will observe the behavior of the model by showing the increase in the lower grades, i.e., from CCC to C. Results are shown below.

Grades	No. of Customers	No of Defaults	Realized PDs	Bayesian Estimates	Implied PDs of Actual Portfolio	Implied PDs in Current Scenario
AAA	34	1	2.94%	4.76%	1.08%	1.08%
AA	56	1	1.79%	4.76%	1.74%	1.74%
A	119	3	2.52%	14.29%	2.33%	2.33%
BBB	257	2	0.78%	9.52%	2.55%	2.55%
BB	191	2	1.05%	9.52%	2.85%	2.85%
B	102	6	5.88%	28.57%	3.90%	3.90%
CCC	100	3	3.00%	14.29%	5.28%	4.61%
CC	68	1	1.47%	4.76%	6.36%	5.16%
C	24	2	8.33%	9.52%	10.46%	7.36%
Total	**951**	**21**				

Table 2.10

We can see that the implied probabilities of default from the grades AAA to B remain unchanged from the actual portfolio. Bayesian estimates and realized probabilities of default both remain unchanged from the previous scenario. That is the reason there was no change in that range. In contrast, the lower grades, i.e., from CCC to C, possess decreasing implied probabilities of default.

4. Open Issues

As this is a very new mechanism for calculating PD, there are a few limitations to be discussed. In our next version, we will come up with further workings, including overcoming limitations.

- The first shortcoming is the decision to select the distributions. Binomial and Poisson distributions were very sophisticated as per the portfolio and the mechanism; however, we can use other distributions as well. It should purely be the practitioner's choice.

- The second shortcoming is the practice to cumulate the PDs of upper grades with a specific grade's PD. According to our mechanism, every grade should have a relation to the performance of other grades. If the PD of a better grade increases, it should impact its comparative lower grade in such a way that the PDs for lower grades increase as well. However, in this case, PDs of the higher grades should remain the same.

5. Conclusion

In this paper, we introduced a new model to estimate the probability of default for low-default portfolios. The methodology is based on an actuarial mechanism named convolution. We calculated Bayesian probability and realized PD for each scenario by using these two estimates. We generated an implied distribution of each scenario with the convolution technique. Besides that, we have developed different scenarios to see the behavior of the model. The model justified its performance very well. This model is very practical and related organizations can use it accordingly.

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