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ACTUARIAL NOTE: RESERVES BY DIFFERENT MORTALITY TABLES

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CECIL J. NESBITT:

It is a coincidence that in the same number of the *Transactions* there should appear discussions of two of the main methods of comparing reserves on different mortality bases. The Lidstone method, as treated by Mr. Baillie, is, I believe, a somewhat more flexible tool than the method discussed by Mr. Gershenson, but both methods are of interest, as the authors have demonstrated, and both contain difficulties. These difficulties were possibly what Mr. Gershenson had in mind when he wrote that he "avoided some theoretical details and limited this note to fundamentals." As pointed out by Dumas (*JIA* LXII, 109), these difficulties appear if one attempts to make statements holding for the complete range of the mortality table. If ω is the limiting age of the mortality table ($l_{\omega-1} \neq 0$, $l_\omega = 0$), statements such as $q'_x = q_x + k/v\ddot{a}_{x+1}$ clearly cannot hold when $x = \omega - 1$.

I believe that the theorems for the note should have been stated for a range of ages excluding a few ages before age ω. It would then be necessary, as indicated by Dumas, to be a little more explicit about k. For example, Theorem I might be stated as:

If, for $\alpha \leq x < \beta$ and $x + t \leq \beta$,

$${}_tV_x = {}_tV'_x$$

then, for $\alpha \leq x < \beta$,

$$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}},$$

where

$$k = \frac{\ddot{a}_\beta}{\ddot{a}'_\beta} - 1 \left(= \frac{\ddot{a}_\alpha}{\ddot{a}'_\alpha} - 1 \right).$$

A statement for Theorem II would then be:

If, for $\alpha \leq x < \beta$,

$$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}},$$

where

$$k = \frac{\ddot{a}_\beta}{\ddot{a}'_\beta} - 1,$$

then, for $\alpha \leq x < \beta$ and $x + t \leq \beta$,

$$_tV_x = {}_tV'_x\,.$$

The restated Theorem II could be proved by the method of the note or by use of Dumas' Theorem III. By the way, Dumas probably had his Theorem III in mind when he spoke of proving the converse of the note's Theorem I "by similar methods."

For the remaining Theorems of the note some similar limitation on the range of ages should be imposed, and some relationship given for θ_x, but I have not had time to thoroughly study the question. Of course, there is the other suggestion of Dumas, namely, to modify the death benefit in the final year, but I prefer his first suggestion about limiting the age range.

Mr. Gershenson's note should serve its purpose of clarifying for students the proofs of its theorems.

THOMAS N. E. GREVILLE:

I think many of us at some time have been quite dissatisfied with Spurgeon's treatment of the problem of equal reserves by different mortality tables, in which he proves a condition necessary and then treats it as sufficient. Mr. Gershenson is to be warmly commended for bringing this matter out into the open and discussing it in some detail.

Besides the paper of Dumas to which the note refers, this question has been discussed by Steffensen in his small book *Some Recent Researches in the Theory of Statistics and Actuarial Science* (pages 9–11) published for the Institute of Actuaries by the Cambridge University Press in 1930, and by Lidstone in *JIA* LXI, 343–5. The reader who is interested in pursuing the subject may wish to consult these references.

I agree that a "proof" of Theorem II by methods similar to those used in proving Theorem I would necessarily involve circular reasoning. It is curious that Dr. Dumas implies such a proof is possible, since its impossibility is almost an obvious consequence of the results he arrives at in the part of his paper immediately following this remark.

EFFECT OF MANNER OF TERMINATING THE MORTALITY TABLE

While I sympathize with Mr. Gershenson's desire to avoid "theoretical details" for the benefit of students, there is one such detail to which I wish he had made at least a brief reference. My point is that the status of his Theorems I and II is somewhat nebulous without some indication as to whether he had in mind mortality tables terminating at a definite limiting age, or tables (such as Makehamized tables, for instance) which, at least in theory, have no limiting age.

If there is a definite limiting age, it is not difficult to see that the equation ${}_tV_x = {}_tV'_x$ cannot hold strictly "for all values of x and t," except in the trivial case in which $k = 0$ and the two mortality tables are identical. If both tables have the same limiting age ω, then we must have $\ddot{a}_{\omega-1} = \ddot{a}'_{\omega-1} = 1$, and it follows from the relation $\ddot{a}_x = (1 + k)\ddot{a}'_x$ that $k = 0$. On the other hand, if the limiting ages are different, and if, for example $\omega' < \omega$, then ${}_{\omega'-x}V'_x = 1$, while ${}_{\omega'-x}V_x < 1$. Also, there will be some attained age or ages beyond ω' at which the unprimed reserves exist and the primed ones do not.

If there is no limiting age, strict equality of all ordinary life reserves can occur in a nontrivial case. However, Steffensen points out that, even in this situation, certain restrictions are imposed on the value of k by the requirement that the values of q'_x must be confined to the interval from 0 to 1. It is easily shown that the condition is equivalent to

$$-vq_x\ddot{a}_{x+1} \leq k \leq a_x \,. \tag{1}$$

The expression $vq_x\ddot{a}_{x+1}$ will be found to have a minimum value near the age at which the minimum value of q_x occurs. If this minimum value is denoted by c, we must have $k \geqslant -c$. If q_x increases with increasing x, a_x of course decreases. Hence the condition $k \leq a_x$ will be satisfied for all values of x if $k \leq a_\infty$. Hence the condition (1) is equivalent to

$$-c \leq k \leq a_\infty \,. \tag{2}$$

As Steffensen points out, it is not mathematically necessary that $\mu_\infty = \infty$ (although this is the case for a mortality table graduated by Makeham's law). He shows, however, that the conditions $\mu_\infty = \infty$, $q_\infty = 1$, and $a_\infty = 0$ are all equivalent. By allowing x to approach ∞ in the relation

$$a_x = vp_x\,(1 + a_{x+1})\,,$$

and solving for a_∞, we find that

$$a_\infty = \frac{vp_\infty}{1 - vp_\infty}\,.$$

It follows that if $q_\infty < 1$, then $a_\infty > 0$; and also, since $p_\infty \leq 1$, we have $a_\infty \leq i$. If $q_\infty = 1$, then $a_\infty = 0$, as previously stated, and k must be negative. Since c is a rather small quantity (of the order of .02 or .03 for modern mortality tables), it is apparent that the conditions (2) rather severely restrict the range of possible values of k.

In Lidstone's remarks previously referred to, which are a commentary on Steffensen's treatment, he seems to have mistakenly assumed that a_∞ and a'_∞ must both be zero, and thus reached the unwarranted conclusion that "no value of k, positive or negative, will fulfill all the necessary conditions."

SUGGESTED RESTATEMENT OF THEOREMS I AND II

The questions we have been discussing, relative to the manner of terminating the mortality table, arise in Mr. Gershenson's note in his proof of Theorem II, at the top of page 70, where he says "Continuing and collecting terms. . . ." A rigorous treatment of this step would require some such discussion as we have just given. However, this difficulty can be entirely avoided by restating Theorems I and II in the following manner (equivalent to Dumas' Theorem III):

Theorem I: If

$$_tV_x = {}_tV_x'$$

for all values of x and t such that $x + t \leq y$, then

$$q_x' = q_x + \frac{k}{v\ddot{a}_{x+}}$$

for all values of x less than y.

Theorem II: If

$$q_x' = q_x + \frac{k}{v\ddot{a}_{x+1}}$$

for all values of x less than y, and if

$$\ddot{a}_y = (1 + k)\,\ddot{a}_y' ,$$

then

$$_tV_x = {}_tV_x'$$

for all values of x and t such that $x + t \leq y$.

These theorems are entirely adequate for all practical purposes, since the fixed age y can be chosen sufficiently high so that reserves at attained ages beyond y are of no practical importance.

The proof of the revised Theorem I does not differ in any important respect from that given by Mr. Gershenson. However, the new Theorem II is most conveniently proved by induction in the following manner:

Proof of Theorem II:

Suppose that, for some age u less than or equal to y,

$$\ddot{a}_u = (1 + k)\,\ddot{a}_u' .$$

Then,

$$1 + vp_{u-1}\ddot{a}_u = 1 + v\left(p_{u-1}' + \frac{k}{v\ddot{a}_u}\right)(1 + k)\,\ddot{a}_u' = (1 + k)(1 + vp_{u-1}'\ddot{a}_u') ,$$

or

$$\ddot{a}_{u-1} = (1 + k)\,\ddot{a}_{u-1}' .$$

Since, by hypothesis,

$$\ddot{a}_y = (1 + k)\,\ddot{a}_y' ,$$

it follows by induction that

$$\ddot{a}_x = (1+k)\,\ddot{a}'_x$$

for all values of x less than y. Therefore,

$${}_tV_x = {}_tV'_x$$

for all values of x and t such that $x + t \leq y$.

UNEQUAL RESERVES

In the case of unequal reserves, fewer difficulties arise. However, if there is a definite limiting age, a strict interpretation of the theorems would require both mortality tables to terminate at the same age; and there may be difficulty, in some instances, in maintaining the increasing or decreasing character of the function θ_x right up to the very end of the table. Moreover, the use of such phrases as "Continuing and collecting terms," and representing a lengthy series by the first three terms followed by three dots, make Mr. Gershenson's proof of Theorem III not fully convincing. For these reasons, I propose the following revised statement:

Theorem III: If, for all values of x less than y,

$$q'_x = q_x + \frac{\theta_x}{v\ddot{a}_{x+1}}$$

and θ_x is an increasing function, and if

$$z_y > \theta_{y-1}\ ,$$

where z_x is defined by the relation

$$\ddot{a}_x = (1+z_x)\,\ddot{a}'_x\ ,$$

then

$${}_tV'_x > {}_tV_x$$

for all values of x and t such that $x + t \leq y$.

Proof:

Since

$$p_x = p'_x + \frac{\theta_x}{v\ddot{a}_{x+1}}\ ,$$

$$(1+z_x)\,\ddot{a}'_x = \ddot{a}_x = 1 + vp_x\ddot{a}_{x+1} = 1 + v\left(p'_x + \frac{\theta_x}{v\ddot{a}_{x+1}}\right)(1+z_{x+1})\,\ddot{a}'_{x+1}$$

$$= 1 + vp'_x(1+z_{x+1})\,\ddot{a}'_{x+1} + \theta_x = \ddot{a}'_x + \theta_x + vp'_x z_{x+1}\ddot{a}'_{x+1}\ .$$

Therefore,

$$z_x\ddot{a}'_x = \theta_x + vp'_x z_{x+1}\ddot{a}'_{x+1}\ , \tag{3}$$

a relation obtained by Mr. Gershenson in a somewhat different manner.

It will now be shown by induction that

$$z_x > \theta_x$$

for all values of x less than y. Suppose that, for some u less than $y-1$,

$$z_{u+1} > \theta_{u+1} .$$

Then, since θ_x is increasing,

$$z_{u+1} > \theta_u .$$

By equation (3),

$$z_u \ddot{a}'_u > \theta_u + v p'_u \theta_u \ddot{a}'_{u+1} , \qquad \text{or} \qquad \theta_u \ddot{a}'_u . \tag{4}$$

Therefore

$$z_u > \theta_u .$$

However, setting $x = y - 1$ in equation (3), since by hypothesis

$$z_y > \theta_{y-1} , \tag{5}$$

it follows from the relations (4) that

$$z_{y-1} > \theta_{y-1} .$$

This completes the induction.

Finally, we obtain from equation (3), in the same manner as Mr. Gershenson, the relation

$$z_x - \theta_x = (z_{x+1} - z_x) a'_x$$

for all values of x less than y, from which the remainder of the proof follows as given by him.

Using this form of the theorem, it is not necessary to restrict the individual values of q'_x for ages y and beyond. They are subject only to the "over-all" restriction that they be such that

$$\ddot{a}'_y < \frac{\ddot{a}_y}{1 + \theta_{y-1}} ,$$

which is equivalent to the condition (5). The general Theorem IV may be restated as follows:

Theorem IV: If

$$q'_x = q_x + \frac{\theta_x}{v \ddot{a}_{x+1}}$$

for all values of x less than y, and if

$$\ddot{a}_y = (1 + z_y) \ddot{a}'_y ,$$

then for all values of x and t such that $x + t \leq y$,

(a) $\quad {}_tV'_x = {}_tV_x \quad$ if $\quad \theta_x$ is constant and $z_y = \theta_{y-1}$;

(b) $\quad {}_tV'_x > {}_tV_x \quad$ if $\quad \theta_x$ is an increasing function and $z_y > \theta_{y-1}$;

(c) $\quad {}_tV'_x < {}_tV_x \quad$ if $\quad \theta_x$ is a decreasing function and $z_y < \theta_{y-1}$.

THE CONTINUOUS CASE

In the last part of the paper by Dumas, he considers the case of a policy with continuous premiums and the sum assured payable at the moment of death. This part of his paper is a masterpiece of confusion (heightened by the fact that he switches here from the ordinary life reserves he has considered in the remainder of the paper to a consideration of endowment reserves), and he arrives at the completely erroneous conclusion that "When the continuous method is used, it is impossible to find two mortality tables producing the same policy values either for whole life or endowment assurances."

By a method entirely analogous to that which applies to the ordinary kind of reserves, it is easily found that equality of continuous ordinary life reserves is equivalent to the condition

$$\bar{a}_x = (1+k)\,\bar{a}'_x \tag{6}$$

for all values of x. Differentiating this relation and simplifying as in the proof of Theorem I, we easily obtain the condition

$$\mu'_x = \mu_x + \frac{k}{\bar{a}_x}. \tag{7}$$

The crucial step in demonstrating the incorrectness of Dumas' conclusion is to reverse the process and show that if the relation (7) holds for all values of x, the equation (6) follows. We have

$$ {}_t p'_x = e^{-\int_0^t \mu'_{x+u}du} = {}_t p_x e^{-k\int_0^t du/\bar{a}_{x+u}}\,.$$

Therefore,

$$\bar{a}'_x = \int_0^\infty v^t\,{}_t p'_x dt = \int_0^\infty v^t\,{}_t p_x e^{-k\int_0^t du/\bar{a}_{x+u}} dt\,.$$

This may be integrated by parts, taking

$$U = e^{-k\int_0^t du/\bar{a}_{x+u}}\,; \qquad dV = v^t\,{}_t p_x dt\,.$$

It follows that

$$dU = -\frac{k\,dt}{\bar{a}_{x+t}}\,e^{-k\int_0^t du/\bar{a}_{x+u}}\,; \qquad V = -\,v^t\,{}_t p_x \bar{a}_{x+t}\,,$$

since

$$v^t\,{}_t p_x \bar{a}_{x+t} = \int_t^\infty v^u\,{}_u p_x du\,.$$

This gives

$$\bar{a}'_x = \Big[-\,v^t\,{}_t p_x \bar{a}_{x+t} e^{-k\int_0^t du/\bar{a}_{x+u}}\Big]_0^\infty - k\int_0^\infty v^t\,{}_t p_x e^{-k\int_0^t du/\bar{a}_{x+u}} dt$$

$$= \bar{a}_x - k\,\bar{a}'_x\,.$$

In other words,

$$\bar{a}_x = (1+k)\,\bar{a}_x' .$$

In case some reader should wonder where the mistake is in Dumas' "proof," it occurs near the top of page 116, where he apparently assumes that, in order for a certain integral to be identically zero, the integrand must be identically zero. This is not at all necessary in the case he considers. The last sentence of his final paragraph (in which he purports to reach the same conclusion by general reasoning) is also a *non sequitur*.

(AUTHOR'S REVIEW OF DISCUSSION)

HARRY GERSHENSON:

Dr. Nesbitt and Dr. Greville have given careful attention to the "theoretical details" which were intentionally avoided in the note. Students who are interested in a thorough study of the matter will find their discussions cogent and stimulating. Although my own preference is to relieve the majority of students of a study of these details, there is no doubt that the material in the discussions is a valuable addition to actuarial literature.

I agree with Dr. Greville's restatement of the general Theorem IV, and wish to add only the comment that a similar general theorem holds for the case of continuous reserves. Dr. Greville is, undoubtedly, familiar with this fact although he limited his discussion of the continuous case to the situation involving equal reserves.

As Dr. Nesbitt points out, the Lidstone method is probably more flexible than the method treated in the note. On the other hand, I wonder whether the complexities of the "Equation of Equilibrium" do not confront the student with so many problems that the flexibility becomes lost in the maze. In any event, the method treated in the note seems to be adequate to handle the problems which come up in practical situations. It is interesting to note that Professor Baillie treats exactly the same problems as does my note (with the exception that he adds an analysis of limited-payment policies).

The two methods have been perhaps the most confusing sections of Spurgeon's textbook, and Dumas' paper has helped create chaos out of disorder. Whether future students prefer the method of the Note or the Lidstone method as clarified by Professor Baillie, their path will probably be smoother than that of previous generations. For this, as for so many other helpful analyses, they are indebted to Dr. Nesbitt and Dr. Greville.