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ACTUARIAL NOTE: THE EQUATION OF EQUILIBRIUM

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CECIL J. NESBITT:

The Lidstone theory concerning the effect on reserves of a change in the interest or mortality basis, as presented in Spurgeon's text, is a fine example of actuarial reasoning. Most students, however, will be somewhat happier with an algebraic discussion such as is presented in this note. We have used such an algebraic presentation at Michigan for several years but have not acquired complete ease of mind concerning the subject. Difficulties exist in regard to the limiting ages in the mortality tables and also in regard to the final year under an endowment insurance.

An example of this latter difficulty is the situation in formula (7) when $t = m - 1$. For this value of t, $1 - {}_{t+1}\mathrm{V} = 1 - {}_{m}\mathrm{V} = 0$; and then from (7), $\mathrm{P}' = \mathrm{P}$ and, for $t < m - 1$, $q' = q$, and the two reserve bases are identical with the possible exception of the rate of mortality in the final year. The same difficulty appears in formula (8), where for $t = m - 1$ a denominator may become zero.

Sometimes the difficulty may be avoided by limiting the range of ages involved, as indicated in my discussion of Mr. Gershenson's note. For example, it is not quite correct to say (as I have sometimes done) that a necessary and sufficient condition for terminal reserves on the two bases to be the same for n years is $R_t = 0$, $(t = 0, 1, 2, \ldots, n - 1)$. (Students will recognize $R_t = 0$ as the famous Equation of Equilibrium). The correct statement would include some proviso about D'_{x+n} being assumed different from 0. Without the proviso the condition may not be sufficient.

For an m-year endowment insurance, with premiums payable for the whole term, some premium inequalities can be obtained. The trivial case where $m = 1$ is excluded.

I. Let us assume that $q' = q$ but $i' < i$. Then $R_t = ({}_t\mathrm{V} + \mathrm{P})(i' - i) + (\mathrm{P}' - \mathrm{P})(1 + i')$ constantly decreases and hence, as shown in the note, R_t must start positive and end negative. From $R_0 > 0$, we have

$$\mathrm{P}\,(i' - i) + (\mathrm{P}' - \mathrm{P})\,(1 + i') > 0$$

or

$$\mathrm{P}' > \frac{\mathrm{P}\,(1 + i)}{1 + i'}\,.$$

From $R_{m-1} < 0$, we have

$$v(i'-i)+(P'-P)(1+i') < 0$$

or

$$P' < P + \frac{i-i'}{(1+i)(1+i')} .$$

Putting these together we obtain the bracketing relation

$$P + \frac{i-i'}{(1+i)(1+i')} > P' > \frac{P(1+i)}{1+i'} .$$

II. For this second case assume that $i' = i$ and $q'_{x+t} = q_{x+t} + h$, $h > 0$. Then $R_t = (P' - P)(1 + i) - h(1 - {}_{t+1}V)$ constantly increases and so R_t must start negative and end positive.

$R_0 < 0$ yields

$$(P'-P)(1+i) - h(1 - {}_1V) < 0$$

so that

$$P' < P + vh(1 - {}_1V) < P + vh .$$

$R_{m-1} > 0$ leads to the obvious relation $P' > P$.

The complete relation then is

$$P + vh > P' > P .$$

The author's note should serve to stimulate thinking on the subject. His treatment of the limited payment case is new to me and will help to round out my ideas.

FRANK A. WECK:

Mr. Baillie's excellent note is of importance because it helps the student to understand and use an important actuarial tool. I should merely like to show that another equation, which may be called the reserve equation, is also useful in dealing with the type of problem discussed by Mr. Baillie, and, incidentally, also discussed by Mr. Gershenson in his Actuarial Note on page 68. For some problems the reserve equation with its varying premium would seem to be an easier tool to use than the equation of equilibrium with its remainder term.

What I have called the reserve equation may be expressed in the following general form, covering policy durations from s to t:

$$\left.\begin{array}{r} {}_s\bar{V} + \delta' \int_0^k {}_{s+h}\bar{V}\,dh + \int_0^k {}_{s+h}\bar{P}'\,dh - \int_0^k \mu'_{s+h}(1 - {}_{s+h}\bar{V})\,dh = {}_{s+k}\bar{V} , \\ (0 \le k \le t - s) \end{array}\right\} (1)$$

(Continuous functions are used here for simplicity of analysis, but the equation can, of course, also be expressed in traditional form using discrete functions.) Primed functions designate a particular mortality and interest assumption which may differ from the mortality and interest assumption used in determining the continuous reserves, $\bar{V}$, in the equation. The equality between the right and left sides of the equation is preserved by letting the premium, ${}_{s+h}\bar{P}'$, vary with duration. The symbol ${}_{s+h}\bar{P}$ represents the annual rate of premium at duration $s+h$.

As the usefulness of this equation is perhaps most easily shown by considering a few specific questions, some examples will be taken based on the whole life policy. The general method of approach, including applications to other questions and to other plans of insurance, should then be apparent.

Accordingly, let the reserves, $\bar{V}$, in equation (1) be for the whole life plan at a particular age at issue, and let s be zero and $t=\omega-x$, so that equation (1) reduces to

$$\delta'\int_0^k {}_h\bar{V}\,dh+\int_0^k {}_h\bar{P}'\,dh-\int_0^k \mu_h'(1-{}_h\bar{V})\,dh={}_k\bar{V}\,,\quad (0\leq k\leq\omega-x)\ (2)$$

where ω is the limiting age of the mortality table.

Here it should be observed that if the mortality and interest assumption in the primed functions is identical with the mortality and interest assumption in the reserves, $\bar{V}$, then ${}_h\bar{P}'$ becomes a constant and equals the net level premium $\bar{P}$ on the reserve assumptions. If the mortality and interest assumption for the primed functions is changed, ${}_h\bar{P}'$ will generally differ from $\bar{P}$ and vary by duration.

As the first example, consider the question of finding a new mortality table, $\mu''\neq\mu$, such that the reserves on a net level premium whole life policy are the same as when calculated on the original table, the interest rate remaining unchanged. To answer the question we need only put $\delta''=\delta$ in equation (2) and determine μ'' so that ${}_h\bar{P}''$ is a constant. Remembering that a consequence of making both $\delta'\equiv\delta$ and $\mu'\equiv\mu$ is that ${}_h\bar{P}'\equiv\bar{P}$, it follows that a necessary and sufficient condition for a mortality table μ'' to produce the same reserves as mortality table μ is that

$$(\mu_h''-\mu_h)(1-{}_h\bar{V})=\text{Constant}\,,\qquad (0\leq h\leq\omega-x)\,.\quad (3)$$

That is, that

$$\mu_h''=\mu_h+c\,\frac{\bar{a}_0}{\bar{a}_h}=\mu_h+\frac{C}{\bar{a}_h}\,,\qquad (0\leq h\leq\omega-x)\,.\ (4)$$

The converse is immediately evident by substitution of the value of μ_h'' from (4) in equation (2).

As another example based on the whole life plan, consider the effect on equation (2) of $\delta'>\delta$ when $\mu'\equiv\mu$. If ${}_h\bar{V}\uparrow$ (using $\uparrow$ to indicate increase with duration and $\downarrow$, decrease) then ${}_h\bar{P}'\downarrow$. This shows that the decreasing premium ${}_h\bar{P}'$ derived from the assumption of an interest rate δ' and mortality table $\mu' \equiv \mu$ produces reserves equal to $\bar{V}$. Therefore, the level premium on the same basis and corresponding to ${}_h\bar{P}'$ must produce net level premium reserves $\bar{V}'$ that are smaller than $\bar{V}$, because reserves accumulated by a premium that decreases with duration are necessarily larger than the reserves accumulated by an equivalent level premium. Therefore, for net level premium whole life policies, the higher the interest rate the smaller the reserve.

Similarly, it may easily be shown that, with respect to level premium reserves generally, if ${}_h\bar{P}'\downarrow$ then $\bar{V}' < \bar{V}$, and if ${}_h\bar{P}'\uparrow$ then $\bar{V}' > \bar{V}$.

Also worth noting is the fact that equation (1) does not require that the reserves $\bar{V}$ be based on any specified mortality and interest assumption. Any continuous function may be used for $\bar{V}$. In particular, asset share values or cash values may be used in place of reserves values. Modifications of equation (1) along these lines are sometimes useful in connection with dividend and surplus analyses.

(AUTHOR'S REVIEW OF DISCUSSION)

DONALD C. BAILLIE:

In the last paragraph of my Note, I put forward the hypothesis that many other readers of Spurgeon (or Lidstone) had worked out the equilibrium equation to their own satisfaction. Besides Dr. Simonsen's published work, there now seem to be masses of other evidence supporting this hypothesis.

First, I learnt that Professor C. W. Jordan of Williams College had done it in the course of writing a textbook on life contingencies. Then I saw Mr. Gershenson's Note, smack alongside my own. Finally, I learn that Dr. Nesbitt has been uneasy about some of my results for a number of years! As he has pointed out so politely, they just aren't true.

When one comes to think of it—rather belatedly—it is perfectly obvious that if two m-year endowment policies, at the same rate of interest, have identical reserves at *all* durations, then they must have identical premiums also. The reserve at duration $m - 1$ is just $v - P$ in each case. It follows that the constant k_z, to which I devoted nearly half a page of the *Transactions*, is unhappily zero. And this is of course the only way that $\ddot{a}_{y:\overline{z-y}|}$ and $\ddot{a}'_{y:\overline{z-y}|}$ could be in a fixed ratio for all ages y, since both annuities equal 1 when $y = z - 1$. Plainly "things are seldom what they seem, skim milk masquerades as cream"—too hastily skimmed in this case. On the other hand, things are seldom as bad as they seem, either, and I may be able to wriggle out of my fraudulent position if Dr. Nesbitt will accept two m-year endowment policies with different premiums and identical reserves at all durations *except* $m-1$.

First, let the ratio $\ddot{a}_{z-2:\overline{2}|}/\ddot{a}'_{z-2:\overline{2}|}$ be called $1+c$, where c is assumed positive. Then $\ddot{a}_{z-3:\overline{3}|}/\ddot{a}'_{z-3:\overline{3}|} = 1+c$ also, provided that

$$\frac{1+vp_{z-3}\ddot{a}_{z-2:\overline{2}|}}{\ddot{a}_{z-2:\overline{2}|}} = \frac{1+vp'_{z-3}\ddot{a}'_{z-2:\overline{2}|}}{\ddot{a}'_{z-2:\overline{2}|}},$$

that is, if

$$\frac{1+i}{\ddot{a}_{z-2:\overline{2}|}} + p_{z-3} = \frac{1+i}{\ddot{a}'_{z-2:\overline{2}|}} + p'_{z-3},$$

that is, if

$$q'_{z-3} - q_{z-3} = \frac{1+i}{\ddot{a}_{z-2:\overline{2}|}}(1+c-1) = \frac{c(1+i)}{\ddot{a}_{z-2:\overline{2}|}}.$$

Similarly

$$\frac{\ddot{a}_{z-4:\overline{4}|}}{\ddot{a}'_{z-4:\overline{4}|}} = 1+c$$

also, provided that

$$q'_{z-4} - q_{z-4} = \frac{c(1+i)}{\ddot{a}_{z-3:\overline{3}|}}.$$

Proceeding in this way we can have $\ddot{a}_{z-n:\overline{n}|}/\ddot{a}'_{z-n:\overline{n}|} = 1+c$ for $n = 2, 3, \ldots, m$. This is a necessary and sufficient set of conditions for ${}_tV'_{x:\overline{m}|}$ to equal ${}_tV_{x:\overline{m}|}$, where $x = z-m$ and $t = 0, 1, 2, \ldots, m-2$. (Q.E.D.)

The connection between q'_{z-2} and q_{z-2} is determined by

$$1+vp_{z-2} = (1+c)(1+vp'_{z-2}),$$

which leads to the odd-looking result

$$q'_{z-2} = \frac{q_{z-2} + c(2+i)}{1+c}$$

The difference $q'_{z-2} - q_{z-2}$ is approximately $c(2+i-q_{z-2})$, rather than the $c(1+i)$ that the other differences would suggest. Finally q'_{z-1} and q_{z-1} can be anything we like since they have no effect on the endowment.

One may ask how closely this compromise situation agrees with the equation $R_t = 0$. Using the definition of R_t on the right side of my original equation (3), we see that all R_t will be zero except those involving $({}_{m-1}V' - {}_{m-1}V)$, namely R_{m-2} and R_{m-1}. Using ${}_{m-1}V' - {}_{m-1}V = P - P'$, we have $R_{m-2} = p'_{z-2}(P-P')$ and $R_{m-1} = (P'-P)(1+i)$. It is easily verified that my original equations (5) and (6) are true for this set of R_t's. As for my equation (8), it has a batting average of .900 for a twenty year endowment.

This compromise can be extended to the whole life case by making

$z - 1$ the last age to which anyone survives in each table. Then we define $(1 + c)$ to be $\ddot{a}_{z-2}/\ddot{a}'_{z-2}$ and, as before, we choose

$$q'_{z-3} = q_{z-3} + \frac{c\,(1+i)}{\ddot{a}_{z-2}}$$

to make $\ddot{a}_{z-3}/\ddot{a}'_{z-3} = 1 + c$ also. When we finally reach $\ddot{a}_x/\ddot{a}'_x$ we have ${}_tV'_x = {}_tV_x$ for $t = 0, 1, 2, \ldots, z - 2 - x$. For $t = z - x - 1$ we have ${}_tV'_x = v - P'_x$ while ${}_tV_x = v - P_x$. As before

$$q'_{z-2} = \frac{q_{z-2} + c\,(2+i)}{1+c},$$

and we can take q_{z-1} and q'_{z-1} to be 1.

Another way to avoid trouble at the end of the table is to have the table with the higher mortality stop one year before the other table. Then q'_{z-2} can be defined as

$$q_{z-2} + \frac{c\,(1+i)}{\ddot{a}_{z-1}},$$

which will be 1 if $c = vp_{z-2}$. We can then proceed as before to construct a table of q_x such that the ratio $\ddot{a}_x/\ddot{a}'_x = 1 + c$ for all ages except $z - 1$, where $\ddot{a}'_{z-1}$ in the denominator is meaningless. Whole life reserves by these two tables will be identical at all ages and durations for which they both exist.

Dr. Nesbitt's premium inequalities are interesting. Of the four that he gives, I think that $P(1 + i) < P'(1 + i')$ is the only one whose truth could not be seen just as readily without the use of R_t. The second inequality in his Case I can be written $P + d - d' > P'$, that is, $1/\ddot{a} > 1/\ddot{a}'$, which follows at once from $i' < i$. The first inequality in his Case II would be an equality if q were constant, in which case $P = vq$ and $P' = vq + vh = P + vh$. Here vh is the constant extra premium to cover the constant extra mortality on an amount at risk equal to unity. The amount at risk is, however, generally less than unity, owing to the positive reserves that develop when q_x increases with x. Hence a constant extra premium of vh is generally more than adequate to cover the extra risk, *i.e.*, $P + vh > P'$.

Mr. Weck has introduced a novel reserve equation with a continuously varying premium. In the first example of its use the result would, I think, follow just as well if the integral equation were written as a differential equation. In the second example, the statement is made that if ${}_hV\uparrow$ then ${}_h\overline{P}'$ must $\downarrow$. Since this conclusion is equivalent to

$$\frac{d}{dh}\left({}_h\overline{P}'\right) < 0,$$

it would seem to me that the integral equation should be differentiated *twice* before we can study the sign of $(d/dh)({}_k\mathrm{P}')$.

One differentiation of Mr. Weck's equation (2) gives

$$\delta'{}_k\bar{\mathrm{V}} + {}_k\bar{\mathrm{P}}' - \mu'(1 - {}_k\bar{\mathrm{V}}) = \frac{d}{dk}({}_k\bar{\mathrm{V}}),$$

which can be compared with the ordinary level premium differential equation,

$$\delta_k\bar{\mathrm{V}} + \bar{\mathrm{P}} - \mu(1 - {}_k\bar{\mathrm{V}}) = \frac{d}{dk}({}_k\bar{\mathrm{V}}),$$

to obtain, if $\mu' = \mu$, the equation $(\delta' - \delta){}_k\bar{\mathrm{V}} + {}_k\bar{\mathrm{P}}' - \bar{\mathrm{P}} = 0$, and the final conclusion then follows after one more differentiation. Had we kept the μ terms we should have in the same way

$${}_k\bar{\mathrm{P}}' = \bar{\mathrm{P}} - (\delta' - \delta){}_k\bar{\mathrm{V}} + (\mu' - \mu)(1 - {}_k\bar{\mathrm{V}}),$$

which equals $\bar{\mathrm{P}}' - \bar{R}_k$, where $\bar{R}_t$ is the continuous Remainder corresponding to R_t in the discrete case. The continuous equation of equilibrium is perhaps most easily developed by writing the ordinary level premium differential equation in both primed and unprimed symbols, subtracting one equation from the other, and multiplying through by D'_{x+t} to obtain

$$\frac{d}{dt}\mathrm{D}'_{x+t}({}_t\bar{\mathrm{V}}' - {}_t\bar{\mathrm{V}}) = \mathrm{D}'_{x+t}\bar{R}_t,$$

where

$$\bar{R}_t = (\delta' - \delta){}_t\bar{\mathrm{V}} + (\bar{\mathrm{P}}' - \bar{\mathrm{P}}) - (\mu' - \mu)(1 - {}_t\bar{\mathrm{V}}).$$

The fact that $\bar{R}_k$ and ${}_k\bar{\mathrm{P}}'$ add to a constant means that $\bar{R}_k\uparrow$ or $\downarrow$ will be equivalent to ${}_k\bar{\mathrm{P}}'\downarrow$ or $\uparrow$, as Mr. Weck has observed.

In my Note the first six equations still hold true for a varying P so that they could be applied to asset shares or cash values about as readily as Mr. Weck's Reserve Equation, I think.

Finally, I am grateful to both Dr. Nesbitt and Mr. Weck for taking the trouble to wade through a pretty unexciting Note and then to write down their observations. Their discussion certainly has clarified my own thinking on the subject.