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A PRESENT VALUE APPROACH TO PROFIT MARGINS AND DIVIDENDS

HARWOOD ROSSER

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ROBERT J. JOHANSEN:

The first thought that comes to mind on reading Mr. Rosser's paper is: "Why has such an approach not previously been set forth?" While similar methods have been mentioned in some previous articles, not until now has there been published a clear exposition of a method which is particularly suited to computation by lower grade clerks. The present value method has the further advantage that if mortality and lapse rates are varied only for large groups of plans, then the computational work of asset shares may be cut considerably by the use of the same discounted persistency factors for several calculations. In addition to permitting changes in expenses, as mentioned by Mr. Rosser, it also facilitates rapid testing of experimental dividend scales.

Before discussing additional uses for the present value method, it is instructive to derive the new formula from the traditional method. While Mr. Rosser has assumed that cash values are accumulated and held, he states that any values can be used in their stead. That this is so is evident from the following derivation of the present value formula from the traditional asset share approach. This means that a company could, for example, hold as funds the valuation reserve plus a desired surplus. Expressing the traditional asset share calculation in equation form, we have, representing asset shares for years 1, 2, 3, etc., as A_1, A_2, A_3, etc.:

$$A_1=\left\{P-E_1-\frac{1000}{1+i}\,q^d_{[x]}-\frac{CV_1}{1+i}\,w_{[x]}-\frac{D_1}{1+i}\right\}\div\left\{\left(1-q^d_{[x]}-w_{[x]}\right)v\right\}$$

$$A_2=\left\{P-E_2-\frac{1000}{1+i}\,q^d_{[x]+1}-\frac{CV_2}{1+i}\,w_{[x]+1}-\frac{D_2}{1+i}+A_1\right\}\div\left\{\left(1-q^d_{[x]+1}-w_{[x]+1}\right)v\right\}$$

$$A_3=\left\{P-E_3-\frac{1000}{1+i}\,q^d_{[x]+2}-\frac{CV_3}{1+i}\,w_{[x]+2}-\frac{D_3}{1+i}+A_2\right\}\div\left\{\left(1-q^d_{[x]+2}-w_{[x]+2}\right)v\right\}$$

and, in general, for year t

$$A_t = \left\{ \mathrm{P} - E_t - \frac{1000}{1+i} q^d_{[x]+t-1} - \frac{\mathrm{CV}_t}{1+i} w_{[x]+t-1} - \frac{D_t}{1+i} + A_{t-1} \right\} \div \left\{ (1 - q^d_{[x]+t-1} - w_{[x]+t-1})\, v \right\}$$

where E_t represents expenses and D_t the dividend of the tth year and P is the gross annual premium.

Multiplying A_1 by $(1 - q^d_{[x]} - w_{[x]})v$, A_2 by $(1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1})v^2$, and A_t by $(1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1}) \ldots (1 - q^d_{[x]+t-1} - w_{[x]+t-1})v^t$, we have

$$A_1 \{ (1 - q^d_{[x]} - w_{[x]})\, v \} = \mathrm{P} - E_1 - \frac{1000}{1+i} q^d_{[x]} - \frac{\mathrm{CV}_1}{1+i} w_{[x]} - \frac{D_1}{1+i}$$

$$A_2 \{ (1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1})\, v^2 \}$$
$$= \left\{ \mathrm{P} - E_2 - \frac{1000}{1+i} q^d_{[x]+1} - \frac{\mathrm{CV}_2}{1+i} w_{[x]+1} - \frac{D_2}{1+i} \right\} \times \{ (1 - q^d_{[x]} - w_{[x]})\, v \} + A_1 \{ (1 - q^d_{[x]} - w_{[x]})\, v \}$$

$$A_3 \{ (1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1})(1 - q^d_{[x]+2} - w_{[x]+2})\, v^3 \}$$
$$= \left\{ \mathrm{P} - E_3 - \frac{1000}{1+i} q^d_{[x]+2} - \frac{\mathrm{CV}_3}{1+i} w_{[x]+2} - \frac{D_3}{1+i} \right\} \times \{ (1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1})\, v^2 \}$$
$$+ A_2 \{ (1 - q^d_{[x]} - w_{[x]})(1 - q^d_{[x]+1} - w_{[x]+1})\, v^2 \}$$

and in general

$$A_t \{ (1 - q^d_{[x]} - w_{[x]}) \ldots (1 - q^d_{[x]+t-2} - w_{[x]+t-2}) \times (1 - q^d_{[x]+t-1} - w_{[x]+t-1})\, v^t \}$$
$$= \left\{ \mathrm{P} - E_t - \frac{1000}{1+i} q^d_{[x]+t-1} - \frac{\mathrm{CV}_t}{1+i} w_{[x]+t-1} - \frac{D_t}{1+i} \right\} \times \{ (1 - q^d_{[x]} - w_{[x]}) \ldots (1 - q^d_{[x]+t-2} - w_{[x]+t-2})\, v^{t-1} \}$$
$$+ A_{t-1} \{ (1 - q^d_{[x]} - w_{[x]}) \ldots (1 - q^d_{[x]+t-2} - w_{[x]+t-2})\, v^{t-1} \},$$

which shows that the present value at issue of an asset share at the end of year t is equal to the present value at issue of the asset share at the end of the $(t-1)$th year plus the present value at issue of contributions of the tth year per entrant into the tth year.

Transposing and combining, we have

$$\Big\{ \mathrm{P} - E_t - \frac{1000 - A_t}{1+i}\, q^d_{[x]+t-1} - \frac{\mathrm{CV}_t - A_t}{1+i}\, w_{[x]+t-1} - \frac{D_t}{1+i} + A_{t-1} - \frac{A_t}{1+i} \Big\}$$

$$\times \{ (1 - q^d_{[x]} - w_{[x]}) \ldots (1 - q^d_{[x]+t-2} - w_{[x]+t-2})\, v^{t-1} \} = 0 .$$

Setting the first term equal to zero and solving for D_t gives Mr. Rosser's formula for the crude dividend provided we define A_t as equal to the cash value CV_t. If $D_t/1+i$ is then multiplied by the second term in brackets, we obtain the value of the dividend at issue.

The general form set forth above shows, as Mr. Rosser states, that cash values need not be the funds held, and it further points out the modifications of the formula required when other funds are held. Taking account of death claim and surrender expenses, immediate payment of claims, return of premiums paid beyond the date of death, and a pro rata dividend paid at death, we have

$$\mathrm{P}(1 - e_t) - E^c_t - \frac{(1+i)^{1/2}(1000 + \tfrac{1}{2}\mathrm{P} + E^D - \tfrac{1}{2}D_t) - A_t}{1+i}\, q^d_{[x]+t-1}$$

$$- \frac{D_t}{1+i} - \frac{\mathrm{CV}_t + E^S - A_t}{1+i}\, w_{[x]+t-1} + A_{t-1} - \frac{A_t}{1+i} = 0 ,$$

from which we can solve for D_t to find the margin as of the end of the policy year. For the tth year, e_t = percentage of premium expense, E^c_t = constant expense per \$1,000 insurance, E^D = death claim expense, E^S = surrender expenses, i = assumed interest rate, P = gross annual premium. This formula assumes that premiums are paid annually and surrenders take place at the end of the policy year. Somewhat more complicated terms are necessary to take account of premiums payable more often than annually or surrenders occurring throughout the year.

Since it may be desired in the dividend formula to provide for contributions to surplus or contingency funds, the method lends itself to several methods of attack. First, the accumulation of surplus may be provided for in the definition of A_t as mentioned earlier. In such a case, the contribution to surplus would be included in the term $(A_t/1 + i - A_{t-1})$. Another method would be to define, say, a twentieth year surplus, obtain its present value at issue, multiply by $1 + i$ and divide by $\sum_{1}^{20} {}_{n-1}p^s_{[x]} \cdot v^{t-1}$ given in the first column of Table 8 of Mr. Rosser's article to obtain the level amount which must be deducted each year from the dividend.

In using the asset share method to produce dividends, particular care should be taken in ascertaining the source and incidence of earnings from which dividends will be paid. In many cases, where early duration earnings are not so large as in the example given, the rather arbitrary method of obtaining dividends outlined in this paper may result in losses because of adverse experience, particularly in regard to withdrawals. It might be preferable to work up asset shares and dividends without taking account of withdrawals and then use these asset shares as funds to be kept on hand (as values of A_t in the above formula), in evaluating the effect of withdrawals on dividends. The effect of withdrawals could be especially significant in early policy years and particularly on decreasing face amount policies where funds in early durations are frequently negative. It is also desirable that there should not be too great a difference between yearly earnings and dividends, since otherwise it may happen that small changes in lapse rates will have a large effect on future earnings, funds and dividends.

The method suggested by Mr. Rosser for revising existing dividend scales has an additional use where "pegging" is to be employed, since it would aid in maintaining equity among several years of issue on the same rates. Since the customary method is to continue a flat dividend (usually the last dividend paid under the old schedule) until the increasing dividends on the new scale first exceed the flat payment, additional amounts are withdrawn from surplus without provision for return. The method outlined in Mr. Rosser's paper can be used to modify future dividends to take account of the "pegged" dividends, in effect borrowing from future dividends to pay larger present dividends.

B. FRANKLIN BLAIR:

Mr. Rosser is to be congratulated on his very practical paper, explaining in some detail how to carry out calculations based on a variation of a formula presented by Mr. E. W. Marshall in *Actuarial Studies No. 6*. Mr. Rosser's paper will certainly be useful to students. I have run across a number of students, particularly those with nonparticipating companies or in consulting work, who have had considerable trouble with the question of how a dividend scale is determined in actual practice. Mr. Rosser has given one very useful answer to that question.

Although we still use the accumulated asset share for many purposes in the Provident Mutual, we have found that a "present value approach," essentially similar to Mr. Rosser's, has advantages in certain situations. It is particularly useful for term insurance studies. Because of the extreme "tontine" effect produced by the high lapse and conversion rates, accumulated asset shares on term insurance are likely to have almost fantastically large absolute values.

As dividends are payable at the end of the policy year and death claims and surrender values are usually assumed to be payable at that time, it may simplify the work to relate all calculations to the *end* of the year and use a persistency and discount factor in the form $v^n{}_{n-1}p^s_{[x]}$ rather than in the form $v^{n-1}{}_{n-1}p^s_{[x]}$ used by Mr. Rosser in Table 1 and elsewhere. Of course, the particular choice will depend on the actual details of how the calculations are to be used.

One word of caution may be advisable if the present value approach is to be limited to a period, such as 20 years, which is shorter than the full term of the policy. In order to avoid a sharp change in the slope of the dividends, the dividends at the end of the test period should be comparatively close to the crude dividend at that point. This criterion has not been followed by Mr. Rosser in making up his illustrative tables. For example, the crude dividend at the end of the 20th year in his Table 1 is \$8.693 per \$1,000. But according to the dividend scales shown in Table 1 and Table 6 (second trial), the 20th year dividends are \$18.50 and \$15.66 respectively. Unless the crude dividend jumps tremendously in the years immediately following the 20th year, one might have considerable trouble finding satisfactory dividends to fit on to those shown for the first 20 years. That Mr. Rosser's illustrations do not follow this criterion is no criticism of his excellent paper, as he was only illustrating the development of dividend scales "in the simplest sort of fashion" (page 191).

EDWARD A. RIEDER:

Mr. Rosser's suggested technique for graduating crude dividends, under control, so that a given asset share will be automatically reproduced at a given point of time, makes interesting reading. The technique can be useful for this purpose.

It should be noted, however, that the demonstration in his Table 1, which performs the graduation in what he terms "the simplest sort of fashion," cannot be relied upon to produce dividends whose incidence will follow the actual emergence of surplus with sufficient accuracy. If it does, it will be purely an accident.

It is necessary to compare the graduated dividends with the crude dividends produced by some other method to see the extent of the departure from true surplus earnings. For example, Rosser's dividend for the 20th year is \$18.50 compared with a crude dividend for that year of \$8.69. What would he allow in the 21st year?

From this point of view, Mr. Rosser has not presented a new approach to dividends but rather a graduation technique, with a control on the over-all distribution but without control on the incidence of the distribution. The crude dividends to which he applies his graduation technique are themselves the result of a retrospective or accumulative approach, so that I question the appropriateness of the title used for the paper.

It is possible to equate, with similar control, the over-all value of the crude dividends with that of the smoothed dividends at any point of time. He has selected the date of issue and doubtless the selection of this date is responsible for the title of his paper. One can just as logically equate the values as at the middle or the end of the period. The resulting dividends would be the same but the paper would have had a different title.

A considerable portion of his paper is devoted to a demonstration of the seemingly obvious over-all equivalence of dividends obtained by various approaches, all having been determined in the first place from the same fundamental experience rates of interest, mortality, withdrawal and expense.

His application of "The Present Value" method to the revision of an existing scale seems to involve one fundamental error. For example, after ten years of operation on one dividend scale, it seems incongruous to assume present day experience factors as applying retroactively to date of issue of the existing policy, in order to calculate revised future dividends reflecting current experience.

KERMIT LANG:

Evidently a satisfactory practical solution to the problem of surplus distribution can be obtained by the "present value approach," but the student of the subject will do well to study carefully the discussion by Prof. Rietz[1] of an earlier description of this method by Mr. Estes[2] in 1932. In his remarks Prof. Rietz stated that "an examination of [the] underlying assumptions and principles of procedure suggests that the assumptions are rather numerous and relatively arbitrary, but that they give a sort of common-sense background of simplicity."

For more information on that portion of Mr. Rosser's paper dealing with the "equivalence" or relationship between the asset share approach and the contribution formula, the student may refer to another earlier paper, that presented by Mr. Bowerman[3] at the Ninth International Congress of Actuaries in 1930.

[1] *RAIA* XXII, 10.

[2] *RAIA* XXI, 220. "Annual Dividends—An Asset-Share Method of Distribution," by E. F. Estes.

[3] *TICA* IX, I, 78. "Contribution of Dividends," by Walter G. Bowerman.

As noted in Mr. Estes' paper and in the discussion by Prof. Rietz, the "present value approach" was first suggested in 1907 by Mr. Ferguson.[4] On looking up this reference we learn that he was searching "to find a comprehensive method of division (or bonus distribution) equally applicable to tontine and annual dividends." This at once suggests one of the principal advantages of the "present value approach," namely, its flexibility. Whereas the earlier asset share approach had been most useful in indicating equitable scales of dividends to be disbursed on deferred dividend or semitontine policies, some modification was necessary to give results which were at the same time both consistent and equitable as between the deferred dividend and the annual dividend series of policies.

Delving into the history of asset shares for a moment, one finds that they are of much older origin than one might think. In fact, they seem to have gone hand in hand with the deferred dividend system and may have been used ever since 1871, as suggested by Mr. Van Cise[5] in his discussion of Mr. Weeks's paper[6] on "An American Method of Allotting Surplus to Deferred Bonus Policies," a paper which was read at the Third International Congress of Actuaries in 1900. The precise term, "asset-share," may first have been used by Mr. Weeks[7] in 1905 in his paper entitled "An Equitable Method of Keeping the Accounts of 'Deferred Dividend,' " wherein he showed an elaborate model of an asset share calculation.

Today the asset share method remains a powerful tool for testing the "general equity of the dividend scale."[8] It is almost indispensable for analyzing the effects of eliminating first year or first and second year dividends, or of introducing extra quinquennial dividends or surrender dividends, not to mention the troublesome problem of modified life policies which provide for a reduced premium and no dividends in the first few years. All of these plans have something in common with the deferred dividend system, which no doubt explains the peculiar appropriateness of the asset share method in these circumstances.

The principal disadvantage of the asset share method is the laboriousness of the calculations, which often has led to a restriction of its use to a few plans and ages at issue and to a period of not longer than twenty policy years from date of issue. This same objection obviously applies to the present value approach. However, medium-sized and larger companies are now availing themselves of the electronic calculator[9] and with this power-

[4] *TASA* X, 359.

[5] *TASA* VII, 46.

[6] *TICA* III, 1121, *TASA* VI, 367.

[7] *TASA* IX, 93.

[8] *Actuarial Studies No. 6, Distribution of Surplus*, 51.

[9] *The National Underwriter*, Life Insurance Edition, January 12, 1951, "Companies Warming Up to 'Magic Brain' Calculators."

ful new tool they should easily be able to multiply their asset share calculations fifty-fold.

It would therefore seem logical to devise dividend scales in the future as in the past, by either a three-factor or a two-factor formula, and then to calculate asset shares in order to test the level of the dividends and the slope of the scale by a comparison of the resulting asset shares with the corresponding cash values.

As an alternative approach, if a three-factor formula is used, the combined interest and mortality return could be calculated and used as a first approximation to the dividend. This would establish the slope of the scale. Then a second approximation could be made by introducing an arbitrary loading return. With these two results available, the proper loading return could be obtained by interpolation in exactly the same way as suggested by Mr. Hoskins[10] for nonparticipating premiums.

Likewise, with a two-factor formula, the proposed interest return could be calculated and this would determine the slope of the scale. Then asset shares could be calculated, first assuming a basic dividend of zero and then some other arbitrary basic dividend. Interpolating between these two results, the correct basic dividend from mortality and loading would be determined, thereby fixing the level of the dividend scale.

The author has stated that one of the advantages of the present value approach is that any change in assumptions for a particular year does not affect the figures for any other year, but only the totals. However, this does not seem to be strictly true, for if the second year persistency were changed, for example, it would appear to change all the "persistency and discount" factors.

To discount each year's profit margin to date of issue, taking into account interest, mortality and the rate of withdrawal, and then to redistribute this average "value at issue," as the author has done in Table 1, is to run the risk of producing fantastically large dividends at the longer durations on long-term policies, such as Ordinary Life. For example, if the author had carried the demonstration in Table 1 to the end of 30 years, the later dividends would have exceeded the gross premium.

The example given in Table 6 of the revision of an existing dividend scale necessitated by experience less favorable than the original assumptions is rather curious because it appears to illustrate the opposite situation. At least the revised dividend scale, labeled "second trial," starts at the same level as the present dividend scale and is steeper rather than flatter thereafter. This may be attributable to the fact that a renewal

[10] *TASA* XXX, 140. "A New Method of Computing Non-Participating Premiums," by James E. Hoskins.

overhead expense of only $0.25 per $1,000 over and above percentage expenses has been assumed, which seems too low for premium-paying policies.

J. ROSS GRAY:

In his paper, Mr. Rosser gives us a demonstration of the not unreasonable proposition that the same results are obtainable by the means of a present value approach as by an accumulation. His technique of accumulating asset shares in Table 2 should be useful, but for the reader it might have helped if he had explained more definitely that the persistency and discount factors incorporate the rate of mortality as well as the lapse rate.

It is when he comes to apply his method to the calculation of dividends in Table 1 that a weakness is displayed. Mr. Rosser makes a point of mentioning that his method is independent of the reserves; he disregards profits after the 20th year, although he says that the period could run longer; he then finds that his dividends are completely flexible except that the present value at issue of the dividends over the 20 year period is fixed. The gradation of those dividends is left completely to the choice of the operator. He has disregarded the choices based on operating directly on the final dividends. Instead, he claims that it is much more logical to operate on the initial values of the profit margins and obtain dividends whose initial values bear some relationship to each other. As an example, he takes initial values which are equal to each other. The dividends increase in the inverse of the ratio by which the persistency and discount factors decrease. It hardly seems that such is theoretically the most defensible basis.

His Table 1 shows that the margin or crude dividend of the 20th year is 8.693. It is to be expected that the dividend of the 21st year will be in the neighbourhood of 8.76. The proposed dividend of 18.50 in the 20th year, dropping to 8.76 in the 21st year cannot be regarded as being satisfactory.

Instead of condemning the method, this may give us what we want—something to pin our dividend scale to, thereby removing the absolute flexibility of the dividend calculation. If we impose the requirement that the dividend of the 20th year will be 8.693 with an initial value of 1.463 and that the initial values of the dividends of the first 20 years will total to 59.181, we have a fairly close control over the rate of the distribution of the margin. Admittedly we are left with the question of whether the initial values will decrease in arithmetical progression, or geometrical progression, or in some other form.

Table A below shows the result of assuming that the initial values of the dividends reduce in arithmetical progression to the 20th year figure of 1.463. The resulting dividends and asset shares are shown. The purpose of having the 20th year dividend equal the earnings of the 20th year has been accomplished; the asset shares exceed the cash values throughout; as in the case of Mr. Rosser's figures some adjustment of the dividends will be needed if we wish the asset shares to cover the Commissioners' reserves, but I doubt whether any company would be prepared to contemplate the increasing and then decreasing scale of dividends unless such is definitely indicated by some other calculation.

Further calculations were done eliminating firstly the excess interest, and secondly both the excess interest and mortality savings, and redis-

TABLE A

Year	Value at Issue of Margin Redistributed	Dividend Scale	Asset Share
1	....	....	$ 1.70
2	$4.767	$6.32	20.82
3	4.583	6.80	39.95
4	4.399	7.31	58.62
5	4.216	7.76	77.03
6	4.032	8.22	94.70
7	3.849	8.60	112.25
8	3.665	8.97	129.66
9	3.482	9.34	146.65
10	3.298	9.59	163.52
11	3.115	9.82	180.09
12	2.931	9.92	196.62
13	2.748	9.97	213.15
14	2.564	9.99	229.65
15	2.381	9.95	246.15
16	2.197	9.85	264.47
17	2.014	9.69	283.00
18	1.830	9.44	301.81
19	1.647	9.12	320.92
20	1.463	8.69	340.55

tributing the balance, using Mr. Rosser's method of operating on the initial values. Satisfactory results were not obtained, in that the final dividends increased and then decreased. The balance which remained after removing both the excess interest and mortality savings was, of course, the expense margin, and if that couldn't be redistributed satisfactorily, it seemed that there was something wrong with the method.

An obvious step was to redistribute the expense margin in arithmetical progression of the final figures of the margin (rather than the initial values). The requirement was applied that the expense margins have the proper total initial value, and that the 20th expense margin be 4.054 at the beginning of the year. Excess interest and mortality savings were added to obtain the typical three-factor dividends shown in the first column of Table B.

While the three-factor method distributes the excess interest and mortality savings as they occur, I submit that there is nothing which makes the distribution of the expense margin as an arithmetical progression the only way of doing so. It can vary from that. If so, why not distribute Mr. Rosser's dividend margin of 59.181 as an arithmetical progression of

TABLE B

Year	Three-Factor Dividends	Dividends in Arithmetical Progression	
		Scale	Asset Shares
1	....	....	$ 1.70
2	$8.60	$8.33	18.64
3	8.64	8.35	35.82
4	8.51	8.37	52.91
5	8.39	8.39	70.03
6	8.09	8.41	86.82
7	8.17	8.43	103.80
8	8.24	8.45	120.95
9	8.30	8.47	138.13
10	8.36	8.49	155.44
11	8.42	8.51	172.78
12	8.47	8.53	190.23
13	8.53	8.55	207.76
14	8.56	8.57	225.34
15	8.60	8.59	242.95
16	8.63	8.61	262.33
17	8.66	8.63	281.80
18	8.67	8.65	301.34
19	8.68	8.67	320.94
20	8.69	8.69	340.55

the final dividends? Dividends on that basis are shown in the second column of Table B, and the resulting asset shares in the third column.

Throughout these calculations Mr. Rosser's method has been a very useful means of doing the work, but each time the shape of the persistency and discount factors has been permitted to have any bearing on the shape of the resulting dividend scale, trouble has resulted. I come to the conclusion that Mr. Rosser's method should be restricted to the matter of making the calculations. The shape of the dividend scale should be determined in other ways.

ROBERT L. BERGSTRESSER:

The present value method of examining a dividend scale or a scale of nonparticipating premiums was used by our Company for the first time several years ago. We used this method because of one feature which seems to me to be a powerful advantage, although this feature is not commented upon in Mr. Rosser's paper. I refer to the fact that, since the experience of each year can be computed independently and discounted back to the issue date, as is done in Column 11 of Table I, it is possible to observe the trend of the margins without calculating all the lines in the table. The work can, therefore, be condensed so that with the calculation of only four or five policy years, the characteristics of the earnings for the entire twenty years can be deduced. For example, the effects of mortality and interest follow a rather steady and predictable course during the twenty years, but the expenses are discontinuous at points where the overhead costs or commission rates change suddenly. In Table I these points of sudden change are at durations 2 and 16, as shown in Column 7, and by examining Column 11 it will be seen that from the third to the fifteenth year there is a steady progression downward in the earnings and that a new progression commences in the sixteenth year at a higher level. If, therefore, we merely calculate the margins for the first three years and the fifteenth, sixteenth and twentieth years, we could forecast the general slope of the margins for the intervening years. The usual asset-share calculation, of course, requires working from each policy year to the next and cannot be telescoped.

Since calculations of this sort are frequently made on an experimental basis, a method which enables the actuary to have a general picture of the entire calculation, but which requires only skeleton values, will save time and expense.

(AUTHOR'S REVIEW OF DISCUSSION)

HARWOOD ROSSER:

A pure mathematician can afford to state that, if he be granted certain assumptions, he can erect quite a scientific edifice upon them. Thus did Euclid. But the actuary may expect his assumptions, and anything resulting from them, to be subjected to the test of practicality.

ATTRIBUTES OF DISTRIBUTION METHODS

Two major requirements of a practical method of surplus distribution are equity and popularity. *Equity* requires that substantially all the excess earnings on any policy should be distributed as dividends to its owner. By *popularity* is meant that the scale should be as pleasing as possible to policyholders and agents. Specifically, it should avoid abrupt changes; and a steady increase is preferable.

In addition, it is highly desirable to have *economy* of calculation method; that is, a method that lends itself readily to obtaining and testing experimental scales, sometimes in skeleton form. Messrs. Bergstresser and Johansen have been kind enough to state that the method of the paper has this characteristic. Mr. Bergstresser has elaborated the point.

On the assumptions made, the crude dividends (as shown in Table 1, Column 10) represent perfect equity. To obtain a practical dividend scale, or popularity, as defined, we must graduate them somehow. Any graduation involves a compromise between fit and smoothness. Here, this means a compromise between equity and popularity.

In the illustration in Table 1, the crude dividends, after discounting for interest and persistency (Column 11), rather than before (Column 10), were selected for graduation. In the graduation process, my intention was to keep the manipulation to a minimum, in order not to divert the stream of thought. Hence the graduated figures (Column 12) were taken as equal to each other, with reproduction of the original total.

This gave the ultimate in smoothness, but ignored fit. A corollary would be difficulty of junction with later dividends. All but one of the discussants commented on this, one at length. Accordingly, the discounted dividends have been regraduated, with more emphasis on fit. The results are shown in Table 9 below. This amounts to a recalculation of certain columns of Table 1. A description of the various steps follows.

RECALCULATION OF DIVIDEND SCALE

Mr. Lang has mentioned that asset share calculations are often restricted to the first twenty years. Similarly, dividend scales are seldom projected beyond twenty years. Indeed, it is often argued that they should not be published, at least, that far in advance.* Usually, then, one is graduating a truncated column of dividend figures. Mr. Gray has rightly suggested essential reproduction of the figure at the highest policy duration shown. This facilitates extension of the scale at a later date.

Originally, to avoid complications, it was assumed that no first year dividend would be payable. However, many companies, including mine, declare such dividends regularly, contingent upon the collection of the second year's premium. A first year dividend, with this reservation, will therefore be granted.

This will simplify the regraduation, since the fixing of the upper end-

* Cf. *RAIA* XXIV, 95–104.

point, in accordance with Mr. Gray's suggestion, produces a very flat scale for the particular example. This can be seen from Mr. Gray's final scale, in his Table B, with a constant difference of $0.02. (His scale is another solution, along different lines, and without a first year dividend.) In fact, if only the first fifteen years were to be used, and the end-point similarly fixed, it would be impossible to avoid some dividend decreases.

In short, my choice of underlying figures has provided, quite unintentionally, a fairly severe test of any method of obtaining a dividend scale that possesses both equity and popularity. Nevertheless, after some experimentation, a satisfactory solution can be reached. Mr. Gray's Table A represents one experiment, which he rejects because it lacks popularity.

TABLE 9

RECALCULATION OF DIVIDEND SCALE AND FINAL MARGINS IN TABLE 1

Year n	Persistency and Discount Factor (1)	Dividend Scale: Value at Issue (Smoothed) (2)	Dividend Scale: End of Year $1.03\times(2)\div(1)$ (3)	Value at Issue of Margin (after Dividend): For Year (Col. 11, Table 1)−(2) (4)	Value at Issue of Margin (after Dividend): Total to Date $(4)+(5)_{n-1}$ (5)
1	1.00000	4.516	5.81*	− 3.193	−3.193
2	.77670	4.942	6.55	12.924	9.731
3	.69375	4.619	6.86	− .588	9.143
4	.61966	4.317	7.18	− .759	8.384
5	.55950	4.035	7.43	− .863	7.521
6	.50518	3.771	7.69	− 1.023	6.498
7	.46103	3.524	7.87	− .953	5.545
8	.42075	3.293	8.06	− .901	4.644
9	.38398	3.078	8.26	− .847	3.797
10	.35416	2.877	8.37	− .780	3.017
11	.32666	2.689	8.48	− .718	2.299
12	.30446	2.513	8.50	− .645	1.654
13	.28376	2.349	8.53	− .575	1.079
14	.26448	2.195	8.55	− .520	.559
15	.24651	2.051	8.57	− .463	.096
16	.22975	1.917	8.59	− .043	.053
17	.21414	1.792	8.62	− .029	.024
18	.19958	1.675	8.64	− .017	.007
19	.18601	1.565	8.67	− .007	.000
20	.17337	1.463	8.69	.000	.000
Total	....	59.181	159.92	0.000	....

* (2) ÷ .77670.

Let us replace the irregular curve of discounted dividends by a geometrical series with the same area under it, and the same end-point. To do this, we need to solve the following equation for r:

$$\frac{r^{20}-1}{r-1}=\frac{59.181}{1.463}=40.452\,.$$

Consulting interest tables, we find that r is close to 1.07. With this ratio, we build up Column 2 of Table 9, starting at the bottom. The top figure is obtained by balancing.

Normally, the adjustment due to using an inexact ratio would be spread over several figures. In this case, however, it is unnecessary. The adjustment indicated by the footnote, to take account of the dependence of the first year dividend on the second year's premium, operates here to offset the first adjustment. Column 3 represents the recalculated dividend scale, and Column 4 the revised margins.

INITIAL VERSUS FINAL VALUES

Mr. Gray's work highlights the necessity of working with crude dividends together with either their present values or else asset shares. Under the traditional approach, the crude dividends are hammered into a tentative scale. This is then tested for equity by computing asset shares. If revisions are necessary, more asset share calculations must be made. This may be repeated several times.

If we choose the other alternative, as I have done, the necessity of calculating asset shares is eliminated. While both the initial and final values of the crude dividends should be kept in mind, the smoothing process is to be applied to the discounted column. The undiscounted figures could be used to the extent of determining key values, and the present values employed as guides in filling in the remainder of the scale. Although the examples utilize arithmetical and geometrical progressions, it is not necessary that any mathematical law be invoked.

The total of the dividend scale will vary according to its shape and slope. But equity requires that its present value should remain fixed, or at least within a very narrow range, as long as the assumptions are unchanged. From one set of assumptions, several dividend scales have been obtained, all with the same present value of \$59.18.* Their totals range from \$159.92, in Column 3 of Table 9, to \$195.59, in Column 13 of Table 1. Mr. Gray's final scale totals \$161.69.

The present value of the dividend scale is not necessarily the total

* For a similar example, see the last page of Mr. Estes' paper, *RAIA* XXI, 220.

amount of the discounted margins. A certain amount of surplus after dividends, at the end of the period, might be specified as desirable for contingencies. Its initial value would be deducted from the total of the discounted margins, giving the present value of the potential dividends. This deduction would be spread over the individual items, so that the resulting discounted crude dividends would add up to the revised total.

Alternatively, this adjustment could be made later, after the graduation was completed. Mr. Johansen has indicated how a level deduction from each dividend, to build up a contingency fund, can be found. Being level, it would not affect the smoothness. Being small, it would hardly complicate extension of the scale to higher durations.

Before or after this adjustment, and after any obvious additional adjustments required, we have a column of discounted dividends to be smoothed, by hand or otherwise. Equity suggests a graduation formula that gives some weight to fit. A Whittaker-Henderson "A" formula meets this requirement, and approximately reproduces totals as well. Normally, if this is applied to the discounted dividends, and the results translated into undiscounted values, little further revision will be needed.

If such a formula were applied instead to undiscounted figures, additional adjustment would be necessary. Its extent is suggested by a comparison of the crude dividend total, $146.97, from Column 10 of Table 1, with the three totals already quoted.

Another variation is possible. One can operate on the crude dividends themselves, as before, to obtain an experimental scale. Then this is tested, not by asset shares, but by taking the present values of this scale. The total is compared with that of the discounted crude dividends. The two must be reasonably close together before the dividend scale is satisfactory. It seems obvious that a lot of trial and error would usually be involved.

ARE ASSET SHARES NECESSARY?

It had been my thought that the paper showed the lack of any general need for actually calculating asset shares. Apparently, however, the concept has taken deep root.

Two reasons are usually given for computing asset shares: to test a set of cash values, and to test a dividend scale. But Column 5 of Table 9, which corresponds to Column 12 of Table 2, constitutes a test of both. This column represents the present value of the accumulated margins after dividends, or the discounted excess of the asset share over the cash value.

Only the first entry in Column 5 is negative. This loses its significance when it is recalled that the first year dividend is contingent upon the next year's premium. In general, however, a negative figure would mean that either the cash value or the dividend is too high, under the assumptions used.

Sometimes it is desired to obtain the effect of varying the assumptions. For instance, one might inquire as to the result of doubling the termination rates, to study the possible impact of a severe depression. Or one might wish, with Mr. Johansen and others before him, to calculate figures using zero surrender rates, for purposes of comparison. In either case, the present value approach provides economy of method. Column 9 of Table 1 remains unchanged, and we put a new set of persistency and discount factors in Column 8. If the dividend scale is to be retained and tested, we can follow the setup of Table 2 instead. Its Column 9 also remains unchanged, and the new factors go into Column 10. It will be necessary to go only as far as Column 12, to apply the test of the previous paragraph.

OTHER POINTS

Mr. Johansen has supplied an additional advantage of the present value method, and an algebraic proof. He has also gone considerably farther than I did under the heading of "Other Variations." However, I am most grateful to him for presenting the only discussion submitted by an Associate. Since I wrote the article partly with students in mind, his reaction was particularly gratifying.

To Mr. Blair I owe a similar debt for a vicarious response on the same score. Also, his suggestion of relating all calculations to the end of the year does produce a more compact worksheet. Finally, he deserves my especial gratitude for indicating that he perceived the defects of the paper but also saw beyond them to a principle.

The point in Mr. Rieder's last paragraph is well taken. To save time, I pressed one set of assumptions into double duty, without adequate explanation. This point helps to show why the revision of a dividend scale, where several years of issue are involved, is so complicated.

Mr. Lang's comments concern both the past and the future. He has presented quite a bit of historical background, and also enlarged the Bibliography. (The extent of his reading is to be envied!) As to what lies ahead, machines will undoubtedly continue to accelerate actuarial calculations. At the same time, I believe a number of years will pass before we can feed assumptions into a machine and have it deliver a dividend scale that is equitable, popular, and competitive.

It is interesting to compare Mr. Lang's "alternative approach," outlined for two-factor and three-factor formulas, with some of Mr. Gray's ideas. Mr. Lang's suggestion of using interpolation is very ingenious. His thinking is geared to a well equipped machine room. Both men are using the contribution approach.

As a matter of history, Occidental borrowed the present value technique, as applied to the estimation of future profits, several years ago from Mr. Marcus Gunn. But for him, this paper might never have been written.

It is naturally gratifying to me that it has elicited so much discussion. I wish to thank all the gentlemen who participated. Their comments have broadened its scope and greatly enhanced its value. They have also stimulated the author, in this reply, to further analysis and to more thorough explanations. Finally, if any reader still doubts the practicality of the present value approach, I commend to him the remarks of Mr. Blair and Mr. Bergstresser.