Exam MFE/3F

Additional Sample Questions and Solutions

The difference between this version and the September 21 version is the correction of a typo in the last line of the solution on page 3.

21. The Cox-Ingersoll-Ross (CIR) interest-rate model has the short-rate process:

$$\mathrm{d}r(t) = a[b - r(t)]\mathrm{d}t + \sigma\sqrt{r(t)}\,\mathrm{d}Z(t),$$

where $\{Z(t)\}$ is a standard Brownian motion.

For $t \le T$, let $P(r,t,T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the CIR model follows an Itô process:

$$\frac{\mathrm{d}P[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t),t,T]\mathrm{d}t - q[r(t),t,T]\mathrm{d}Z(t) \qquad t \le T.$$

You are given $\alpha(0.05, 7, 9) = 0.06$.

Calculate $\alpha(0.04, 11, 13)$.

(A) 0.042

(B) 0.045

(C) 0.048

(D) 0.050

(E) 0.052

Solution to (21) Answer: (C)

As pointed out on pages 782 and 783 of McDonald (2006), the condition of no riskless arbitrages implies that the Sharpe ratio does not depend on *T*,

$$\frac{\alpha(r,t,T)-r}{q(r,t,T)}=\phi(r,t). \tag{24.17}$$

(Also see Section 20.4.) This result may not seem applicable because we are given an α for $t = 7$ while asked to find an α for $t = 11$.

Now, equation (24.12) in McDonald (2006) is

$$q(r,t,T)=-\sigma(r)P_r(r,t,T)/P(r,t,T)=-\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)],$$

the substitution of which in (24.17) yields

$$\alpha(r,t,T)-r=-\phi(r,t)\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)].$$

In the CIR model (McDonald 2006, p. 787), $\sigma(r)=\sigma\sqrt{r}$, $\phi(r,t)=\frac{\bar{\phi}}{\sigma}\sqrt{r}$ with $\bar{\phi}$ being a constant, and $\frac{\partial}{\partial r}\ln[P(r,t,T)]=-B(t,T)$. Thus,

$$\begin{aligned}\alpha(r,t,T)-r &= -\phi(r,t)\sigma(r)\frac{\partial}{\partial r}\ln[P(r,t,T)]\\ &= -\frac{\bar{\phi}}{\sigma}\sqrt{r}\times\sigma\sqrt{r}\times\left[-B(t,T)\right]\\ &= \bar{\phi}rB(t,T),\end{aligned}$$

or

$$\frac{\alpha(r,t,T)}{r}=1+\bar{\phi}B(t,T).$$

Because $B(t,T)$ depends on *t* and *T* through the difference $T-t$, we have, for $T_1-t_1=T_2-t_2$,

$$\frac{\alpha(r_1,t_1,T_1)}{r_1}=\frac{\alpha(r_2,t_2,T_2)}{r_2}.$$

Hence,

$$\alpha(0.04,11,13)=\frac{0.04}{0.05}\alpha(0.05,7,9)=0.8\times0.06=0.048.$$

Remarks: (i) In earlier printings of McDonald (2006), the minus sign in (24.1) was given as a plus sign. Hence, there was no minus sign in (24.12) and $\bar{\phi}$ would be a negative constant. However, these changes would not affect the answer to this question. (ii) What McDonald calls Brownian motion is usually called *standard* Brownian motion by other authors.

22. You are given:

(i) The short-rate $r(t)$ follows the Itô process:

$$dr(t) = [0.09 - 0.5r(t)]dt + 0.3dZ(t),$$

where $\{Z(t)\}$ is a standard Brownian motion.

(ii) The risk-neutral process of the short-rate is given by

$$dr(t) = [0.15 - 0.5r(t)]dt + \sigma(r(t))d\tilde{Z}(t),$$

where $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral measure.

(iii) $g(r, t)$ denotes the price of an interest-rate derivative at time t, if the short-rate at that time is r.

(iv) $g(r(t), t)$ satisfies

$$dg(r(t), t) = \mu(r(t), g(r(t), t))dt - 0.4g(r(t), t)dZ(t).$$

Determine $\mu(r, g)$.

(A) $(r - 0.09)g$

(B) $(r - 0.08)g$

(C) $(r - 0.03)g$

(D) $(r + 0.08)g$

(E) $(r + 0.09)g$

Solution to (22) Answer: (D)

Formula (24.2) of McDonald (2006),

$$\mathrm{d}r(t) = a(r(t))\mathrm{d}t + \sigma(r(t))\mathrm{d}Z(t)\,,$$

is the stochastic differential equation for $r(t)$ under the actual probability measure, while formula (24.19),

$$\mathrm{d}r(t) = \left[a(r(t)) + \sigma(r(t))\phi(r(t),t)\right]\mathrm{d}t + \sigma(r(t))\mathrm{d}\tilde{Z}(t)\,,$$

is the stochastic differential equation for $r(t)$ under the risk-neutral probability measure, where $\phi(r, t)$ is the Sharpe ratio. Hence,

$$\sigma(r) = 0.3,$$

and

$$\begin{aligned} 0.15 - 0.5r &= a(r) + \sigma(r)\phi(r, t) \\ &= [0.09 - 0.5r] + \sigma(r)\phi(r, t) \\ &= [0.09 - 0.5r] + 0.3\phi(r, t). \end{aligned}$$

Thus, $\phi(r, t) = 0.2$.

Now, for the model to be arbitrage free, the Sharpe ratio of the interest-rate derivative should also be given by $\phi(r, t)$. Rewriting (iv) as

$$\frac{\mathrm{d}g(r(t),t)}{g(r(t),t)} = \frac{\mu(r(t), g(r(t),t))}{g(r(t),t)}\mathrm{d}t - 0.4\mathrm{d}Z(t) \qquad \text{[cf. equation (24.13)]}$$

we see that

$$\frac{\dfrac{\mu(r, g(r,t))}{g(r,t)} - r}{0.4} = \phi(r, t) \qquad \text{[cf. equation (24.17)]}$$

$$= 0.2.$$

Thus,

$$\mu(r, g) = (r + 0.08)g.$$

Remark: $\mathrm{d}\tilde{Z}(t) = \mathrm{d}Z(t) - \phi(r(t),t)\mathrm{d}t$

23. Consider a European call option on a nondividend-paying stock with exercise date T, $T > 0$. Let $S(t)$ be the price of one share of the stock at time t, $t \geq 0$. For $0 \leq t \leq T$, let $C(s,t)$ be the price of one unit of the call option at time *t*, if the stock price is *s* at that time. You are given:

(i) $\dfrac{\mathrm{d}S(t)}{S(t)} = 0.1\mathrm{d}t + \sigma \mathrm{d}Z(t)$, where σ is a positive constant and $\{Z(t)\}$ is a Brownian motion.

(ii) $\dfrac{\mathrm{d}C(S(t),t)}{C(S(t),t)} = \gamma(S(t),t)\mathrm{d}t + \sigma_C(S(t),t)\mathrm{d}Z(t), \qquad 0 \leq t \leq T$

(iii) $C(S(0),0) = 6$

(iv) At time $t = 0$, the cost of shares required to delta-hedge one unit of the call option is 9.

(v) The continuously compounded risk-free interest rate is 4%.

Determine $\gamma(S(0),0)$.

(A) 0.10

(B) 0.12

(C) 0.13

(D) 0.15

(E) 0.16

Solution to (23) Answer: (C)

As explained in Section 20.4 of McDonald (2006), the no-arbitrage condition implies that, at each instant in time, the assets have the same Sharpe ratio. Equating the Sharpe ratio of the stock with that of the option at time *t* yields

$$\frac{0.1-0.04}{\sigma}=\frac{\gamma(S(t),t)-0.04}{\sigma_C(S(t),t)},$$

or

$$\gamma(S(t),t)=0.04+\frac{\sigma_C(S(t),t)}{\sigma}\times 0.06. \tag{1}$$

By (12.9) [or by (21.20)], $\dfrac{\sigma_C(S(t),t)}{\sigma}=\left|\Omega(S(t),t)\right|$.

By (12.8), $\Omega(S(t),t)=\dfrac{S(t)\times\Delta(S(t),t)}{C(S(t),t)}$. Thus,

$$\Omega(S(0),0)=\frac{S(0)\times\Delta(S(0),0)}{C(S(0),0)}=\frac{9}{6}=1.5.$$

By (1), the option's expected instantaneous rate of return at time 0 is

$$\gamma(S(0),0)=0.04+1.5\times 0.06=0.13.$$

24. Consider the stochastic differential equation:

$$\mathrm{d}X(t) = \lambda[\alpha - X(t)]\mathrm{d}t + \sigma\,\mathrm{d}Z(t), \qquad t \geq 0,$$

where λ, α and σ are positive constants, and $\{Z(t)\}$ is a standard Brownian motion. The value of $X(0)$ is known.

Find a solution.

(A) $X(t) = X(0)\,e^{-\lambda t} + \alpha(1-e^{-\lambda t})$

(B) $X(t) = X(0) + \int_0^t \alpha\,\mathrm{d}s + \int_0^t \sigma\,\mathrm{d}Z(s)$

(C) $X(t) = X(0) + \int_0^t \alpha X(s)\,\mathrm{d}s + \int_0^t \sigma X(s)\,\mathrm{d}Z(s)$

(D) $X(t) = X(0) + \alpha(e^{\lambda t} - 1) + \int_0^t \sigma e^{\lambda s}\,\mathrm{d}Z(s)$

(E) $X(t) = X(0)\,e^{-\lambda t} + \alpha(1-e^{-\lambda t}) + \int_0^t \sigma e^{-\lambda(t-s)}\,\mathrm{d}Z(s)$

Solution to (24) Answer: (E)

The given stochastic differential equation is (20.9) in McDonald (2006).

Rewrite the equation as

$$\mathrm{d}X(t) \ + \ \lambda\, X(t)\mathrm{d}t \ = \ \lambda\alpha\mathrm{d}t + \sigma\,\mathrm{d}Z(t).$$

If this were an ordinary differential equation, we would solve it by the *method of integrating factors*. (Students of life contingencies have seen the method of integrating factors in Exercise 4.22 on page 129 and Exercise 5.5 on page 158 of *Actuarial Mathematics*, 2nd edition.) Let us give this a try. Multiply the equation by the integrating factor $e^{\lambda t}$, we have

$$e^{\lambda t}\,\mathrm{d}X(t) \ + \ e^{\lambda t}\lambda X(t)\mathrm{d}t \ = \ e^{\lambda t}\lambda\alpha\mathrm{d}t \ + \ e^{\lambda t}\sigma\mathrm{d}Z(t). \qquad (*)$$

We hope that the left-hand side is exactly $\mathrm{d}[e^{\lambda t}X(t)]$. To check this, consider $f(x, t) = e^{\lambda t}x$, whose relevant derivatives are $f_x(x, t) = e^{\lambda t}$, $f_{xx}(x, t) = 0$, and $f_t(x, t) = \lambda e^{\lambda t}x$. By Itô's Lemma,

$$\mathrm{d}f(X(t), t) \ = \ e^{\lambda t}\,\mathrm{d}X(t) \ + \ 0 \ + \ \lambda e^{\lambda t}\,X(t)\mathrm{d}t,$$

which is indeed the left-hand side of (*). Now, (*) can be written as

$$\mathrm{d}[e^{\lambda s}X(s)] \ = \ \lambda\alpha e^{\lambda s}\mathrm{d}s + \sigma e^{\lambda s}\mathrm{d}Z(s).$$

Integrating both sides from $s = 0$ to $s = t$, we have

$$e^{\lambda t}X(t) - e^{\lambda 0}X(0) = \lambda\alpha\int_0^t e^{\lambda s}\mathrm{d}s + \sigma\int_0^t e^{\lambda s}\mathrm{d}Z(s) = \alpha(e^{\lambda t} - 1) + \sigma\int_0^t e^{\lambda s}\mathrm{d}Z(s)\,,$$

or

$$e^{\lambda t}X(t) \ = \ X(0) \ + \ \alpha(e^{\lambda t} - 1) \ + \sigma\int_0^t e^{\lambda s}\mathrm{d}Z(s)\,.$$

Multiplying both sides by $e^{-\lambda t}$ and rearranging yields

$$X(t) \ = \ X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) \ + \ \sigma e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s)$$

$$= \ X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) \ + \ \sigma\int_0^t e^{-\lambda(t-s)}\mathrm{d}Z(s)\,,$$

which is (E).

Remarks: This question is the same as Exercise 20.9 on page 674. In the above, the solution is derived by solving the stochastic differential equation, while in Exercise 20.9, you are asked to use Itô's Lemma to verify that (E) satisfies the stochastic differential equation.

If $t = 0$, then the right-hand side of (E) is $X(0)$.

If $t > 0$, we differentiate (E). The first and second terms on the right-hand side are not random and have derivatives $-\lambda X(0)e^{-\lambda t}$ and $\alpha\lambda e^{-\lambda t}$, respectively. To differentiate the stochastic integral in (E), we write

$$\int_0^t e^{-\lambda(t-s)}\mathrm{d}Z(s) = e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s),$$

which is a product of a deterministic factor and a stochastic factor. Then,

$$\begin{aligned}
\mathrm{d}\left(e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s)\right) &= (\mathrm{d}e^{-\lambda t})\int_0^t e^{\lambda s}\mathrm{d}Z(s) + e^{-\lambda t}\mathrm{d}\int_0^t e^{\lambda s}\mathrm{d}Z(s) \\
&= (\mathrm{d}e^{-\lambda t})\int_0^t e^{\lambda s}\mathrm{d}Z(s) + e^{-\lambda t}[e^{\lambda t}\mathrm{d}Z(t)] \\
&= -\left(\lambda e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s)\right)\mathrm{d}t + \mathrm{d}Z(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{d}X(t) &= -\lambda X(0)e^{-\lambda t}\mathrm{d}t + \alpha\lambda e^{-\lambda t}\mathrm{d}t - \sigma\left(\lambda e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s)\right)\mathrm{d}t + \sigma\mathrm{d}Z(t) \\
&= -\lambda\left[X(0)e^{-\lambda t} + \sigma e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z(s)\right]\mathrm{d}t + \alpha\lambda e^{-\lambda t}\mathrm{d}t + \sigma\mathrm{d}Z(t) \\
&= -\lambda[X(t) - \alpha(1-e^{-\lambda t})]\mathrm{d}t + \alpha\lambda e^{-\lambda t}\mathrm{d}t + \sigma\mathrm{d}Z(t) \\
&= -\lambda[X(t) - \alpha]\mathrm{d}t + \sigma\mathrm{d}Z(t),
\end{aligned}$$

which is the same as the given stochastic differential equation.

25. Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of $100.

The chooser option price is $20 at time $t = 0$.

The stock price is $95 at time $t = 0$. Let $C(T)$ denote the price of a European call option at time $t = 0$ on the stock expiring at time $T, T > 0$, with a strike price of $100.

You are given:

(i) The risk-free interest rate is 0.

(ii) $C(1) = \$4$.

Determine $C(3)$.

(A) $ 9

(B) $11

(C) $13

(D) $15

(E) $17

Solution to (25) Answer: (B)

Let $C(S(t), t, T)$ denote the price at time-*t* of a European call option on the stock, with exercise date *T* and exercise price $K = 100$. So,

$$C(T) = C(95, 0, T).$$

Similarly, let $P(S(t), t, T)$ denote the time-*t* put option price.

At the choice date $t = 1$, the value of the chooser option is

$$\text{Max}[C(S(1), 1, 3), P(S(1), 1, 3)],$$

which can expressed as

$$C(S(1), 1, 3) + \text{Max}[0, P(S(1), 1, 3) - C(S(1), 1, 3)]. \qquad (1)$$

Because the stock pays no dividends and the interest rate is zero,

$$P(S(1), 1, 3) - C(S(1), 1, 3) = K - S(1)$$

by put-call parity. Thus, the second term of (1) simplifies as

$$\text{Max}[0, K - S(1)],$$

which is the payoff of a European put option. As the time-1 value of the chooser option is

$$C(S(1), 1, 3) + \text{Max}[0, K - S(1)],$$

its time-0 price *must* be

$$C(S(0), 0, 3) + P(S(0), 0, 1),$$

which, by put-call parity, is

$$C(S(0), 0, 3) + [C(S(0), 0, 1) + K - S(0)]$$
$$= C(3) + [C(1) + 100 - 95] = C(3) + C(1) + 5.$$

Thus,

$$C(3) = 20 - (4 + 5) = 11.$$

Remark: The problem is a modification of Exercise 14.20.b.

26. Consider European and American options on a nondividend-paying stock. You are given:

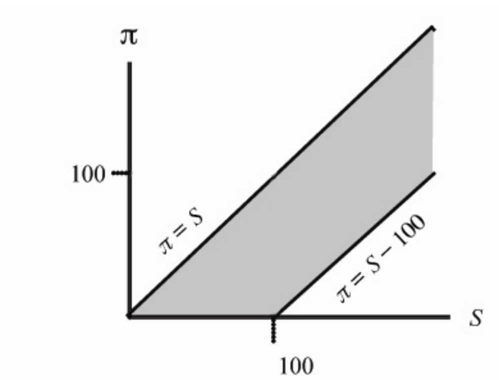
(i) All options have the same strike price of 100.

(ii) All options expire in six months.

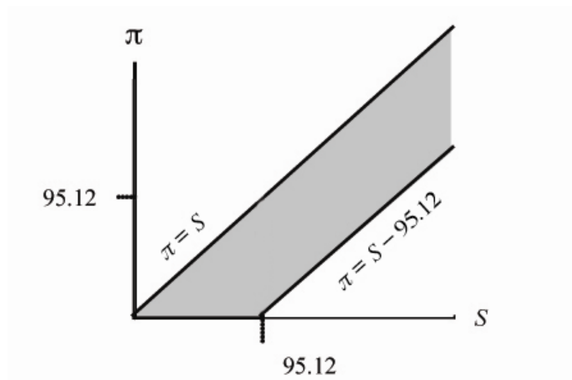
(iii) The continuously compounded risk-free interest rate is 10%.

You are interested in the graph for the price of an option as a function of the current stock price. In each of the following four charts I–IV, the horizontal axis, S, represents the current stock price, and the vertical axis, π, represents the price of an option.

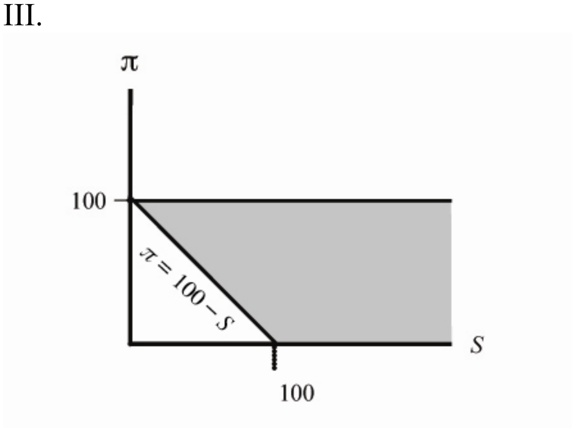
I.



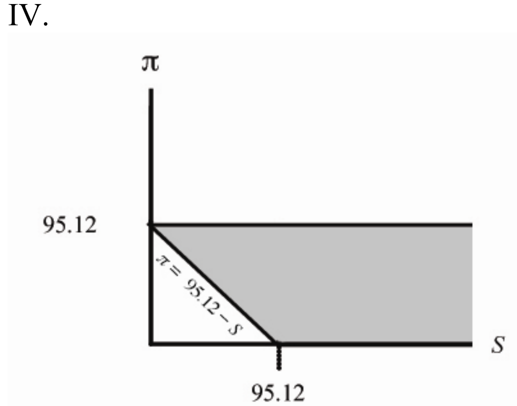
II.



III.



IV.



26. Continued

Match the option with the shaded region in which its graph lies. If there are two or more possibilities, choose the chart with the smallest shaded region.

	European Call	American Call	European Put	American Put
(A)	I	I	III	III
(B)	II	I	IV	III
(C)	II	I	III	III
(D)	II	II	IV	III
(E)	II	II	IV	IV

Solution to (26) Answer: (D)

$T = \frac{1}{2};\quad \mathrm{PV}_{0,T}(K) = Ke^{-rT} = 100e^{-0.1/2} = 100e^{-0.05} = 95.1229 \approx 95.12.$

By (9.9) on page 293 of McDonald (2006), we have

$$S(0) \ge C_{Am} \ge C_{Eu} \ge \mathrm{Max}[0, F_{0,T}^P(S) - \mathrm{PV}_{0,T}(K)].$$

Because the stock pays no dividends, the above becomes

$$S(0) \ge C_{Am} = C_{Eu} \ge \mathrm{Max}[0, S(0) - \mathrm{PV}_{0,T}(K)].$$

Thus, the shaded region in II contains C_{Am} and C_{Eu}. (The shaded region in I also does, but it is a larger region.)

By (9.10) on page 294 of McDonald (2006), we have

$$\begin{aligned} K \ge P_{Am} \ge P_{Eu} &\ge \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - F_{0,T}^P(S)] \\ &= \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S(0)] \end{aligned}$$

because the stock pays no dividends. However, the region bounded above by $\pi = K$ and bounded below by $\pi = \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S]$ is not given by III or IV.

Because an American option can be exercised immediately, we have a tighter lower bound for an American put,

$$P_{Am} \ge \mathrm{Max}[0, K - S(0)].$$

Thus,

$$K \ge P_{Am} \ge \mathrm{Max}[0, K - S(0)],$$

showing that the shaded region in III contains P_{Am}.

For a European put, we can use put-call parity and the inequality $S(0) \ge C_{Eu}$ to get a tighter upper bound,

$$\mathrm{PV}_{0,T}(K) \ge P_{Eu}.$$

Thus,

$$\mathrm{PV}_{0,T}(K) \ge P_{Eu} \ge \mathrm{Max}[0, \mathrm{PV}_{0,T}(K) - S(0)],$$

showing that the shaded region in IV contains P_{Eu}.

Remarks:

(i) It turns out that II and IV can be found on page 156 of Capiński and Zastawniak (2003) *Mathematics for Finance: An Introduction to Financial Engineering*, Springer Undergraduate Mathematics Series.

(ii) The last inequality in (9.9) can be derived as follows. By put-call parity,

$$\begin{aligned} C_{Eu} &= P_{Eu} + F_{0,T}^{P}(S) - e^{-rT}K \\ &\geq F_{0,T}^{P}(S) - e^{-rT}K \qquad \text{because } P_{Eu} \geq 0. \end{aligned}$$

We also have

$$C_{Eu} \geq 0.$$

Thus,

$$C_{Eu} \geq \text{Max}(0, F_{0,T}^{P}(S) - e^{-rT}K).$$

(iii) An alternative derivation of the inequality above is to use *Jensen's Inequality* (see, in particular, page 883).

$$\begin{aligned} C_{Eu} &= \text{E}^*\left[e^{-rT}\text{Max}(0, S(T) - K)\right] \\ &\geq e^{-rT}\text{Max}(0, \text{E}^*[S(T) - K]) \quad \text{because of Jensen's Inequality} \\ &= \text{Max}(0, \text{E}^*\left[e^{-rT}S(T)\right] - e^{-rT}K) \\ &= \text{Max}(0, F_{0,T}^{P}(S) - e^{-rT}K). \end{aligned}$$

Here, E* signifies risk-neutral expectation.

(iv) The fact that $C_{Eu} = C_{Am}$ for nondividend-paying stocks can also be seen as a consequence of Jensen's Inequality.

27. You are given the following information about a securities market:

(i) There are two nondividend-paying stocks, X and Y.

(ii) The current prices for X and Y are both \$100.

(iii) The continuously compounded risk-free interest rate is 10%.

(iv) There are three possible outcomes for the prices of X and Y one year from now:

Outcome	X	Y
1	\$200	\$0
2	\$50	\$0
3	\$0	\$300

Let C_X be the price of a European call option on X, and P_Y be the price of a European put option on Y. Both options expire in one year and have a strike price of \$95.

Calculate $P_Y - C_X$.

(A) \$4.30

(B) \$4.45

(C) \$4.59

(D) \$4.75

(E) \$4.94

Solution to (27) Answer: (A)

We are given the price information for three securities:

$$B: \qquad e^{-0.1} \left\langle \begin{array}{l} 1 \\ 1 \\ 1 \end{array} \right.$$

$$X: \qquad 100 \left\langle \begin{array}{l} 200 \\ 50 \\ 0 \end{array} \right.$$

$$Y: \qquad 100 \left\langle \begin{array}{l} 0 \\ 0 \\ 300 \end{array} \right.$$

The problem is to find the price of the following security

$$?? \left\langle \begin{array}{l} -10 \\ 95 \\ 0 \end{array} \right.$$

The time-1 payoffs come from:

$(95 - 0)_+ - (200 - 95)_+ = 95 - 105 = -10$

$(95 - 0)_+ - (50 - 95)_+ = 95 - 0 = 95$

$(95 - 300)_+ - (0 - 95)_+ = 0 - 0 = 0$

So, this is a *linear algebra* problem. We can take advantage of the 0's in the time-1 payoffs. By considering linear combinations of securities *B* and *Y*, we have

$$B - \frac{Y}{300}: \qquad e^{-0.1} - \tfrac{1}{3} \left\langle \begin{array}{l} 1 \\ 1 \\ 0 \end{array} \right.$$

We now consider linear combinations of this security and *X*. Then, the answer for the problem is

$$(100, \quad e^{-0.1} - \tfrac{1}{3})\begin{pmatrix} 200 & 1 \\ 50 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -10 \\ 95 \end{pmatrix}$$

$$= \frac{1}{200-50}(100, \quad e^{-0.1} - \tfrac{1}{3})\begin{pmatrix} 1 & -1 \\ -50 & 200 \end{pmatrix}\begin{pmatrix} -10 \\ 95 \end{pmatrix}$$

$$= \frac{1}{150}(100, \quad 0.571504085)\begin{pmatrix} -105 \\ 19500 \end{pmatrix}$$

$$= 4.295531 \ \approx \ 4.30.$$

Remarks: (i) We have priced the security without knowledge of the real probabilities. This is analogous to pricing options in the Black-Scholes framework without the need to know α, the continuously compounded expected return on the stock.

(ii) We have used the following inversion formula for 2-by-2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ab - cd}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(iii) Matrix calculations can also be used to derive some of the results in Chapter 10 of McDonald (2006). The price of a security that pays C_u when the stock price goes up and pays C_d when the stock price goes down is

$$(S \quad 1)\begin{pmatrix} uSe^{\delta h} & e^{rh} \\ dSe^{\delta h} & e^{rh} \end{pmatrix}^{-1}\begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= \frac{1}{uSe^{(\delta+r)h} - dSe^{(\delta+r)h}}(S \quad 1)\begin{pmatrix} e^{rh} & -e^{rh} \\ -dSe^{\delta h} & uSe^{\delta h} \end{pmatrix}\begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= \frac{1}{(u-d)e^{(\delta+r)h}}(e^{rh} - de^{\delta h} \quad ue^{\delta h} - e^{rh})\begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= e^{-rh}(\frac{e^{(r-\delta)h} - d}{u-d} \quad \frac{u - e^{(r-\delta)h}}{u-d})\begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$= e^{-rh}(p* \quad 1-p*)\begin{pmatrix} C_u \\ C_d \end{pmatrix}.$$

28. Assume the Black-Scholes framework. You are given:

(i) $S(t)$ is the price of a nondividend-paying stock at time t.

(ii) $S(0) = 10$

(iii) The stock's volatility is 20%.

(iv) The continuously compounded risk-free interest rate is 2%.

At time $t = 0$, you write a one-year European option that pays 100 if $[S(1)]^2$ is greater than 100 and pays nothing otherwise.

You delta-hedge your commitment.

Calculate the number of shares of the stock for your hedging program at time $t = 0$.

(A) 20

(B) 30

(C) 40

(D) 50

(E) 60

Solution to (28) Answer: (A)

Note that $[S(1)]^2 > 100$ is equivalent to $S(1) > 10$.
Let $I(.)$ denote the *indicator function*, i.e.,

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Then, the option payoff is

$$100 \times I(S(1) > 10).$$

Now, the time-0 price for a European option with a time-T payoff of $I(S(T) > K)$ is

$$e^{-rT} N(d_2).$$

One way to obtain this result is to note that the time-T payoff of a *gap call option* with strike price K_1 and payment trigger K_2 is

$$[S(T) - K_1]\, I(S(T) > K_2)$$
$$= S(T) I(S(T) > K_2) - K_1 I(S(T) > K_2).$$

Because K_1 is arbitrary, the coefficient of K_1 in the gap call option price formula (14.15) gives the result.

To find the number of shares in the hedging program, we differentiate the price formula with respect to S,

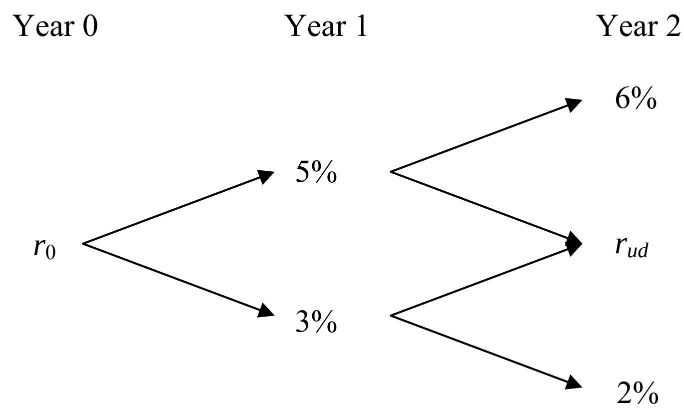
$$\frac{\partial}{\partial S} 100 e^{-rT} N(d_2)$$
$$= 100 e^{-rT} N'(d_2) \frac{\partial d_2}{\partial S} = 100 e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}}.$$

With $T = 1$, $r = 0.02$, $\delta = 0$, $\sigma = 0.2$, $S = S(0) = 10$, $K = K_2 = 10$, we have $d_2 = 0$ and

$$100 e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}} = 100 e^{-0.02} N'(0) \frac{1}{2}$$
$$= 100 e^{-0.02} \frac{e^{-0^2/2}}{\sqrt{2\pi}} \frac{1}{2}$$
$$= \frac{50 e^{-0.02}}{\sqrt{2\pi}} = 19.55.$$

Remark: An option with payoff $I(S(T) > K)$ is sometimes called a *cash-or-nothing call*. An option with payoff $S(T)I(S(T) > K)$ is sometimes called an *asset-or-nothing call*.

29. The following is a Black-Derman-Toy binomial tree for effective annual interest rates.



Compute the volatility in year 1 of the 3-year zero-coupon bond generated by the tree.

(A) 14%

(B) 18%

(C) 22%

(D) 26%

(E) 30%

Solution to (29) Answer: (D)

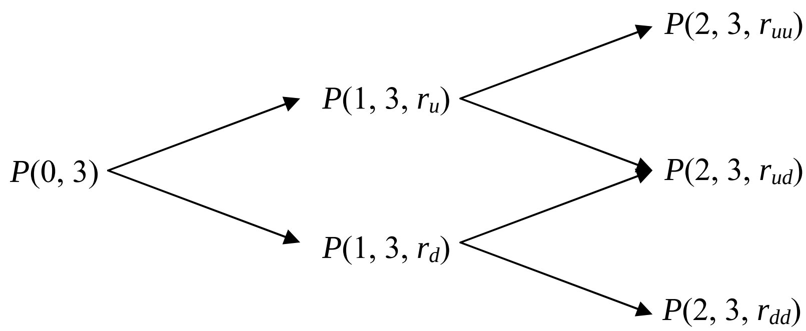
According to formula (24.48) in McDonald (2006), the "volatility in year 1" of an n-year zero-coupon bond in a Black-Derman-Toy model is the number κ such that

$$y(1, n, r_u) = y(1, n, r_d)\, e^{2\kappa},$$

where y, the yield to maturity, is defined by

$$P(1, n, r) = \left(\frac{1}{1+y(1,n,r)}\right)^{n-1}.$$

Here, $n = 3$ [and hence κ is given by the right-hand side of (24.53)]. To find $P(1, 3, r_u)$ and $P(1, 3, r_d)$, we use the method of *backward induction*.



$$P(2, 3, r_{uu}) = \frac{1}{1+r_{uu}} = \frac{1}{1.06},$$

$$P(2, 3, r_{dd}) = \frac{1}{1+r_{dd}} = \frac{1}{1.02},$$

$$P(2, 3, r_{du}) = \frac{1}{1+r_{ud}} = \frac{1}{1+\sqrt{r_{uu} \times r_{dd}}} = \frac{1}{1.03464},$$

$$P(1, 3, r_u) = \frac{1}{1+r_u}[\tfrac{1}{2} P(2, 3, r_{uu}) + \tfrac{1}{2} P(2, 3, r_{ud})] = 0.909483,$$

$$P(1, 3, r_d) = \frac{1}{1+r_d}[\tfrac{1}{2} P(2, 3, r_{ud}) + \tfrac{1}{2} P(2, 3, r_{dd})] = 0.945102.$$

Hence,

$$e^{2\kappa} = \frac{y(1,3,r_u)}{y(1,3,r_d)} = \frac{[P(1,3,r_u)]^{-1/2} - 1}{[P(1,3,r_d)]^{-1/2} - 1} = \frac{0.048583}{0.028633},$$

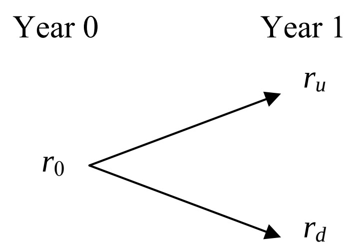
resulting in κ = 0.264348 ≈ 26%.

Remark: The term "year n" can be ambiguous. In the Exam MFE/3L textbook *Actuarial Mathematics*, it usually means the n-th year, depicting a period of time. However, in many places in McDonald (2006), it means time n, depicting a particular instant in time.

30. You are given the following market data for zero-coupon bonds with a maturity payoff of \$100.

Maturity (years)	Bond Price (\$)	Volatility in Year 1
1	94.34	N/A
2	88.50	10%

A 2-period Black-Derman-Toy interest tree is calibrated using the data from above:



Calculate r_d, the effective annual rate in year 1 in the "down" state.

(A) 5.94%

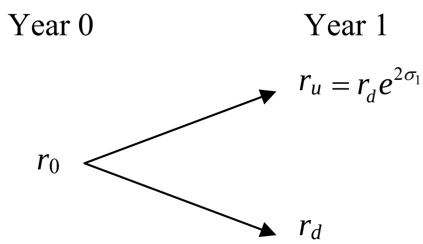
(B) 6.60%

(C) 7.00%

(D) 7.27%

(E) 7.33%

Solution to (30) Answer: (A)



In a BDT interest rate model, the risk-neutral probability of each "up" move is ½.

Because the "volatility in year 1" of the 2-year zero-coupon bond is 10%, we have

$$\sigma_1 = 10\%.$$

This can be seen from simplifying the right-hand side of (24.51).

Now,

$$\begin{aligned} P(0, 2) &= P(0, 1)[\tfrac{1}{2}P(1, 2, r_u) + \tfrac{1}{2}P(1, 2, r_d)] \\ &= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_u} + \frac{1}{2}\frac{1}{1+r_d}\right] \\ &= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_d e^{0.2}} + \frac{1}{2}\frac{1}{1+r_d}\right], \end{aligned}$$

or

$$\frac{1}{1+r_d e^{0.2}} + \frac{1}{1+r_d} = \frac{2\times 0.8850}{0.9434} = 1.8762.$$

Thus, we have

$$2 + r_d(1+e^{0.2}) = 1.8762[1 + r_d(1+e^{0.2}) + r_d^{\,2} e^{0.2}],$$

which is equivalent to

$$1.8762e^{0.2} r_d^{\,2} + 0.8762(1+e^{0.2})r_d - 0.1238 = 0.$$

The solution set of the quadratic equation is $\{0.0594, \ -0.9088\}$. Thus,

$$r_d \approx 5.94\%.$$

31. You compute the delta for a 50-60 bull spread with the following information:

(i) The continuously compounded risk-free rate is 5%.

(ii) The underlying stock pays no dividends.

(iii) The current stock price is \$50 per share.

(iv) The stock's volatility is 20%.

(v) The time to expiration is 3 months.

How much does the delta change after 1 month, if the stock price does not change?

(A) increases by 0.04

(B) increases by 0.02

(C) does not change, within rounding to 0.01

(D) decreases by 0.02

(E) decreases by 0.04

Solution to (31) Answer: (B)

Assume that the bull spread is constructed by buying a 50-strike call and selling a 60-strike call. (You may also assume that the spread is constructed by buying a 50-strike put and selling a 60-strike put.)

The delta for the bull spread is equal to

(delta for the 50-strike call) – (delta for the 60-strike call).

The delta will be identical if put options are used instead of call options.

Call option delta = $N(d_1)$, where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}\times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = 0.175, \quad N(d_1) \approx N(0.18) = 0.5714$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2}\times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = -1.6482, \quad N(d_1) \approx N(-1.65)$$

$$= 1 - 0.9505 = 0.0495$$

Delta of the bull spread = 0.5714 – 0.0495 = 0.5219.

After one month, 50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}\times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = 0.1429, \qquad N(d_1) \approx N(0.14) = 0.5557$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2}\times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = -2.0901, \qquad N(d_1) \approx N(-2.09)$$

$$= 1 - 0.9817 = 0.0183$$

Delta of the bull spread = 0.5557 – 0.0183 = 0.5374.

The change in delta = 0.5374 – 0.5219 = 0.0155 ≈ 0.02.