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ACTUARIAL NOTE: ON AVERAGE AGE AT DEATH PROBLEMS

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JOHN C. MAYNARD:

The authors of this paper are not the first to be intrigued by the series of problems involving average age at death which arise from considering different sections of a stationary population. The examination committees on life contingencies have long realized that these problems afford a useful test of reasoning power and of understanding of the basic symbolism of actuarial mathematics.

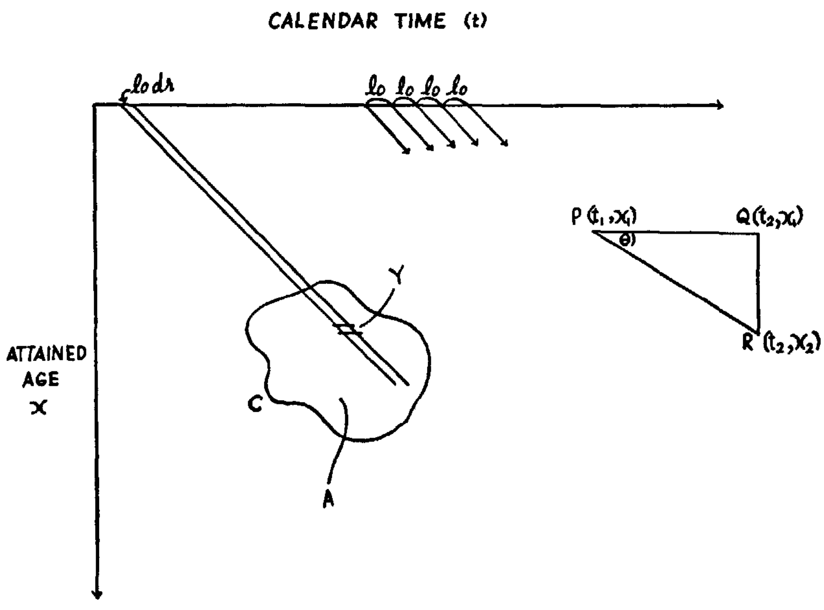

FIG. 1

The paper expresses the relevant principle in the formula (10). It is then shown to apply in several particular cases. It is the purpose of this discussion to give a general proof of the principle, and in the process to develop exact general formulas which will lead directly to a solution in a particular case.

It is necessary first to have a mathematical model for the stationary population. This is provided by the two co-ordinate systems of calendar time (t) and attained age (x) (see Fig. 1). It is assumed that persons enter the diagram along the birth line at age 0, at the uniform rate of l_0 per year. Survivors move along diagonal paths as age and time increase. The only withdrawals are by death as called for by the underlying mortality table.

It will be noted that all of the problems examined in the paper may be represented on the diagram as specific closed areas bounded by straight lines. For example, Fig. 2 represents problem (c) page 72, Fig. 3 represents (a) page 72 and Fig. 4 represents (b) page 73. The average age at death is

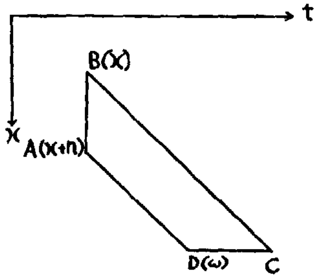

FIG. 2

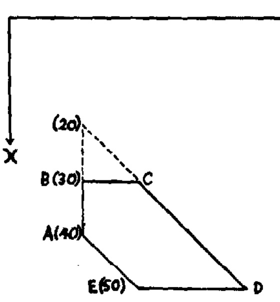

FIG. 3

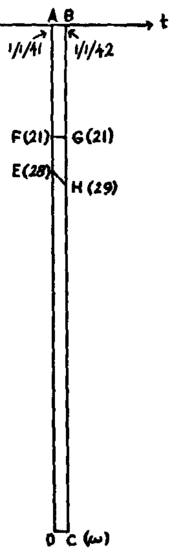

FIG. 4

found by dividing the total of ages at death of those who die within the area by the number of persons who die within the area. It will, therefore, be sufficiently general to assume an area (A) of arbitrary shape bounded by a closed curve (C) and to examine the expression for the average age at death.

Any part of the population may be identified by the co-ordinates (r, x) as well as by (t, x), where r is the calendar time at birth. The connecting relation between the two sets of co-ordinates is $r + x = t$. The survivors of $l_0 dr$ entrants after x years will number $l_x dr$ and the number of deaths among them in the instant dx after attaining that age is $(l_x dr)(\mu_x dx)$. This is the number of deaths in the elemental area Y in Fig. 1. The total of ages at death for the same persons is $x(l_x dr)(\mu_x dx)$. The average age at death is obtained by integrating these expressions over the area A, and by remembering that l_x and F_x are functions of x only, such that

$$dl_x = -l_x \mu_x dx$$

$$dF_x = -x l_x \mu_x dx\,.$$

Thus

$$a = \frac{\tau}{\eta} = \frac{\int\int_A x l_x \mu_x dx\,dr}{\int\int_A l_x \mu_x dx\,dr} = \frac{\int\int_A dr\,(-dF_x)}{\int\int_A dr\,(-dl_x)} \tag{i}$$

$$= \frac{\oint_C F_x dr}{\oint_C l_x dr} \tag{ii}$$

$$= \frac{\oint_C (F_x dt - F_x dx)}{\oint_C (l_x dt - l_x dx)} \tag{iii}$$

$$= \frac{\oint_C F_x dt}{\oint_C l_x dt}\,. \tag{iv}$$

The expressions (ii), (iii), (iv) are line integrals such that the integrand and differentials take their values on the curve C. The integration is performed around the curve clockwise. The form (iv) arises because F_x, l_x are functions of x only and the integrals $\int F_x dx$ and $\int l_x dx$ taken around any closed path are zero.

Most of the problems which arise involve areas bounded by straight lines and it is useful to apply the formula (iii) to examples of this kind. To do this it is necessary to evaluate the line integrals over succeeding segments of straight lines.

It will be recalled that

$$T_x = \int_x^\infty l_y dy \tag{v}$$

$$G_x = \int_x^\infty F_y dy\,. \tag{vi}$$

Then if the line P(t_1, x_1), R(t_2, x_2) is inclined at an angle θ to the t-axis and Q(t_2, x_1) is the other corner of the right angled triangle PQR (Fig. 1); and if $h(x)$ is any function of x:

$$\int_P^R [h(x)\,dt - h(x)\,dx] = (\cot\theta - 1)\int_{x_1}^{x_2} h(x)\,dx = (1 - \cot\theta) \times \left[\int_{x_2}^{\infty} h(x)\,dx - \int_{x_1}^{\infty} h(x)\,dx\right] \quad \text{where } \theta \neq 0 \tag{vii}$$

and

$$\int_P^Q [h(x)\,dt - h(x)\,dx] = h(x_1)\int_{t_1}^{t_2} dt = (t_2 - t_1)\,h(x_1) \tag{viii}$$

which is the general form where $\theta = 0$.

When l_x and F_x are substituted for $h(x)$ we get the corresponding forms

$$\text{where } \theta \neq 0: \int_P^R (l_x dt - l_x dx) = (1 - \cot\theta)\,[T_{x_2} - T_{x_1}] \tag{ix}$$

$$\int_P^R (F_x dt - F_x dx) = (1 - \cot\theta)\,[G_{x_2} - G_{x_1}] \tag{x}$$

$$\text{and where } \theta = 0: \int_P^Q (l_x dt - l_x dx) = (t_2 - t_1)\,l_{x_1} \tag{xi}$$

$$\int_P^Q (F_x dt - F_x dx) = (t_2 - t_1)\,F_{x_1}. \tag{xii}$$

Formulas (ix) and (x) simplify further in the important case when the line segment is parallel to the x-axis ($\theta = 90°$, $\cot\theta = 0$).

From equation (vii) it is seen that the line integral is zero when taken over any segment of a diagonal ($\theta = 45°$, $\cot\theta = 1$). This is a useful property because many of the problems are represented by diagrams which include diagonal segments. It is this property which makes formula (iii) easier to apply to an actual example than formula (iv) which is simpler in appearance.

It is now clear from formulas (iii) and (ix) to (xii) that any portion of the stationary population which can be represented by a closed polygon gives rise to a definite expression for the average age at death. This expression has for denominator a linear combination of certain values of l_x and T_x, and the numerator is obtained from the denominator by replacing l_x by F_x and T_x by G_x. This is exactly what the authors state in their equations (7) to (10). The above formulas show the exact values taken by the constants a_i and b_i in the paper.

The principle underlying formulas (10) and (ii), (iii), (iv) is that the numerator is derived from the F_x function in the same way as the denominator is derived from the l_x function. It is noted, however, that the latter three formulas extend this principle to a general section of the stationary population and not merely to a section which may be represented by a closed polygon, as demanded by the conditions underlying formula (10).

As an application of the foregoing, it is possible in a given problem to write down at once the expression for the average age at death as soon as the appropriate diagram has been drawn. To show this we use formula (iii) and its derivations (ix) to (xii). Line integrals are evaluated over the successive segments AB, BC, CD, etc.

(c) page 72 (Fig. 2):

$$a=\frac{(G_x-G_{x+n})+0+0+0}{(T_x-T_{x+n})+0+0+0}.$$

(a) page 72 (Fig. 3):

$$a=\frac{(G_{30}-G_{40})+10F_{30}+0-20F_{50}+0}{(T_{30}-T_{40})+10l_{30}+0-20l_{50}+0}.$$

(b) page 73 (Fig. 4):

Here the formulas are applied to area ABCD and then 1/10 of the contributions to numerator and denominator from area EFGH are deducted.

$$a=\frac{F_0+(0-G_0)+0+(G_0-0)-\frac{1}{10}[(G_{21}-G_{28})+F_{21}+(G_{29}-G_{21})+0]}{l_0+(0-T_0)+0+(T_0-0)-\frac{1}{10}[(T_{21}-T_{28})+l_{21}+(T_{29}-T_{21})+0]}.$$

The writer wishes to thank Mr. W. B. Waugh for reading this discussion and making several suggestions.

(AUTHORS' REVIEW OF DISCUSSION)

WALTER L. GRACE AND CECIL J. NESBITT:

It is something of a tribute to the mathematical interests and abilities of members of our profession that in discussion and correspondence in regard to our note we received two elegant and quite independent formal proofs of our formula (10).

A proof using single integrals was suggested by Mr. L. H. Longley-Cook. Our adaptation of his proof begins with the noting of the following relations for l_x and T_y:

$$l_x=\int_x^\infty l_z\mu_z d\,z \tag{1}$$

$$\left.\begin{aligned} T_y&=\int_y^\infty l_z d\,z\\ &=-yl_y+\int_y^\infty zl_z\mu_z d\,z\\ &=\int_y^\infty (z-y)\,l_z\mu_z d\,z\end{aligned}\right\} \tag{2}$$

and the corresponding relations for F_x and G_y:

$$F_x = \int_x^{\infty} z l_z \mu_z d z \tag{3}$$

$$\left.\begin{aligned} G_y &= \int_y^{\infty} F_z d z \\ &= -yF_y + \int_y^{\infty} z^2 l_z \mu_z d z \\ &= \int_y^{\infty} z(z-y) l_z \mu_z d z . \end{aligned}\right\} \tag{4}$$

We next assume that the total number of deaths in the group we are studying may be represented by

$$\eta = \int_{z_0}^{\infty} g_z l_z \mu_z d z . \tag{5}$$

Here $g_z l_z \mu_z dz$ is a quite general expression for the number of deaths that occur in the moment of age z to $z + dz$ among members of the group. The function g_z need not be continuous but should be such that $g_z l_z \mu_z$ is integrable. The corresponding integral for τ is immediately written down as

$$\tau = \int_{z_0}^{\infty} z g_z l_z \mu_z d z . \tag{6}$$

In the note we were interested in the case where η is expressible as

$$\sum_{i=1}^{r} a_i l_{x_i} + \sum_{j=1}^{s} b\, T_{y_j} .$$

To be more precise, we were interested in the situation

$$\eta = \int_{z_0}^{\infty} g_z l_z \mu_z d z \equiv \sum_{i=1}^{r} a_i l_{x_i} + \sum_{j=1}^{s} b_j T_{y_j} \tag{7}$$

where the identity sign indicates that equality holds *no matter what mortality table is employed.* It may then be shown that the identity (7) holds if and only if z_0 is the least of the x_i, y_j and [for almost all z within the effective domain]

$$g_z = \sum_i a_i + \sum_j b_j (z - y_j) \tag{8}$$

with summation over i such that $x_i \leq z$ and summation over j such that $y_j < z$. The "if" part of this statement may be seen by applying relations (1) and (2) to bring the right hand member of (7) into the form of the integral $\int_{z_0}^{\infty} g_z l_z \mu_z dz$. The "only if" part depends on the fact that the relation (7) holds for any mortality table and hence for a quite arbitrary choice of the "curve of deaths" function $l_z \mu_z$.

Applying equation (8) in formula (6), we obtain

$$\tau = \int_{z_0}^{\infty} z\Big[\sum_i a_i + \sum_j b_j\,(z - y_j)\Big] l_z \mu_z d\,z\,. \tag{9}$$

By use of the relations (3) and (4) equation (9) may be reduced to

$$\begin{aligned}\tau &= \sum_{i=1}^{r} a_i F_{x_i} + \sum_{j=1}^{s} b_j\,G_{y_j}\\ &= \varphi\,(F_{x_i}, G_{y_j})\,,\end{aligned}$$

which completes this proof of formula (10) of the note.

Mr. Maynard's proof uses the more powerful tools of double and line integrals. His methods are directly related to those indicated by Professor James Glover in *United States Life Tables 1890, 1901, 1910* and *1901–1910*, Part VI. For stationary population problems his discussion elegantly develops the basis of our formula (10) and supplies a very efficient way of applying it. It is most interesting to us to see his diagram representations of the problems of our note. He may be interested to know that one of the authors first developed formula (10) by means of diagrams of an entirely different nature which were, in fact, related to the Venn diagrams of Boolean algebra. Our original approach was algebraic rather than analytic but to obtain full generality the latter approach seems necessary. However, for some of the problems of the note, including the last and most complicated one, simple verbal solutions, based on stationary population concepts, may be given.

It should be noted that formula (10) was not developed for stationary population groups only, but was to apply to problems in regard to survivorship groups as well. Mr. Maynard's discussion is given in reference to stationary population models only. It seems possible, however, that average age at death problems in connection with some model other than a stationary population could usually be related to a corresponding stationary population problem and thus permit the use of Mr. Maynard's methods.

We wish to thank Mr. Longley-Cook and Mr. Maynard for the ideas and mathematical interest that they have added to our note.