TRANSACTIONS OF SOCIETY OF ACTUARIES
1950 REPORTS VOL. 2 NO. 4

CALCULATION OF APPROXIMATE ANNUITY VALUES ON A MORTALITY BASIS THAT PROVIDES FOR FUTURE IMPROVEMENTS IN MORTALITY

CHARLES M. STERNHELL

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E. WARD EMERY:

We have had two papers on methods of applying the new mortality basis with Projection Scale B suggested in the Jenkins-Lew paper (*TSA* I, 369). Mr. Sternhell has approached the problem from one angle. Messrs. Fassel and Noback have approached from another angle. These authors are to be congratulated on having brought out and implemented their methods.

In this discussion I shall approach the problem from still another angle. My aim has been to produce a compact set of tables from which values may be computed with the same ease as applies to the method of Fassel and Noback but with a closeness of fit comparable to that obtained by Sternhell. Because closeness of fit seems to me to be even more important than ease of computation, this method is being presented as a discussion of Sternhell's paper. Actually it should be regarded as a discussion of both. For brevity the method is applied for male lives only with interest at $2\frac{1}{2}\%$.

My method is really two methods blended into one by empirical means. At the older ages Projection Scale B is applied in a straightforward manner for the births of 1885 and 1895. For the younger ages a new age setback method which resembles the Fassel and Noback method was used. The decision to blend the two methods at age 55 in 1950 was empirical as were also the means applied to form the juncture.

Exhibit I explains how the method works and includes the necessary basic tables. Exhibit II offers proof that it does work by actually comparing annuity values. The theory of why it works is given in the remainder of this discussion.

This paper by Sternhell has demonstrated that a forecast annuity in the year $1950 + k$ can be expressed as a forecast annuity in 1950 plus k times an annual increment. This demonstration is a noteworthy advance in this field. In this discussion the values in essentially the same form are obtained by interpolating on exactly two values to be established for each age at issue.

Accepting this advance as adequately demonstrated, it seems apparent that first difference interpolation based on a small set of year-of-birth mortality tables which exactly apply the basic assumptions, *i.e.*, the 1949 Annuity Table mortality with improvements according to Projection Scale B, should give a satisfactory fit. Apparently this method was rejected without adequate testing as not giving a sufficiently compact set of tables. I shall demonstrate, by actually comparing approximate annuity values so obtained with exact ones, that above age 60 only two such tables are necessary to give an adequate fit. These tables were chosen so as to give exact values at age 65 for the years 1950 and 1960 and consequently are for the births of 1885 and 1895. Probably the best explanation of why no more tables are needed is that Projection Scale B provides comparatively little improvement in mortality at the older ages.

Thus the values of q_x shown on the third page of Table I were obtained by applying formula (1) with $y = 65$, and the values of q_x shown on the second page of Table I for the births of 1895 were obtained using $y = 55$.

$$q_x = (1 - s_x)^{x-y} q''_x \qquad (1)$$

The double primed functions are ultimate values for the 1949 Annuity Table without projection (*TSA* I, 386–7). The s_x are the Projection Scale B factors (*TSA* I, 417) expressed as decimals, with intermediate values not shown by Jenkins and Lew supplied by first difference interpolation. The l_x and d_x shown on those two pages are those for the 1949 Annuity Table without projection for ages 90 and over and are logical consequences of these and the q_x for younger ages.

Extension of the above two tables to younger ages would give years of issue prior to 1950 from which values subsequent to 1950 could be obtained by extrapolation. This procedure was used for 6 years for the births of 1885 but the errors would become substantial as the period of extrapolation increased. Further exact tables like those just described could also be constructed for lives younger than 55 in 1950. Perhaps those who do not consider compactness of tables any great virtue will wish to adopt that procedure. The amount of additional space which would be required I leave to others. Certainly the requirements would be much greater than the additional 45 values of q_x which are used here. No loss of ease in computation is caused by the device introduced and very little loss of accuracy.

Note that a life born in $1895 + mf$ reaches the age $55 + m$ in the year $1950 + m(f + 1)$. An immediate or deferred annuity on such a life which is still younger than $55 + m$ can always be expressed as a temporary annuity terminating at that age plus an annuity of the form of the left side of (2) below times a reduction factor for the discounted probability of surviving to that age. For the low mortality rates which currently are applicable to younger age lives the value of this temporary annuity and this benefit of survivorship factor will not vary widely from those which would result from assuming zero mortality. This subject was discussed somewhat by Jenkins and Lew in order to indicate the maximum effect which could result from improvements in mortality at the younger ages. It is not intended to suggest that any such crude assumption might be considered, but rather to point out that a considerable variety of assumptions with regard to younger age mortality rates could be made without changing the values of the temporary annuity and the benefit of survivorship factor significantly.

It is convenient to use the notation of Jenkins' earlier paper (*TASA* XLVII, 265) where a number in a square bracket superscript indicates the duration from the base date at the time a life reaches the subscript age for mortality functions which recognize continuous improvement in mortality rates. It is clearly possible to find a value of f which exactly fulfills the requirements of formula (2) for any particular value of n and m. For the mortality basis under study and $m = 1$ the value of f is approximately 9.8 for $n = 0$ and increases to about 10.4 for $n = 10 - m$. For $m = 2$ the value of f increases from about 10.2 to 10.8 for those values of n. Evidently if $f = 10$ and the right side is exact, the error in the left side will not be large by assuming the forecast value to be defined by

$$|\, a_{55+m}^{[m(f+1)]} = {}_n|\, a_{55}^{[0]} \,. \tag{2}$$

The procedure adopted here is to attach at age 55 a younger age section to the exact births of 1895 table whose construction was described above. This composite table with an m year setback in age is then described as also applying to the births of $1895 + mf$ or more definitely to the births of $1895 + 10m$ since it has already been noted that a value of $f = 10$ is approximately right for (2). This younger age section is a plausible approximation to the mortality rates which formula (1) would define by setting y equal to the 1950 age for each year of birth and consequently it defines the younger age temporary annuity and the benefit of survivorship factor referred to above quite accurately. Its justification is that the method of construction is consistent with the age setback assumption and consequently the approximation is reasonable not only for some of the years of birth but for all of those used.

The age setback method by which the younger age section of the births of 1895 table was computed may be related to the base year mortality by means of formula (3) where h is a constant equal to $f/(f+1)$. The double prime superscripts used there indicate the base year mortality functions, *i.e.*, those for the 1949 Annuity Table without projection, and the number in the square bracket superscript is the time from the base date to the attainment of the lower right subscript age for the mortality functions which recognize continuous reductions in the mortality rates. By integration of (3) with respect to t between the limits of s and $n+s$, formula (4) is obtained.

$$\mu^{[t]}_{x+t} = \mu''_{x+ht} \tag{3}$$

$$\operatorname{colog} {}_n p^{[s]}_{x+s} = \frac{1}{h} \operatorname{colog} {}_{nh} p''_{x+sh} \tag{4}$$

Intermediate functions are defined by single prime superscripts by means of formulae (5) and (6) which can be seen to be consistent. A is a constant of the integration which produced (4) and arises because the different mortality tables do not have the same radix. An l_{55} for the births of 1895 has already been determined so as to cause the exact table to merge at age 90 with the base table. Using this as l'_{55} and $f = 10$ the single prime table was defined by formula (6) with a value for A of $-.3826519$.

$$ {}_n p^{[s]}_{x+s} = {}_{nh} p'_{x+sh} \tag{5}$$

$$\log l'_x = \frac{1}{h} \log l''_x + A \tag{6}$$

A table of $\log l''_x$ for integral ages 13 to 56 was constructed first, and then, using the Lagrange interpolation formula based on the four adjacent integral ages, values were obtained second for $\log l''_{55-sh}$ for $h = 10/11$ and s varying integrally from 0 to 45. Application of formula (6) to this second set of values, followed by differencing of adjacent values, produced the third set $\log {}_h p'_{55-sh}$. One minus the antilogarithm of this third set of values was then obtained and noted to be ${}_h q'_{55-sh}$ or q_{55-s} for the births of 1895. These values appear on the first page of Table I as this latter set of mortality rates. Although, for reasons to be given later, births of 1895 are not mentioned on this page, the age labels which apply may be easily deduced. The l_x and d_x shown on that page are logical consequences of those values of q_x and the l_{55} for the births of 1895. If antilogarithms had been taken of the log l'_{55-sh} obtained in one of the above steps the variations of these l'_x from the l_x actually obtained would have been slight due to the effects of dropped decimals. In fact values of l'_x were used as controls at ages 15, 25, 35, and 45.

Note that when $t = m(f+1) + s$ the right side age in formula (3) becomes $x + mf + sh$. Consequently by substituting y for $x + mf$ formula (3) may also be written as (7). This latter formula shows why if f is an integer the age setback method gives exactly a series of year of birth mortality tables spaced f years apart.

$$\mu_{y+m+s}^{[m(f+1)+s]} = \mu''_{y+sh} = \mu_{y+s}^{[s]} \qquad (7)$$

The elementary functions shown on the first three pages of Table I were obtained in the manner just described. The commutation functions D_x and C_x were obtained by applying v^x and v^{x+1} to these elementary functions in such a way as to cause them to merge with the 1949 Annuity Table without projection at age 90. These functions and also the derived functions N_x, M_x, and R_x make up the second three pages of Table I. In addition to as brief a set of instructions as it seemed reasonable to consider complete for a new method, Exhibit I also includes an example in Table II of how the method works for a complete set of ages. The method of using Table I resembles closely the method of using the Progressive Annuity Mortality Table which Fassel and Noback have constructed. This close resemblance may not be immediately apparent because the problem is visualized here as one of obtaining values at integral ages and only certain years of issue rather than at every year but at fractional ages. Intermediate values are then obtained by interpolation on years of issue rather than on age. The fact that year of birth 1885 is an entirely independent table made this approach necessary rather than optional as would be the case for an unmodified age setback table.

The following illustration of the age, year of issue method of labeling which is used in this discussion applied to Example 3 of Fassel and Noback will show how the two methods of labeling lead to the same answer with the same arithmetic steps for an unmodified age setback table. Example 3 is: "Using the Progressive Annuity Mortality Table and 2 percent interest, derive the value in 1955 of a nonrefund immediate annuity of one per annum to a male annuitant born in 1878." The authors note that such a life is aged 77 in 1955 and that his equivalent age as a birth of 1900 is 77.88 and hence using the annuity values shown below the answer is $.88 \times 6.725 + .12 \times 7.147 = 6.776$. The age, year of issue method applies several labels simultaneously, as is shown below, depending on whether the table is to be regarded as for the births of 1950, 1925, 1900, 1875, or 1850. The label (77,1977) appears on the line with the 7.147 annuity value and the label (77,1952) appears on the line with the 6.725

annuity value. Using this method an annuity value for a life aged 77 in 1955 is obtained by interpolating to 1955 from the 1952 and 1977 values and hence the arithmetic steps are the same as those shown above.

ILLUSTRATION OF AGE, YEAR OF ISSUE METHOD OF LABELING AN AGE SETBACK TABLE APPLIED TO THE PROGRESSIVE ANNUITY MORTALITY TABLE

(AGE, YEAR OF ISSUE) BY YEAR OF BIRTH					a_x AT 2% FOR MALE LIVES	AGE FOR BIRTHS OF 1900 GIVEN BY FASSEL AND NOBACK
1950	1925	1900	1875	1850		
....	....	....	....	....	....	....
....	....	....	....	....	....	....
(79,2029)	(78,2003)	(77,1977)	(76,1951)	(75,1925)	7.147	77
(80,2030)	(79,2004)	(78,1978)	(77,1952)	(76,1926)	6.725	78
(81,2031)	(80,2005)	(79,1979)	(78,1953)	(77,1927)	6.317	79
....	....	....	....	....	....	....
....	....	....	....	....	....	....
....	....	....	....	....	....	....

In the above illustration for the Progressive Annuity Mortality Table, note that if a value for age 77 in 1950 were required the interpolation would be based on a different set of values. Following the instructions of the table would be equivalent to interpolating between the values labeled (77,1952) and (77,1927) and hence the value would be $.92 \times 6.725 + .08 \times 6.317 = 6.692$. However, if extrapolation based on the values used in Example 3 were used the value would be $1.08 \times 6.725 - .08 \times 7.147 = 6.691$. The difference here is slight but it does exist. Since Sternhell's demonstration that a forecast annuity in the year $1950 + k$ can be adequately approximated by the 1950 forecast annuity plus k times an annual increment was regarded as an advance in the field, a conscious attempt was made to obtain that feature. The principle adopted was that interpolation should include extrapolation and be based on exactly two years of issue for each age. The years of issue on which this interpolation is to be based are those which occur in the period 1950 to 1969, since this is generally the period in which one is most interested in close approximations. An empirical adjustment extended the births of 1885 back to the year of issue 1944, since the births of 1905 table did not give as good a fit at the higher ages. For ages 75 and over complete values for these years of issue are not available but the two values from the births of 1885 and 1895 tables which are available for those ages give a very good fit.

In order to simplify the application of the principle outlined above, the notation device of putting the age and year of issue in parentheses was adopted. Under the convention used here, only ages shown in parentheses are to be used as ages at issue. In order to assist in deferred annuity and like calculations, ages older than those which are acceptable as ages at issue for a given year of birth are shown without a year of issue or parentheses. The situation is slightly different for ages 90 and over, since the 1949 Annuity Table without projection which is reproduced here for those ages is applicable for all years of issue.

The recommended procedure is to obtain first tabular values for the years of issue supplied by the tables. Secondly, values for the years of issue 1950 and 1960 are to be obtained. These are described as pivotal values. Other values would then be obtained by a second interpolation based on the pivotal values. Ordinarily one would need only the pivotal values; this double interpolation is intended as a matter of convenience so that tabular values would not also be necessary except as an intermediate step on the worksheets. Table II illustrates the basic elements required for computing 1950 and 1960 values by applying the method to immediate nonrefund life annuities for all ages from 40 to 75.

In conclusion allow me to summarize. The fact that a compact set of tables with the computational advantages of the Fassel and Noback table have been obtained should be apparent. The advantages of application of the two factor formula, *i.e.*, a 1950 value and an annual increment, suggested by Sternhell have been retained. The fact that the close fit obtained by Sternhell is also retained required testing and is demonstrated in Exhibit II.

Exhibit I

BASIC TABLES FOR APPROXIMATING ANNUITIES BASED ON $2\frac{1}{2}\%$ INTEREST AND THE 1949 ANNUITY TABLE (WITH PROJECTION PER SCALE B) ALSO INSTRUCTIONS AS TO THEIR USE—MALE LIVES ONLY

While the table given here is like an ordinary mortality table in many respects, it is also different in other respects from any other table. Consequently it is requested that the following explanations, instructions, and examples be read carefully before making calculations with it. This Exhibit I is complete in the sense that any annuity value, or other single life contingency function, can be computed for any year of issue from the tables given here without reference to any other material or to the balance of this discussion. This is in contrast, for example, to Mr. Sternhell's tables which are auxiliary to and can be used only in conjunction with the functions of the base year mortality tables.

The term *year of issue* is used as a convenience and is not meant to imply that the method is only applicable to annuities being issued. Since the method is equally applicable to valuation, the term *year of valuation* is to be regarded as synonymous to year of issue.

Table I consists of three pages of elementary functions with corresponding immediate annuities and three pages of corresponding commutation functions with interest at $2\frac{1}{2}\%$. The elementary functions for the births of 1885 and 1895 have been constructed from the q_x obtained in accordance with the main discussion so that the l_x and d_x merge at age 90 with the 1949 Annuity Table without projection. Similarly the commutation functions D_x and C_x were obtained by applying v^x and v^{x+1} to the elementary functions in such a way as to effect this merger for them also. For these purposes an age column for the births of 1895 is to be understood for the first and fourth pages (Table I-1 and I-4) although as is pointed out in the next paragraph this column is not shown because by 1950 the births of that year had attained all the ages which would show on those pages.

Age columns appear on the second and fifth pages for the six years of birth 1945, 1935, 1925, 1915, 1905, and 1895. Since no ages for years of issue prior to 1950 appear on the first and fourth pages the age columns are partly blank on those pages and the births of 1895 are not mentioned. Where the year of issue, *i.e.*, the age at issue plus the year of birth, is 1950 to 1969 inclusive the ages on the first and fourth pages are shown with the year of issue in parentheses. These are the important ages and the only reason for showing any other ages is to assist in deferred annuity calculations. Variations appear on the other pages in the sense that the births of 1895 table is always shown with a year of issue for age 55 and over, the births of 1885 table is always shown with a year of issue except that at ages 90 and over any year of issue is understood so that labels have been omitted. The births of 1905 table is not shown with a year of issue at ages 59 and over since this would then provide three years of issue instead of the exactly two years of issue which do occur with each age. The decision to start the births of 1885 table at age 59 was empirical.

Annuity values computed directly from the tables using the formulae applicable to any ordinary mortality table are described as tabular values. The two tabular values of the required annuity are to be computed for that age for the years of issue shown in parentheses with that age (see Example 1 below). Using first difference interpolation, pivotal values are to be computed from the tabular values for the pivotal years 1950 and 1960 (see Example 2 below). The tabular values are to be to two more places of decimals than the pivotal values. Values for any year of issue other than the pivotal years are to be computed from the pivotal values

TABLE I-1

1949 Annuity Table—Male Lives—Interest at 2½% with Approximate Scale B Projections for Certain Years of Birth

(Age, Year of Issue) or Age by Year of Birth					l_x	d_x	$1{,}000\,q_x$	a_x
1945	1935	1925	1915	1905				
(15, '60)	...	...	...	...	824.8509	.4330	.525	31.07838
(16, '61)	(15, '50)	...	...	...	824.4179	.4419	.536	30.87207
(17, '62)	(16, '51)	...	...	...	823.9760	.4524	.549	30.66084
(18, '63)	(17, '52)	...	...	...	823.5236	.4636	.563	30.44462
(19, '64)	(18, '53)	...	...	...	823.0600	.4757	.578	30.22332
(20, '65)	(19, '54)	...	...	...	822.5843	.4894	.595	29.99682
(21, '66)	(20, '55)	...	...	...	822.0949	.5048	.614	29.76504
(22, '67)	(21, '56)	...	...	...	821.5901	.5209	.634	29.52791
(23, '68)	(22, '57)	...	...	...	821.0692	.5386	.656	29.28531
(24, '69)	(23, '58)	...	...	...	820.5306	.5580	.680	29.03715
25	(24, '59)	...	...	...	819.9726	.5789	.706	28.78333
26	(25, '60)	...	...	...	819.3937	.6023	.735	28.52375
27	(26, '61)	(25, '50)	...	...	818.7914	.6272	.766	28.25835
28	(27, '62)	(26, '51)	...	...	818.1642	.6545	.800	27.98702
29	(28, '63)	(27, '52)	...	...	817.5097	.6859	.839	27.70966
30	(29, '64)	(28, '53)	...	...	816.8238	.7196	.881	27.42625
31	(30, '65)	(29, '54)	...	...	816.1042	.7549	.925	27.13669
32	(31, '66)	(30, '55)	...	...	815.3493	.7950	.975	26.84087
33	(32, '67)	(31, '56)	...	...	814.5543	.8382	1.029	26.53874
34	(33, '68)	(32, '57)	...	...	813.7161	.8853	1.088	26.23023
35	(34, '69)	(33, '58)	...	...	812.8308	.9372	1.153	25.91527
36	35	(34, '59)	...	...	811.8936	.9938	1.224	25.59381
37	36	(35, '60)	...	...	810.8998	1.0550	1.301	25.26581
38	37	(36, '61)	(35, '50)	...	809.8448	1.1233	1.387	24.93119
39	38	(37, '62)	(36, '51)	...	808.7215	1.1961	1.479	24.58997
40	39	(38, '63)	(37, '52)	...	807.5254	1.2759	1.580	24.24205
41	40	(39, '64)	(38, '53)	...	806.2495	1.3634	1.691	23.88742
42	41	(40, '65)	(39, '54)	...	804.8861	1.4593	1.813	23.52608
43	42	(41, '66)	(40, '55)	...	803.4268	1.5627	1.945	23.15803
44	43	(42, '67)	(41, '56)	...	801.8641	1.6831	2.099	22.78324
45	44	(43, '68)	(42, '57)	...	800.1810	1.8372	2.296	22.40195
46	45	(44, '69)	(43, '58)	...	798.3438	2.0342	2.548	22.01483
47	46	45	(44, '59)	...	796.3096	2.2711	2.852	21.62285
48	47	46	(45, '60)	...	794.0385	2.5449	3.205	21.22681
49	48	47	(46, '61)	(45, '50)	791.4936	2.8525	3.604	20.82744
50	49	48	(47, '62)	(46, '51)	788.6411	3.1908	4.046	20.42534
51	50	49	(48, '63)	(47, '52)	785.4503	3.5581	4.530	20.02103
52	51	50	(49, '64)	(48, '53)	781.8922	3.9509	5.053	19.61494
53	52	51	(50, '65)	(49, '54)	777.9413	4.3681	5.615	19.20742
54	53	52	(51, '66)	(50, '55)	773.5732	4.8062	6.213	18.79877
55	54	53	(52, '67)	(51, '56)	768.7670	5.2630	6.846	18.38920
56	55	54	(53, '68)	(52, '57)	763.5040	5.7370	7.514	17.97887
57	56	55	(54, '69)	(53, '58)	757.7670	6.2258	8.216	17.56786
58	57	56	55	(54, '59)	751.5412	6.7278	8.952	17.15622
59	58	57	56	(55, '60)	744.8134	7.2403	9.721	16.74397

TABLE I-2

1949 ANNUITY TABLE—MALE LIVES—INTEREST AT 2½% WITH APPROXIMATE SCALE B PROJECTIONS FOR CERTAIN YEARS OF BIRTH

(Age, Year of Issue) or Age by Year of Birth						l_x	d_x	1,000 q_x	a_x
1945	1935	1925	1915	1905	1895*				
60	59	58	57	(56, '61)	(55, '50)	737.5731	7.7925	10.565	16.33105
61	60	59	58	(57, '62)	(56, '51)	729.7806	8.2837	11.351	15.91806
62	61	60	59	(58, '63)	(57, '52)	721.4969	8.7727	12.159	15.50334
63	62	61	60	59	(58, '53)	712.7242	9.2604	12.993	15.08652
64	63	62	61	60	(59, '54)	703.4638	9.7458	13.854	14.66725
65	64	63	62	61	(60, '55)	693.7180	10.2289	14.745	14.24514
66	65	64	63	62	(61, '56)	683.4891	10.7369	15.709	13.81979
67	66	65	64	63	(62, '57)	672.7522	11.2834	16.772	13.39135
68	67	66	65	64	(63, '58)	661.4688	11.8681	17.942	12.96028
69	68	67	66	65	(64, '59)	649.6007	12.4925	19.231	12.52699
70	69	68	67	66	(65, '60)	637.1082	13.1569	20.651	12.09194
71	70	69	68	67	(66, '61)	623.9513	13.8748	22.237	11.65558
72	71	70	69	68	(67, '62)	610.0765	14.6339	23.987	11.21868
73	72	71	70	69	(68, '63)	595.4426	15.4339	25.920	10.78175
74	73	72	71	70	(69, '64)	580.0087	16.2721	28.055	10.34537
75	74	73	72	71	(70, '65)	563.7366	17.1438	30.411	9.91008
76	75	74	73	72	(71, '66)	546.5928	18.0725	33.064	9.47644
77	76	75	74	73	(72, '67)	528.5203	19.0294	36.005	9.04549
78	77	76	75	74	(73, '68)	509.4909	20.0052	39.265	8.61792
79	78	77	76	75	(74, '69)	489.4857	20.9882	42.878	8.19439
80	79	78	77	76	(75, '70)	468.4975	21.9646	46.883	7.77553
81	80	79	78	77	(76, '71)	446.5329	22.9665	51.433	7.36195
82	81	80	79	78	(77, '72)	423.5664	23.9315	56.500	6.95516
83	82	81	80	79	(78, '73)	399.6349	24.8345	62.143	6.55595
84	83	82	81	80	(79, '74)	374.8004	25.6472	68.429	6.16510
85	84	83	82	81	(80, '75)	349.1532	26.3377	75.433	5.78341
86	85	84	83	82	(81, '76)	322.8155	26.8702	83.237	5.41165
87	86	85	84	83	(82, '77)	295.9453	27.2080	91.936	5.05057
88	87	86	85	84	(83, '78)	268.7373	27.3118	101.630	4.70096
89	88	87	86	85	(84, '79)	241.4255	27.1442	112.433	4.36359
90	89	88	87	86	(85, '80)	214.2813	26.6718	124.471	4.03925
91	90	89	88	87	(86, '81)	187.6095	25.8676	137.880	3.72884
92	91	90	89	88	(87, '82)	161.7419	24.7164	152.814	3.43333
93	92	91	90	89	(88, '83)	137.0255	23.2172	169.437	3.15394
94	93	92	91	90	(89, '84)	113.8083	21.3878	187.928	2.89229

* Use values shown under births of 1885 for births of 1895 for ages 90 and over. For other years of birth use the age setback method shown above.

TABLE I-3

1949 ANNUITY TABLE—MALE LIVES—INTEREST AT 2½% WITH APPROXIMATE SCALE B PROJECTIONS FOR CERTAIN YEARS OF BIRTH

(AGE, YEAR OF ISSUE) OR AGE BY YEAR OF BIRTH 1885	l_x	d_x	$1{,}000q_x$	a_x
(59, '44)	774.7564	12.1164	15.639	14.24766
(60, '45)	762.6400	12.6880	16.637	13.83587
(61, '46)	749.9520	13.2659	17.689	13.42171
(62, '47)	736.6861	13.8843	18.847	13.00498
(63, '48)	722.8018	14.5442	20.122	12.58617
(64, '49)	708.2576	15.2445	21.524	12.16574
(65, '50)	693.0131	15.9850	23.066	11.74419
(66, '51)	677.0281	16.7646	24.762	11.32201
(67, '52)	660.2635	17.5828	26.630	10.89972
(68, '53)	642.6807	18.4385	28.690	10.47787
(69, '54)	624.2422	19.3253	30.958	10.05705
(70, '55)	604.9169	20.2387	33.457	9.63779
(71, '56)	584.6782	21.1823	36.229	9.22070
(72, '57)	563.4959	22.1414	39.293	8.80649
(73, '58)	541.3545	23.1034	42.677	8.39585
(74, '59)	518.2511	24.0557	46.417	7.98938
(75, '60)	494.1954	24.9811	50.549	7.58773
(76, '61)	469.2143	25.8894	55.176	7.19150
(77, '62)	443.3249	26.7356	60.307	6.80176
(78, '63)	416.5893	27.4936	65.997	6.41923
(79, '64)	389.0957	28.1351	72.309	6.04464
(80, '65)	360.9606	28.6278	79.310	5.67868
(81, '66)	332.3328	28.9385	87.077	5.32205
(82, '67)	303.3943	29.0333	95.695	4.97543
(83, '68)	274.3610	28.8781	105.256	4.63948
(84, '69)	245.4829	28.4424	115.863	4.31489
(85, '70)	217.0405	27.7000	127.626	4.00235
(86, '71)	189.3405	26.6343	140.669	3.70258
(87, '72)	162.7062	25.2398	155.125	3.41640
(88, '73)	137.4664	23.5260	171.140	3.14476
(89, '74)	113.9404	21.5199	188.870	2.88893
90	92.42050	19.26829	208.485	2.651
91	73.15221	16.61960	227.192	2.433
92	56.53261	13.98232	247.332	2.226
93	42.55029	11.44433	268.960	2.032
94	31.10596	9.08661	292.118	1.849
95	22.01935	6.97648	316.834	1.677
96	15.04287	5.16154	343.122	1.517
97	9.881330	3.665707	370.973	1.366
98	6.215623	2.488437	400.352	1.227
99	3.727186	1.607159	431.199	1.097

TABLE I-3—*Continued*

(Age, Year of Issue) or Age by Year of Birth 1885	l_x	d_x	$1{,}000q_x$	a_x
100	2.120027	.982452	463.415	.977
101	1.137575	.565227	496.870	.865
102	.5723480	.3041394	531.389	.763
103	.2682086	.1520091	566.757	.669
104	.1161995	.0700351	602.714	.583
105	.04616440	.02949702	638.956	.503
106	.01666738	.01125286	675.143	.428
107	.00541452	.00384917	710.898	.352
108	.00156535	.00116747	745.822	.248
109	.00039788	.00039788	1000.000	0

TABLE I-4

1949 ANNUITY TABLE—MALE LIVES—INTEREST AT 2½% WITH APPROXIMATE SCALE B PROJECTIONS FOR CERTAIN YEARS OF BIRTH

(Age, Year of Issue) or Age by Year of Birth					D_x	N_x	C_x	M_x	R_x
1945	1935	1925	1915	1905					
(15, '60)	....	....	....	....	644.3722	20670.4134	.33001	140.21577	8149.27064
(16, '61)	(15, '50)	....	....	....	628.3258	20026.0412	.32858	139.88576	8009.05487
(17, '62)	(16, '51)	....	....	....	612.6722	19397.7154	.32818	139.55718	7869.16911
(18, '63)	(17, '52)	....	....	....	597.4008	18785.0432	.32810	139.22900	7729.61193
(19, '64)	(18, '53)	....	....	....	582.5019	18187.6424	.32845	138.90090	7590.38293
(20, '65)	(19, '54)	....	....	....	567.9661	17605.1405	.32967	138.57245	7451.48203
(21, '66)	(20, '55)	....	....	....	553.7836	17037.1744	.33175	138.24278	7312.90958
(22, '67)	(21, '56)	....	....	....	539.9450	16483.3908	.33398	137.91103	7174.66680
(23, '68)	(22, '57)	....	....	....	526.4416	15943.4458	.33691	137.57705	7036.75577
(24, '69)	(23, '58)	....	....	....	513.2646	15417.0042	.34053	137.24014	6899.17872
25	(24, '59)	....	....	....	500.4054	14903.7396	.34467	136.89961	6761.93858
26	(25, '60)	....	....	....	487.8558	14403.3342	.34985	136.55494	6625.03897
27	(26, '61)	(25, '50)	....	....	475.6070	13915.4784	.35543	136.20509	6488.48403
28	(27, '62)	(26, '51)	....	....	463.6514	13439.8714	.36186	135.84966	6352.27894
29	(28, '63)	(27, '52)	....	....	451.9810	12976.2200	.36997	135.48780	6216.42928
30	(29, '64)	(28, '53)	....	....	440.5871	12524.2390	.37868	135.11783	6080.94148
31	(30, '65)	(29, '54)	....	....	429.4624	12083.6519	.38757	134.73915	5945.82365
32	(31, '66)	(30, '55)	....	....	418.6001	11654.1895	.39820	134.35158	5811.08450
33	(32, '67)	(31, '56)	....	....	407.9921	11235.5894	.40960	133.95338	5676.73292
34	(33, '68)	(32, '57)	....	....	397.6315	10827.5973	.42206	133.54378	5542.77954
35	(34, '69)	(33, '58)	....	....	387.5111	10429.9658	.43591	133.12172	5409.23576
36	35	(34, '59)	....	....	377.6237	10042.4547	.45096	132.68581	5276.11404
37	36	(35, '60)	....	....	367.9624	9664.8310	.46705	132.23485	5143.42823
38	37	(36, '61)	(35, '50)	....	358.5207	9296.8686	.48516	131.76780	5011.19338
39	38	(37, '62)	(36, '51)	....	349.2911	8938.3479	.50400	131.28264	4879.42558
40	39	(38, '63)	(37, '52)	....	340.2678	8589.0568	.52451	130.77864	4748.14294
41	40	(39, '64)	(38, '53)	....	331.4441	8248.7890	.54681	130.25413	4617.36430
42	41	(40, '65)	(39, '54)	....	322.8133	7917.3449	.57100	129.70732	4487.11017
43	42	(41, '66)	(40, '55)	....	314.3688	7594.5316	.59655	129.13632	4357.40285
44	43	(42, '67)	(41, '56)	....	306.1047	7280.1628	.62684	128.53977	4228.26653
45	44	(43, '68)	(42, '57)	....	298.0119	6974.0581	.66754	127.91293	4099.72676
46	45	(44, '69)	(43, '58)	....	290.0758	6676.0462	.72109	127.24539	3971.81383
47	46	45	(44, '59)	....	282.2797	6385.9704	.78543	126.52430	3844.56844
48	47	46	(45, '60)	....	274.6094	6103.6907	.85866	125.73887	3718.04414
49	48	47	(46, '61)	(45, '50)	267.0529	5829.0813	.93897	124.88021	3592.30527
50	49	48	(47, '62)	(46, '51)	259.6005	5562.0284	1.02471	123.94124	3467.42506
51	50	49	(48, '63)	(47, '52)	252.2440	5302.4279	1.11480	122.91653	3343.48382
52	51	50	(49, '64)	(48, '53)	244.9769	5050.1839	1.20768	121.80173	3220.56729
53	52	51	(50, '65)	(49, '54)	237.7942	4805.2070	1.30264	120.59405	3098.76556
54	53	52	(51, '66)	(50, '55)	230.6917	4567.4128	1.39833	119.29141	2978.17151
55	54	53	(52, '67)	(51, '56)	223.6668	4336.7211	1.49388	117.89308	2858.88010
56	55	54	(53, '68)	(52, '57)	216.7176	4113.0543	1.58871	116.39920	2740.98702
57	56	55	(54, '69)	(53, '58)	209.8431	3896.3367	1.68202	114.81049	2624.58782
58	57	56	55	(54, '59)	203.0430	3686.4936	1.77331	113.12847	2509.77733
59	58	57	56	(55, '60)	196.3174	3483.4506	1.86185	111.35516	2396.64886

TABLE I-5

1949 ANNUITY TABLE—MALE LIVES—INTEREST AT $2\frac{1}{2}\%$ WITH APPROXIMATE SCALE B PROJECTIONS FOR CERTAIN YEARS OF BIRTH

(Age, Year of Issue) or Age by Year of Birth						D_x	N_x	C_x	M_x	R_x
1945	1935	1925	1915	1905	1895*					
60	59	58	57	(56,'61)	(55,'50)	189.6673	3287.1332	1.95497	109.49331	2285.29370
61	60	59	58	(57,'62)	(56,'51)	183.0863	3097.4659	2.02751	107.53834	2175.80039
62	61	60	59	(58,'63)	(57,'52)	176.5933	2914.3796	2.09483	105.51083	2068.26205
63	62	61	60	59	(58,'53)	170.1913	2737.7863	2.15736	103.41600	1962.75122
64	63	62	61	60	(59,'54)	163.8829	2567.5950	2.21506	101.25864	1859.33522
65	64	63	62	61	(60,'55)	157.6707	2403.7121	2.26816	99.04358	1758.07658
66	65	64	63	62	(61,'56)	151.5569	2246.0414	2.32273	96.77542	1659.03300
67	66	65	64	63	(62,'57)	145.5377	2094.4845	2.38142	94.45269	1562.25758
68	67	66	65	64	(63,'58)	139.6066	1948.9468	2.44373	92.07127	1467.80489
69	68	67	66	65	(64,'59)	133.7578	1809.3402	2.50956	89.62754	1375.73362
70	69	68	67	66	(65,'60)	127.9858	1675.5824	2.57857	87.11798	1286.10608
71	70	69	68	67	(66,'61)	122.2857	1547.5966	2.65294	84.53941	1198.98810
72	71	70	69	68	(67,'62)	116.6502	1425.3109	2.72984	81.88647	1114.44869
73	72	71	70	69	(68,'63)	111.0752	1308.6607	2.80885	79.15663	1032.56222
74	73	72	71	70	(69,'64)	105.5572	1197.5855	2.88917	76.34778	953.40559
75	74	73	72	71	(70,'65)	100.0935	1092.0283	2.96970	73.45861	877.05781
76	75	74	73	72	(71,'66)	94.68244	991.93477	3.054217	70.488914	803.599203
77	76	75	74	73	(72,'67)	89.31890	897.25233	3.137494	67.434697	733.110289
78	77	76	75	74	(73,'68)	84.00290	807.93343	3.217932	64.297203	665.675592
79	78	77	76	75	(74,'69)	78.73611	723.93053	3.293709	61.079271	601.378389
80	79	78	77	76	(75,'70)	73.52201	645.19442	3.362866	57.785562	540.299118
81	80	79	78	77	(76,'71)	68.36592	571.67241	3.430498	54.422696	482.513556
82	81	80	79	78	(77,'72)	63.26796	503.30649	3.487453	50.992198	428.090860
83	82	81	80	79	(78,'73)	58.23739	440.03853	3.530775	47.504745	377.098662
84	83	82	81	80	(79,'74)	53.28619	381.80114	3.557384	43.973970	329.593917
85	84	83	82	81	(80,'75)	48.42914	328.51495	3.564058	40.416586	285.619947
86	85	84	83	82	(81,'76)	43.68389	280.08581	3.547431	36.852528	245.203361
87	86	85	84	83	(82,'77)	39.07100	236.40192	3.504417	33.305097	208.350833
88	87	86	85	84	(83,'78)	34.61363	197.33092	3.431987	29.800680	175.045736
89	88	87	86	85	(84,'79)	30.33741	162.71729	3.327733	26.368693	145.245056
90	89	88	87	86	(85,'80)	26.26974	132.37988	3.190068	23.040960	118.876363
91	90	89	88	87	(86,'81)	22.43894	106.11014	3.018421	19.850892	95.835403
92	91	90	89	88	(87,'82)	18.87323	83.67120	2.813747	16.832471	75.984511
93	92	91	90	89	(88,'83)	15.59916	64.79797	2.578611	14.018724	59.152040
94	93	92	91	90	(89,'84)	12.64008	49.19881	2.317492	11.440113	45.133316

* Use values shown under births of 1885 for births of 1895 for ages 90 and over. For other years of birth use the age setback method shown above.

TABLE I-6

1949 Annuity Table—Male Lives—Interest at 2½% with Approximate Scale B Projections for Certain Years of Birth

(Age, Year of Issue) or Age by Year of Birth / 1885	D_x	N_x	C_x	M_x	R_x
(59, '44)	180.4917	2752.0765	2.75386	113.36786	2006.28962
(60, '45)	173.3356	2571.5848	2.81344	110.61400	1892.92176
(61, '46)	166.2944	2398.2492	2.86984	107.80056	1782.30776
(62, '47)	159.3686	2231.9548	2.93036	104.93072	1674.50720
(63, '48)	152.5512	2072.5862	2.99476	102.00036	1569.57648
(64, '49)	145.8357	1920.0350	3.06240	99.00560	1467.57612
(65, '50)	139.2163	1774.1993	3.13284	95.94320	1368.57052
(66, '51)	132.6880	1634.9830	3.20549	92.81036	1272.62732
(67, '52)	126.2462	1502.2950	3.27993	89.60487	1179.81696
(68, '53)	119.8871	1376.0488	3.35567	86.32494	1090.21209
(69, '54)	113.6073	1256.1617	3.43128	82.96927	1003.88715
(70, '55)	107.4052	1142.5544	3.50581	79.53799	920.91788
(71, '56)	101.2797	1035.1492	3.57977	76.03218	841.37989
(72, '57)	95.22971	933.86952	3.650589	72.452405	765.347709
(73, '58)	89.25645	838.63981	3.716292	68.801816	692.895304
(74, '59)	83.36317	749.38336	3.775097	65.085524	624.093488
(75, '60)	77.55482	666.02019	3.824704	61.310427	559.007964
(76, '61)	71.83853	588.46537	3.867091	57.485723	497.697537
(77, '62)	66.21929	516.62684	3.896085	53.618632	440.211814
(78, '63)	60.70810	450.40755	3.908825	49.722547	386.593182
(79, '64)	55.31858	389.69945	3.902467	45.813722	336.870635
(80, '65)	50.06688	334.38087	3.873958	41.911255	291.056913
(81, '66)	44.97178	284.31399	3.820490	38.037297	249.145658
(82, '67)	40.05442	239.34221	3.739517	34.216807	211.108361
(83, '68)	35.33797	199.28779	3.628807	30.477290	176.891554
(84, '69)	30.84726	163.94982	3.486886	26.848483	146.414264
(85, '70)	26.60800	133.10256	3.313045	23.361597	119.565781
(86, '71)	22.64598	106.49456	3.107885	20.048552	96.204184
(87, '72)	18.98575	83.84858	2.873332	16.940667	76.155632
(88, '73)	15.64935	64.86283	2.612908	14.067335	59.214965
(89, '74)	12.65475	49.21348	2.331806	11.454427	45.147630
90	10.01430	36.55873	2.036908	9.122621	33.693203
91	7.733137	26.544427	1.714056	7.085713	24.570582
92	5.830468	18.811290	1.406889	5.371657	17.484869
93	4.281373	12.980822	1.123433	3.964768	12.113212
94	3.053517	8.699449	.8702313	2.8413348	8.1484445
95	2.108809	5.645932	.6518464	1.9711035	5.3071097
96	1.405528	3.537123	.4705051	1.3192571	3.3360062
97	.9007421	2.1315946	.3260010	.8487520	2.0167491
98	.5527718	1.2308525	.2159056	.5227510	1.1679971
99	.3233839	.6780807	.1360418	.3068454	.6452461

TABLE I-6—*Continued*

(AGE, YEAR OF ISSUE) OR AGE BY YEAR OF BIRTH 1885	D_x	N_x	C_x	M_x	R_x
100	.1794547	.3546968	.08113364	.17080355	.33840066
101	.09394413	.17524213	.04553954	.08966991	.16759711
102	.04611327	.08129800	.02390642	.04413037	.07792720
103	.02108213	.03518473	.01165702	.02022395	.03379683
104	.00891091	.01410260	.00523974	.00856693	.01357288
105	.00345383	.00519169	.00215302	.00332719	.00500595
106	.00121657	.00173786	.00080133	.00117417	.00167876
107	.00038557	.00052129	.00026740	.00037284	.00050459
108	.00010875	.00013572	.00007913	.00010544	.00013175
109	.00002697	.00002697	.00002631	.00002631	.00002631

by first difference interpolation (see Example 3 below). Since commutation functions are not life contingency functions, the above process of interpolation is not to be used directly on them.

As a general rule one would wish to establish pivotal values for exactly two years so that the purpose of the double interpolation is to avoid the inconvenience of needing also to establish tabular values on a published basis. For experimental work a single interpolation direct from the tabular values would be adequate. The choice of 1950 and 1960 as pivotal years is arbitrary and others could be chosen should they prove more convenient.

Example: Compute the value of an immediate annuity with payments guaranteed for 20 years certain and life thereafter to be issued at age 40 for the years of issue shown below.

1. 1965. Note that a tabular value occurs in 1965 for the births of 1925. Reading age labels from that column $N_{61} = 2737.7863$ and $D_{40} = 322.8133$. The ratio 8.48094 is the deferred annuity. To this add 15.58916 for the period certain annuity as may be seen from any standard work on interest. The tabular value in 1965 is the sum or 24.07010. For the final value see 3 below.
2. 1950. As for 1 above, the tabular value in 1955 is 23.75662. By extrapolation from this and the 1965 tabular value the pivotal value in 1950 is 23.600.
3. 1965. As for 2 above, the pivotal value in 1960 is 23.913. By extrapolation from this and the 1950 pivotal value the value in 1965 is 24.070.

TABLE II

ILLUSTRATION OF METHOD OF COMPUTATION OF ANNUITIES WITH THE TABLES OF THIS DISCUSSION APPLIED TO IMMEDIATE NONREFUND LIFE ANNUITIES

Age	Tabular Values of N_{x+1}/D_x				Yearly Increase in Tabular Values	Years from 1950 to Early Year	Pivotal Values by Interpolation		Age
	Early Year (1)	Early Year Value (2)	Late Year (3)	Late Year Value (4)	(5)	(6)	Issues of 1950 (7)	Issues of 1960 (8)	
40..	1955	23.15803	1965	23.52608	.036805	5	22.974	23.342	..40
41..	1956	22.78324	1966	23.15803	.037479	6	22.558	22.933	..41
42..	1957	22.40195	1967	22.78324	.038129	7	22.135	22.516	..42
43..	1958	22.01483	1968	22.40195	.038712	8	21.705	22.092	..43
44..	1959	21.62285	1969	22.01483	.039198	9	21.270	21.662	..44
45..	1950	20.82744	1960	21.22681	.039937	0	20.827	21.227	..45
46..	1951	20.42534	1961	20.82744	.040210	1	20.835	20.787	..46
47..	1952	20.02103	1962	20.42534	.040431	2	19.940	20.344	..47
48..	1953	19.61494	1963	20.02103	.040609	3	19.493	19.899	..48
49..	1954	19.20742	1964	19.61494	.040752	4	19.044	19.452	..49
50..	1955	18.79877	1965	19.20742	.040865	5	18.594	19.003	..50
51..	1956	18.38920	1966	18.79877	.040957	6	18.143	18.553	..51
52..	1957	17.97887	1967	18.38920	.041033	7	17.692	18.102	..52
53..	1958	17.56786	1968	17.97887	.041101	8	17.239	17.650	..53
54..	1959	17.15622	1969	17.56786	.041164	9	16.786	17.197	..54
55..	1950	16.33105	1960	16.74397	.041292	0	16.331	16.744	..55
56..	1951	15.91806	1961	16.33105	.041299	1	15.877	16.290	..56
57..	1952	15.50334	1962	15.91806	.041472	2	15.420	15.835	..57
58..	1953	15.08652	1963	15.50334	.041682	4	14.961	15.378	..58
59..	1944	14.24766	1954	14.66725	.041959	− 6	14.499	14.919	..59
60..	1945	13.83587	1955	14.24514	.040927	− 5	14.041	14.450	..60
61..	1946	13.42171	1956	13.81979	.039808	− 4	13.581	13.979	..61
62..	1947	13.00498	1957	13.39135	.038637	− 3	13.121	13.507	..62
63..	1948	12.58617	1958	12.96028	.037411	− 2	12.661	13.035	..63
64..	1949	12.16574	1959	12.52699	.036135	− 1	12.202	12.563	..64
65..	1950	11.74419	1960	12.09194	.034775	0	11.744	12.092	..65
66..	1951	11.32201	1961	11.65558	.033357	1	11.289	11.622	..66
67..	1952	10.89972	1962	11.21868	.031896	2	10.836	11.155	..67
68..	1953	10.47787	1963	10.78175	.030388	3	10.387	10.691	..68
69..	1954	10.05705	1964	10.34537	.028832	4	9.942	10.230	..69
70..	1955	9.63779	1965	9.91008	.027229	5	9.502	9.774	..70
71..	1956	9.22070	1966	9.47644	.025574	6	9.067	9.323	..71
72..	1957	8.80649	1967	9.04549	.023900	7	8.639	8.878	..72
73..	1958	8.39585	1968	8.61792	.022207	8	8.218	8.440	..73
74..	1959	7.98938	1969	8.19439	.020501	9	7.805	8.010	..74
75..	1960	7.58773	1970	7.77553	.018780	10	7.400	7.588	..75

Further examples of the above type for individual ages are shown in Exhibit II.

Table II gives an example of how the method works for a complete set of ages. The immediate annuity values shown in Table I are tabular values obtained by dividing N_{x+1} by D_x. These tabular values are also shown in Table II for ages 40 to 75 inclusive. In addition pivotal values for 1950 and 1960 are also shown along with the method of computation.

Exhibit II

This exhibit consists of three tables in which exact and approximate annuity values are compared, and comments upon the errors of approximation. In every case the values are for male lives on the basis of the 1949 Annuity Mortality Table (Ultimate) with Projection Scale B and interest at $2\frac{1}{2}\%$. The purpose of exhibiting these values is to show for the important types of annuities that the method proposed in this discussion provides a closeness of fit comparable to that obtained in the paper.

For the most part the tests of fit shown here are limited to those for which tests were made in the paper. However, since these are all for ages ending in 5, it was felt that some question might arise about the effects of interpolation, so that values for ages ending in 0 are also shown in Table III. As might have been expected the errors progress quite smoothly with age. I have not calculated any values by the methods other than those given here but am indebted to Mr. Lew for having additional values prepared for inclusion here. This includes some of Table III and all of Part B of Table V.

Immediate annuities with guaranteed periods of 0, 10, and 20 years are compared in Table III for issues of 1950 and 1960. Of course, in so far as errors are concerned, this is equivalent to comparing annuities deferred for the same periods. The errors by the method of this discussion may be seen to be generally considerably the smaller at the younger ages but at the important older ages the errors on nonrefund issues of 1950 are slightly the larger. Above age 60 the errors are always small, however. The appearance at age 65 of no error by the method of this discussion is slightly misleading since exact values in the years 1950 and 1960 were guaranteed by the method of construction; consequently let it be noted that the errors for issues in 1955 are .004 by the method of the paper and −.002 by the method of this discussion. The apparent lack of conservatism at the younger ages, as evidenced by the errors being negative there, does not appear to be a matter for concern since values which allow for future improvements in mortality are inherently conservative.

Immediate annuities with a guaranteed period of 9 years are compared in Table IV for issues of 1970 and 1980. This is intended as a measure of

TABLE III

Age at Issue	Annuities Issued in 1950					Annuities Issued in 1960				
	Exact Value	Approx. Value by Method of		Error by Method of		Exact Value	Approx. Value by Method of		Error by Method of	
		Paper	Discussion	Paper	Discussion		Paper	Discussion	Paper	Discussion
	Immediate Nonrefund Life Annuities									
15..	30.917	31.018	30.872	.101	−.045	31.134	31.298	31.078	.164	−.056
25..	28.296	28.370	28.258	.074	−.038	28.574	28.704	28.524	.130	−.050
35..	24.962	25.005	24.931	.043	−.031	25.307	25.401	25.266	.094	−.041
40..	23.001	23.028	22.974	.027	−.027	23.381	23.454	23.342	.073	−.039
45..	20.849	20.867	20.827	.018	−.022	21.263	21.319	21.227	.056	−.036
50..	18.605	18.615	18.594	.010	−.011	19.039	19.077	19.003	.038	−.036
55..	16.330	16.336	16.331	.006	.001	16.759	16.785	16.744	.026	−.015
60..	14.043	14.045	14.041	.002	−.002	14.442	14.458	14.450	.016	.008
65..	11.744	11.744	11.744	.000	.000	12.092	12.100	12.092	.008	.000
70..	9.498	9.498	9.502	.000	.004	9.775	9.779	9.774	.004	−.001
75..	7.396	7.395	7.400	−.001	.004	7.588	7.590	7.588	.002	.000
80..	5.518	5.522	5.522	.000	.004	5.625	5.626	5.626	.001	.001
85..	3.927	3.927	3.929	.000	.002	3.965	3.965	3.965	.000	.000
	Immediate Life Annuities with 10 Year Certain Period									
15..	30.944	31.044	30.899	.100	−.045	31.157	31.320	31.105	.163	−.052
25..	28.337	28.410	28.299	.073	−.038	28.610	28.739	28.563	.129	−.047
35..	25.042	25.084	25.011	.042	−.031	25.378	25.470	25.340	.092	−.038
40..	23.129	23.157	23.103	.028	−.026	23.494	23.566	23.459	.072	−.035
45..	21.082	21.099	21.061	.017	−.021	21.469	21.521	21.437	.052	−.032
50..	18.990	19.000	18.983	.010	−.007	19.379	19.414	19.361	.035	−.018
55..	16.912	16.916	16.912	.004	.000	17.276	17.296	17.287	.020	.011
60..	14.885	14.886	14.884	.001	−.001	15.198	15.207	15.203	.009	.005
65..	12.979	12.979	12.979	.000	.000	13.219	13.220	13.219	.001	.000
70..	11.323	11.323	11.324	.000	.001	11.474	11.474	11.475	.000	.001
75..	10.055	10.055	10.055	.000	.000	10.125	10.124	10.125	−.001	.000
80..	9.255	9.255	9.255	.000	.000	9.273	9.273	9.273	.000	.000
85..	8.882	8.881	8.882	−.001	.000	8.883	8.883	8.883	.000	.000
	Immediate Life Annuities with 20 Year Certain Period									
15..	31.011	31.111	30.971	.100	−.040	31.216	31.378	31.174	.162	−.042
25..	28.456	28.526	28.423	.070	−.033	28.713	28.839	28.679	.126	−.034
35..	25.331	25.371	25.305	.040	−.026	25.634	25.717	25.608	.083	−.026
40..	23.614	23.639	23.600	.025	−.014	23.925	23.984	23.913	.059	−.012
45..	21.860	21.873	21.864	.013	.004	22.162	22.196	22.178	.034	.016
50..	20.159	20.163	20.172	.004	.013	20.427	20.441	20.474	.014	.047
55..	18.603	18.602	18.603	−.001	.000	18.814	18.813	18.876	−.001	.062
60..	17.297	17.295	17.297	−.002	.000	17.433	17.427	17.434	−.006	.001
65..	16.354	16.353	16.354	−.001	.000	16.418	16.414	16.418	−.004	.000
70..	15.827	15.827	15.827	.000	.000	15.845	15.844	15.845	−.001	.000
75..	15.632	15.632	15.632	.000	.000	15.635	15.634	15.635	−.001	.000

TABLE IV

Age at Issue	Annuities Issued in 1970					Annuities Issued in 1980				
	Exact Value	Approx. Value by Method of		Error by Method of		Exact Value	Approx. Value by Method of		Error by Method of	
		Paper	Discussion	Paper	Discussion		Paper	Discussion	Paper	Discussion
	Immediate Life Annuities with 9 Year Certain Period									
35..	25.679	25.844	25.656	.165	−.023	25.972	26.232	25.986	.260	.014
45..	21.798	21.910	21.775	.112	−.023	22.145	22.337	22.155	.192	.010
55..	17.539	17.594	17.569	.055	.030	17.880	17.987	17.951	.107	.071
65..	13.262	13.274	13.269	.012	.007	13.506	13.534	13.526	.028	.020
75..	9.772	9.770	9.771	−.002	−.001	9.856	9.854	9.855	−.002	−.001

TABLE V

Age at Issue (x)	Annuities Issued in 1950					Annuities Issued in 1960				
	Exact Value	Approx. Value by Method of		Error by Method of		Exact Value	Approx. Value by Method of		Error by Method of	
		Paper	Discussion	Paper	Discussion		Paper	Discussion	Paper	Discussion
	Part A—Retirement Income without Death Benefit Values of $_{n\|}a_x$ where $n=65-x$									
25........	4.078	4.117	4.098	.039	.020	4.254	4.321	4.293	.067	.039
35........	5.036	5.061	5.047	.025	.011	5.269	5.320	5.297	.051	.028
45........	6.271	6.284	6.274	.013	.003	6.573	6.607	6.589	.034	.016
55........	8.160	8.164	8.160	.004	.000	8.524	8.544	8.535	.020	.011
	Part B—Retirement Income with Death Benefit during Deferred Period Values of $_{n\|}a_x+3A^{1}_{x:\overline{n\|}}$ where $n=65-x$									
25........	4.318	4.345	4.361	.027	.043	4.468	4.513	4.538	.045	.070
35........	5.348	5.366	5.379	.018	.031	5.548	5.583	5.605	.035	.057
45........	6.654	6.664	6.670	.010	.016	6.915	6.939	6.957	.024	.042
55........	8.514	8.517	8.514	.003	.000	8.840	8.856	8.867	.016	.027

the effect of the choice of methods on the determination of settlement option values. It will be found that the fit of the method of this discussion is always substantially the better. Without making exact calculations for the other methods close enough approximations were made to show that in these years the method given here is probably even more favored for nonrefund annuities than for those with the 9 year guaranteed period.

Annuities deferred to age 65 are compared in Part A of Table V. This is intended as a measure of the effect of the choice of method for a retirement income without a death benefit. Again it will be found that the method of the discussion gives somewhat the better fit. In group annuities the employee's contributions are usually returned at death, so that there is a death benefit payable during the deferred period which might average $3 per $1 of annual retirement income. The single premium values for a $3 temporary death benefit payable from issue to age 65 have been added to the Part A values and are shown as Part B of Table V. In this case the errors by the method of this discussion are somewhat larger than those by the method of the paper. In actual practice the retirement income would usually be payable monthly and the death benefit would probably be continued for decreasing amounts until the retirement income payments equaled the original death benefit, but the errors of using either approximate method for these adjustments were investigated and found to be small. This relatively poor fit for death benefits by the method of this discussion arises because it is essentially a fitted method in which immediate and deferred annuities are fitted. If conservatism rather than close fit is to be sought at any point, however, it seems appropriate to have it in the death benefit.

(AUTHOR'S REVIEW OF DISCUSSION)

CHARLES M. STERNHELL:

I would like to thank Mr. Emery for his discussion of this paper. Mr. Emery has tackled the problem of taking account of improving mortality from a somewhat different point of view and has come up with an interesting solution. He states that his method is really two methods blended into one by empirical means. In view of the empirical nature of his method, I think that it is important to examine carefully each of the two parts of his method separately, as well as the junction of the two parts. I have not attempted this careful review in the short time available, but I would like to comment briefly on some of the points that might require further investigation.

First, I would like to say that I do not see any theoretical objection to the part of Mr. Emery's method that deals with the calculation of approximate annuity values at ages 59 and over. This part is based on the same underlying principle as the method presented in the paper; namely, that the approximate values of forecast annuities issued in various calendar years at a particular age may be assumed to lie on a straight line, *i.e.*, to have constant first differences. The paper shows how an approximate value of a forecast annuity in any year may be calculated directly by using some supplementary commutation columns. Mr. Emery's discussion indicates how an approximate value of a forecast annuity in any year may be obtained by first difference interpolation or extrapolation from the exact values of forecast annuities in two particular calendar years. These two exact values may be calculated by using standard commutation columns which were constructed for two specific year-of-birth mortality tables based on the Annuity Table for 1949 with Projection Scale B.

For the calculation of approximate annuity values at ages under 59, Mr. Emery recommends a method that is based on an entirely different principle than the method used at ages 59 and over. A single year-of-birth mortality table is used for this part of Mr. Emery's method, with the mortality rates at ages 55 and over exact and the mortality rates at ages under 55 approximate. Approximate values of forecast annuities in any year are obtained by first difference interpolation or extrapolation from the approximate values of forecast annuities at two particular ages, based on the assumption that a one year setback in age on this year-of-birth mortality table is equivalent to a ten year advance in the calendar year in which the annuity is issued. The approximate annuity values for the two particular ages may be calculated by using standard commutation columns based on the one approximate year-of-birth mortality table.

I believe that this part of Mr. Emery's method requires a particularly careful examination because various attempts in the past have indicated that an age setback method will not accurately reproduce the effect of a constant annual percentage reduction in the mortality rate. While Mr. Emery presented extensive tests of the annuity values produced by his method, I do not believe that any of these tests provide a valid check on the accuracy of the mortality rates that are produced at the younger ages by his method. This statement is based on the fact that all of the annuity values tested in the discussion depend largely on the level of mortality at the older ages and are only slightly influenced by errors in the mortality rates at the younger ages.

Mr. Emery's discussion provides one clue to the fact that the mortality rates produced by his method at the younger ages are not too accurate. He notes that the addition of a death benefit to the retirement income annuity values tested in his Table V produces a significant increase in the errors resulting from his method. Mr. Emery explains this as follows:

"This relatively poor fit for death benefits by the method of this discussion arises because it is essentially a fitted method in which immediate and deferred annuities are fitted." I think that this explanation misses the main point because Mr. Emery will find that the method he uses for annuities issued at the older ages results in an excellent fit for death benefits. The real explanation of why Mr. Emery's method produced a relatively poor fit for death benefits is that the death benefits tested in Table V depend largely on the mortality rates at the younger ages where his method produces relatively inaccurate mortality rates. Mr. Emery's discussion does not explicitly show the errors in the value of the death benefit, but by subtracting the values in Part A of Table V from those in Part B of Table V it may be seen that, for contracts issued in 1960, his method results in errors that range from 15% of the exact value of the death benefit at age 25 to 5% at age 55.

The following brief table provides a more direct measure of the relative accuracy of the mortality rates produced by Mr. Emery's method at the younger ages.

PERCENTAGE EXCESS OF q_x PRODUCED BY INDICATED METHOD OVER q_x BASED ON ANNUITY TABLE FOR 1949 WITH PROJECTION SCALE B—MALES

Age x	In 1955		In 1960		In 1965	
	Method of		Method of		Method of	
	Paper	Discussion	Paper	Discussion	Paper	Discussion
15	−.2%	5.3%	−.8%	10.8%	−2.0%	16.7%
35	−.2	2.9	−.8	6.0	−1.9	9.2
55	−.2	2.1	−.8	4.1	−1.9	5.9

The above table clearly indicates why the method proposed by Mr. Emery produces relatively inaccurate values at the younger ages for the death benefits provided with group annuity contracts and cash refund annuities. In fact, I do not think that Mr. Emery's method provides sufficient accuracy to justify its use for the calculation of any life contingency benefit that depends primarily on mortality rates at the younger ages.

I believe that the above table also raises a serious objection to the use of Mr. Emery's approximate annuity values for valuation purposes because it indicates that at ages below 60 the mortality profits or losses on annuity contracts will be significantly distorted if annuity reserves are based on his approximate method. This is due to the fact that any error in the progression of the approximate annuity reserves from one year to the next would be reflected in the mortality profits or losses, which are calculated by using the approximate annuity values. For example, the 6% error in the mortality rate at age 35 in 1960 indicates that if immediate or deferred annuity reserves are based on Mr. Emery's approximate annuity values, then no mortality profit would be shown for age 35 in 1960 even though the actual mortality rate experienced was 6% higher than the mortality rate based on the Annuity Table for 1949 with Projection Scale B.

The junction at age 59 of the two distinct methods proposed by Mr. Emery also requires careful examination. At ages below 59, annuity values are calculated from a single mortality table by an age setback method while, at ages 59 and over, they are calculated by interpolating or extrapolating from annuity values calculated from two separate mortality tables. It is not surprising, therefore, that annuity values calculated by Mr. Emery's method will exhibit some discontinuities at age 59. One indication of these discontinuities is provided by Table II, where the approximate annuity values for issues of 1950 exhibit the following first and second differences:

Age	First Difference	Second Difference
55	−.454	−.003
56	−.457	−.002
57	−.459	−.003
58	−.462	+.004
59	−.458	−.002
60	−.460	

The following example indicates how the reserves on an immediate life annuity contract issued at age 55 in 1960 would progress from year to year if Mr. Emery's approximate annuity values were used.

Age x	Year $1950+k$	${}^{1950+k}a_x$	First Difference	Second Difference
55	1960	16.744	−.413	.000
56	1961	16.331	−.413	−.002
57	1962	15.918	−.415	−.001
58	1963	15.503	−.416	−.017
59	1964	15.087	−.433	−.003
60	1965	14.654	−.436	
61	1966	14.218		

The mortality rates underlying this progression of approximate annuity values may be obtained by solving for ${}^{1950+k}p_x$ in the following formula:

$${}^{1950+k}a_x = v\,({}^{1950+k}p_x)\,(1 + {}^{1950+k+1}a_{x+1})\,.$$

This produces the following mortality rates:

Age x	Year	$1,000q_x$	First Difference	Second Difference
55	1960	9.721	.843	−.056
56	1961	10.564	.787	.041
57	1962	11.351	.828	−.844
58	1963	12.179	−.016	.805
59	1964	12.163	.789	.091
60	1965	12.952	.880	
61	1966	13.832		

It is apparent from the preceding table that the discontinuities in Mr. Emery's approximate annuity values reflect more abrupt discontinuities in the underlying mortality rates. These discontinuities are due primarily to the fact that, in calculating the approximate values of annuities at ages below 59, the age setback method is applied to the mortality rates at ages 59 and over as well as to the mortality rates below age 59. An entirely different procedure is used in calculating the approximate values of annuities at ages 59 and over. This means that for a given age and calendar year, two entirely different mortality rates may be used depending upon whether the annuity value is being calculated for an age below 59 or an age over 59. For example, the value of $1,000q_x$ at age 60 in 1965 is in effect assumed to be 13.854 for the calculation of the approximate value of an annuity at age 55 in 1960 and 12.952 for the calculation of the approximate value of an annuity at age 60 in 1965.

The above examples were selected at random merely to indicate that if Mr. Emery's method were considered for premium or valuation purposes, the problems of smoothness and consistency would require special investigation. Due to the fact that Mr. Emery's method involves the empirical junction of two different methods, his commutation columns cannot be depended upon to produce the smooth and consistent results that standard commutation columns usually produce.

In closing my reply to the written discussion submitted by Mr. Emery, I would like to state that while his method may have certain drawbacks for premium or valuation purposes, I believe that his empirical tables do provide accurate estimates of immediate and deferred annuity values and might be useful for preliminary experimental studies in connection with selecting a basis for premium rates. In case someone else might attempt to construct similar empirical mortality tables based on other mortality assumptions, the empirical steps in Mr. Emery's method, such as the junction at age 55 or the use of 10 for the value of f, might require some modifications. I also suspect that the errors produced in the mortality rates at the younger ages by the age setback method might be considerably larger if the mortality basis involved a projection scale with varying rates of decrease at the younger ages, such as Projection Scale A in the Jenkins-Lew paper.

I would also like to take this opportunity to discuss briefly an interesting question that was raised by Mr. W. M. Anderson in a letter he wrote me. The paper stated that the supplementary commutation columns will produce annuity values that are *approximately* equal to corresponding annuity values calculated on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale B and $2\frac{1}{2}\%$ interest. Mr. Anderson, in effect, asked me to determine a Projection Scale X such that the annuity values produced by the supplementary commutation columns in the paper would be *exactly* equal to corresponding annuity values calculated on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale X and $2\frac{1}{2}\%$ interest. The table at the end of this discussion compares the annual rates of decrease in mortality in Projection Scale X with those presented in Projection Scale B of the Jenkins-Lew paper and indicates the extent of the approximation that is implicitly introduced by the method presented in my paper.

As the only approximation introduced in deriving all of the formulae in the paper is the assumption that the basic formula (14) in the paper is exact, the problem reduces itself to one of determining what annual rates of decrease would have to be assumed in Projection Scale X to really make formula (14) exact. The basic formula (14) may be expressed as follows:

$$^{1950+k}{}_{n}p_x \doteq {}_{n}p_x\,[1 + kf_x + (k+1)\,f_{x+1} + \ldots + (k+n-1)\,f_{x+n-1}]$$

where all of the symbols are defined in the paper. If we use primes to indicate the exact values of probabilities based on Projection Scale X, then the following equation must be satisfied for all values of k, x, and n.

$$^{1950+k}{}_{n}p'_x = {}_{n}p_x\,[1 + kf_x + (k+1)\,f_{x+1} + \ldots + (k+n-1)\,f_{x+n-1}]$$

This equation produces the following values for a life aged x in the year $1950+k$.

$$^{1950+k}p'_x = p_x\,[1 + kf_x]$$

$$^{1950+k+1}p'_{x+1} = p_{x+1}\left[\frac{1 + kf_x + (k+1)\,f_{x+1}}{1 + kf_x}\right]$$

. . . .

$$^{1950+k+n-1}p'_{x+n-1}$$

$$= p_{x+n-1}\left[\frac{1 + kf_x + (k+1)\,f_{x+1} + \ldots + (k+n-1)\,f_{x+n-1}}{1 + kf_x + (k+1)\,f_{x+1} + \ldots + (k+n-2)\,f_{x+n-2}}\right]$$

. . . .

From these values, it is easy to obtain the values of ${}^{1950+k+t}q'_{x+t} = 1 - {}^{1950+k+t}p'_{x+t}$, where ${}^{1950+k+t}q'_{x+t}$ specifies the future mortality rates that must be assumed for a life aged x in the year $1950+k$. Similar sets of future mortality rates can be determined for any values of k and x.

This means that, for each calendar year $1950+k$, a set of future mortality rates varying by age and calendar year is completely specified. It is relatively easy to determine, for any calendar year $1950+k$, the annual rates of decrease that would have to be assumed in Projection Scale X in order to produce the corresponding set of future mortality rates. The set of annual rates of decrease in Projection Scale X for the calendar year $1950+k$ may be designated by the symbol ${}^{1950+k}_{t}s_x$, where

if $t = 0$, ${}^{1950+k}_{0}s_x =$ the constant annual rate of decrease at attained age x between the years 1950 and $1950+k$, and

if $t > 0$, ${}^{1950+k}_{t}s_x =$ the annual rate of decrease at attained age x between the years $1950+k+t-1$ and $1950+k+t$.

Although Projection Scale X contains a different set of annual rates of decrease for each value of k, *i.e.*, for each calendar year as of which annuity values are calculated, the table at the end of this discussion indicates that a change in the value of k has a relatively slight effect on the annual rates of decrease.

While the various values of ${}^{1950+k}_{t}s_x$ may be computed by the procedure indicated above, the following general formulae prove useful in calculating specimen values at isolated points.

For $t = 0$,

$${}^{1950+k}_{0}s_x = 1 - \left(1 - k\,\frac{p_x f_x}{q_x}\right)^{1/k}$$

For $t > 0$,

$${}^{1950+k}_{t}s_x = \frac{p_x\,({}^{k}A^{t}_{x} - {}^{k}A^{t-1}_{x})}{1 - p_x\,({}^{k}A^{t-1}_{x})}$$

where

$${}^{k}A^{t}_{x} = \frac{1 + k\,(\mathrm{F}_{x-t} - \mathrm{F}_{x+1}) + \mathrm{G}_{x-t} - \mathrm{G}_{x+1} - (t+1)\,\mathrm{F}_{x+1}}{1 + k\,(\mathrm{F}_{x-t} - \mathrm{F}_{x}) + \mathrm{G}_{x-t} - \mathrm{G}_{x} - t\mathrm{F}_{x}}.$$

These general formulae may be derived by following the procedure indicated above and then substituting the appropriate supplementary commutation columns for each series of f_x terms.

Specimen values of ${}^{1950+k}_{t}s_x$ in Projection Scale X are compared in the following table with corresponding values of s_x in Projection Scale B, *i.e.*, the varying annual rates of decrease that must be assumed to produce exact annuity values equal to the approximate values of the paper are compared with the constant annual rates of decrease presentedinProjection Scale B of the Jenkins-Lew paper. These specimen values are compared for annuity values in 1950 and 1960 on male lives at ages 20, 40, 60, and 80. These comparisons indicate that the use of the approximate annuity values produced by the method described in the paper is equivalent to assuming annual rates of decrease in mortality that differ only slightly from those presented in Projection Scale B.

COMPARISON OF ANNUAL RATES OF DECREASE IN MORTALITY
JENKINS-LEW PROJECTION SCALE B VS. PROJECTION SCALE X*
MALE LIVES

ATTAINED AGE x	CONSTANT ANNUAL RATE OF DECREASE AT AGE x IN PROJECTION SCALE B	PROJECTION SCALE X FOR ANNUITY VALUES IN 1950 — Annual Rate of Decrease at Age x between the Indicated Years			
	s_x	1950–1951 ${}^{1950}_{1}s_x$	1960–1961 ${}^{1950}_{11}s_x$	1970–1971 ${}^{1950}_{21}s_x$	1980–1981 ${}^{1950}_{31}s_x$
20	.0125	.0128	.0147	....	....
40	.0125	.0128	.0147	.0172	.0202
60	.0120	.0120	.0133	.0148	.0166
80	.0050	.0050	.0049	.0045	.0039
		PROJECTION SCALE X FOR ANNUITY VALUES IN 1960 — Annual Rate of Decrease at Age x between the Indicated Years			
	s_x	1950–1960 ${}^{1960}_{0}s_x$	1960–1961 ${}^{1960}_{1}s_x$	1970–1971 ${}^{1960}_{11}s_x$	1980–1981 ${}^{1960}_{21}s_x$
20	.0125	.0136	.0147	....	....
40	.0125	.0136	.0147	.0166	.0199
60	.0120	.0127	.0134	.0147	.0166
80	.0050	.0051	.0050	.0044	.0039

* Exact annuity values calculated on the basis of the Annuity Table for 1949 (ultimate) with Projection Scale X and $2\frac{1}{2}$% interest are equal to the corresponding approximate annuity values produced by using the supplementary commutation columns presented in the paper.