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POLYNOMIAL INTERPOLATION IN TERMS OF SYMBOLIC OPERATORS

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INTRODUCTION

IN *TSA* I, 343, one of the authors of this paper described a symbolic method of deriving discrete interpolation formulas based on an analogy between interpolation and graduation previously pointed out by the other author (*JIA* LXXII, 482). By a discrete interpolation formula is meant one which does not produce interpolated values for all arguments, but only for certain specified arguments at equal intervals, the interval between interpolated values being an exact divisor of that between given values. Thus, the interpolation "formula" takes the form of a discrete set of coefficients to be applied to the given values to obtain the interpolated values.

There are, however, certain situations in which it is desirable to interpolate by a method which produces interpolated values for a continuum of values of the argument.[1] This is the case when interpolated values are desired for only a small number of particular arguments, which also may not divide the intervals between given values in a commensurable ratio, and further when it is desired to estimate the values of derivatives of the tabulated function. Such situations arise frequently in physics and astronomy and occasionally in actuarial work. Moreover, for purposes of research and comparison of formulas—in considering smooth junction, for example—it is advantageous to consider interpolation from the continuous point of view.

In *TASA* XLV, 202, one of the authors gave a systematic treatment of the more usual types of continuously defined smooth-junction interpolation formulas. However, the methods developed there are somewhat cumbersome to apply, and there are certain types of formulas (including a few well-known and useful ones) to which they are not readily applicable. We were therefore led to investigate whether the symbolic short

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[1] We shall refer to such a formula as "continuously defined." "Continuous interpolation formula" might suggest that the curve of interpolated values is everywhere continuous, which is not the case for some formulas we wish to consider. This point is discussed more fully on page 453.

cuts which resulted in a marked simplification in the theory of discrete interpolation could be applied in an analogous manner to continuously defined formulas. While it has unexpectedly turned out that the analogy does not strictly hold in certain respects, and that a rigorous mathematical justification of the procedures presents greater difficulties than in the discrete case, we have succeeded in developing a symbolic method which is not difficult to apply and seems to have substantial advantages over other approaches.[2]

The mathematical development has been greatly influenced by the work of two eminent contemporary mathematicians, Professors I. J. Schoenberg, of the University of Pennsylvania, and Laurent Schwartz, of the Institut Henri Poincaré, Paris. Schoenberg, in his remarkable paper, "Contributions to the problem of approximation of equidistant data by analytic functions" (*Quarterly of Applied Mathematics*, IV, 45 and 112), has developed a very elegant theory of interpolation based on the use of the Fourier transform. We have taken over much of his useful terminology, and a number of the results obtained in this paper were previously given by him. Our approach differs from his in two main respects. First, by using symbolic operators rather than Fourier integrals, we believe we have considerably reduced the extent of the mathematical knowledge required of the reader, making the results accessible to a larger group. Secondly, our main interest has been in developing a straightforward method for constructing an interpolation formula of finite range with desired properties stipulated in advance. Schoenberg did not have this objective primarily in view, and his methods do not seem to us to lend themselves quite as readily for this purpose. This seems especially true of his criterion for determining the degree of polynomial maintained or reproduced by an interpolation formula.

A rigorous development of a continuous theory of interpolation from the point of view of symbolic operators seems to require some extension of the usual notion of the derivative of a function, to include the case of functions having finite discontinuities. This is provided by the Theory of Distributions recently developed by Schwartz.[3] Our indebtedness to both

[2] A few of the results obtained in this paper have previously appeared (in Portuguese) in the *Revista Brasileira de Estatística*, XIV (1953), 209.

[3] L. Schwartz, *Théorie des Distributions*, Tomes I et II (Nos. 1091 and 1122 of the series, *Actualités Scientifiques et Industrielles*), Hermann & Cie, Paris, 1950 and 1951.

In this connection, it may be of interest to mention that at the International Congress of Mathematicians held at Cambridge, Massachusetts, in 1950, Professor Schwartz received, in recognition of his work in developing this new branch of mathematics, one of the two Fields medals awarded at each international mathematical congress for outstanding mathematical achievement.

Schoenberg and Schwartz is so extensive that it has not seemed practicable to make specific acknowledgment in every instance. A paper of Professor Haskell B. Curry[4] of the Pennsylvania State University has also influenced our general approach. While no direct application has been made of Curry's results, our reliance on operational methods is probably more thoroughgoing than would have been the case had we not read his paper. Mention should also be made of recent papers by Michalup (*JIA* LXXIX, 74) and Vaughan (*JIA* LXXX, 63) having some relationship to the subject of this paper.

The principle utilized in this paper is capable of wider development. It appears that it can be applied at least to mathematical forms which are the solution of a linear difference equation, *i.e.*, such forms as exponentials and certain sine and cosine series. It is not, however, the present object to deal with this wider aspect, and this paper has in view mainly the application to polynomial interpolation as used in actuarial work. In the main part of the paper, the principal results and methods are stated and described without proof. Full mathematical demonstrations are given in an appendix.

THE BASIC FUNCTION OF AN INTERPOLATION FORMULA

In the kind of interpolation generally used in actuarial work, the same interpolation process is applied in successive interpolation intervals, the interpolated values in each interval being calculated from a certain number of neighboring given values. As we pass from one interpolation interval to the next, the formula is unchanged, but each given value entering into it is replaced by its immediate neighbor. Schoenberg was the first to point out that an interpolation process of this type can be completely characterized by a certain mathematical function which he calls the basic function of the interpolation formula. Such a process implies that when the formula for a particular interpolated value (corresponding, let us say, to the argument x) is expressed in linear compound form in terms of the given ordinates, the coefficient of a particular given value (corresponding to the argument a, for example) can be expressed as a function of the difference, $x - a$. In other words, this coefficient is unchanged when we pass to a corresponding point in another interpolation interval, so that x and a both change, but the difference $x - a$ remains unchanged. Therefore, we call this coefficient $L(x - a)$, and we define the *basic function* of the interpolation formula as the function $L(y)$ which gives, for the entire range of possible values of $y = x - a$, the value of the appropriate coeffi-

[4] "Abstract differential operators and interpolation formulas," *Portugaliae Mathematica*, X (1951), 135.

cient in the formula. Whenever y is sufficiently large in absolute value so that the given value at the argument a does not enter into the computation of the interpolated value for the argument x, we consider that $L(y) = 0$. It will be seen that the basic function plays the same role in continuous interpolation as the discrete set of coefficients which characterizes a discrete interpolation formula.

All this can be stated much more simply algebraically in the form[5]

$$v_x = \sum_{n=-\infty}^{\infty} L(x-n)\, u_n\,, \tag{1}$$

where the u's are the given values and v_x is the interpolated value corresponding to the argument x.

A few specific examples may help to clarify the concept of basic function. In the case of plain central-difference interpolation to third differences, $L(x)$ represents the "Lagrange" coefficients, and is given by the expressions:

$$\begin{array}{ll}
0 & \text{for} \quad x \le -2 \\
\frac{1}{6}(x+3)(x+2)(x+1) & \text{for} \quad -2 \le x \le -1 \\
-\frac{1}{2}(x+2)(x+1)(x-1) & \text{for} \quad -1 \le x \le 0 \\
\frac{1}{2}(x+1)(x-1)(x-2) & \text{for} \quad 0 \le x \le 1 \\
-\frac{1}{6}(x-1)(x-2)(x-3) & \text{for} \quad 1 \le x \le 2 \\
0 & \text{for} \quad x \ge 2
\end{array}$$

For Karup's formula, another four-point formula, the basic function $L(x)$ takes the form:

$$\left.\begin{array}{ll}
0 & \text{for } x \le -2 \\
\frac{1}{2}(x+1)(x+2)^2 & \text{for } -2 \le x \le -1 \\
-\frac{1}{2}(x+1)(3x^2+2x-2) & \text{for } -1 \le x \le 0 \\
\frac{1}{2}(x-1)(3x^2-2x-2) & \text{for } 0 \le x \le 1 \\
-\frac{1}{2}(x-1)(x-2)^2 & \text{for } 1 \le x \le 2 \\
0 & \text{for } x \ge 2
\end{array}\right\} \tag{2}$$

[5] While an infinite summation has been indicated in equation (1) as a matter of algebraic convenience, the summation is actually finite for the cases considered in this paper. Basic functions of infinite range are possible, and are considered by Schoenberg. For example, differential-equation interpolation could be included in the system, and could give rise to such a basic function.

It is evident from these examples that a typical basic function differs from the usual functions encountered in elementary mathematics in that it changes abruptly at certain points from one algebraic expression to another. However, in developing a unified theory of continuous interpolation, it is a great convenience to regard the basic function as a single function, notwithstanding its rather complicated definition.

The graphs of these two basic functions are shown in Figure 1. It will be noted that the curve for plain central third-difference interpolation changes its direction sharply at the points corresponding to integral values of x. The Karup curve follows the plain third-difference curve rather closely, but has a continuous first derivative throughout. Also plotted is the basic function of Jenkins' four-point smoothing third-difference formula, an extreme type of "modified" formula with marked smoothing properties, which is given by:

$$\begin{array}{ll}
0 & \text{for } x \le -2 \\
\frac{1}{4}(x+2)^2 & \text{for } -2 \le x \le -1 \\
-\frac{1}{4}(x^2-2) & \text{for } -1 \le x \le 1 \\
\frac{1}{4}(x-2)^2 & \text{for } 1 \le x \le 2 \\
0 & \text{for } x \ge 2
\end{array}$$

In this paper, consideration will be limited to those interpolation formulas which involve only a finite number of equidistant given values, all situated within a specified distance of the interpolation interval in which we are working. This is tantamount to saying that the basic function $L(x)$ vanishes outside a certain range of values of x, as it does in the three examples already given. We shall also be chiefly interested in interpolation formulas which are *symmetrical*—that is, have the property that reversing the order of the sequence of given values merely reverses the order of the interpolated values in each interval, without changing their numerical values. This will be the case if and only if the basic function $L(x)$ is an even function of x—that is, $L(-x) = L(x)$ for all values of x.

CLASSES OF SYMBOLIC OPERATORS

In addition to the usual displacement operator E of the Calculus of Finite Differences, we shall use the symbol D to denote differentiation, and also Sheppard's central-difference notation, wherein $\delta = E^{1/2} - E^{-1/2}$ and $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$. We shall also introduce a new operator M defined by

$$Mf(x) = \int_{-1/2}^{1/2} f(x+t)\,dt\,,$$

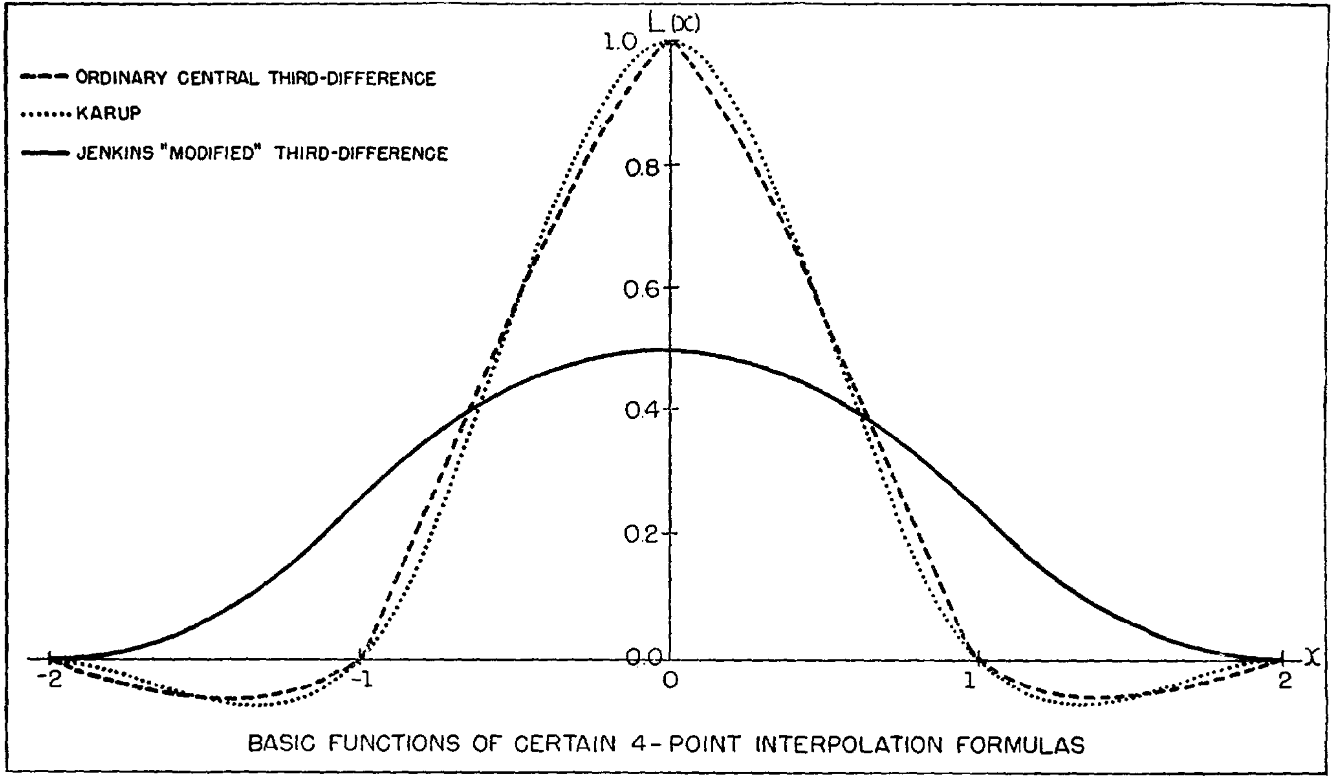
L(x)
1.0
0.8
0.6
0.4
0.2
0.0
ORDINARY CENTRAL THIRD-DIFFERENCE
KARUP
JENKINS "MODIFIED" THIRD-DIFFERENCE
-2
-1
0
1
2
x
BASIC FUNCTIONS OF CERTAIN 4-POINT INTERPOLATION FORMULAS

FIG. 1

or, symbolically,

$$M = \int_{-1/2}^{1/2} E^t dt .$$

It will appear later that this operator plays a role here analogous to that of $[m]$ in the theory of discrete interpolation. It is easily verified that it satisfies the symbolic equation

$$DM = \delta .$$

It is important for our purpose to distinguish certain general classes of operators. Any operator of the form

$$J = \sum_{i=1}^{N} c_i E^{-x_i} , \tag{3}$$

where the c's and x's are any real numbers, will be called a *discrete operator*. Similarly, any operator of the form

$$K = \int_a^b g(t) E^{-t} dt , \tag{4}$$

where $g(t)$ is an integrable function of t, will be called a *continuous operator*. It may be well to explain that, in both cases, the minus sign in the exponent of E has been introduced for algebraic convenience in later developments, and is not an essential part of the definition.

Obvious examples of discrete operators are δ and μ, and all positive integral powers of these operators. M and its positive integral powers are examples of continuous operators, as are also $\delta^k M^l$ and $\mu\delta^k M^l$, k and l being positive integers.

For our purposes, it will be convenient to modify the definition (4) by defining a function $f(t)$ so that

$$f(t) = \begin{cases} 0 & \text{for} \quad t < a \\ g(t) & \text{for} \quad a \leq t \leq b \\ 0 & \text{for} \quad t > b \end{cases}$$

We may now rewrite equation (4) in the form:

$$K = \int_{-\infty}^{\infty} f(t) E^{-t} dt .$$

We shall call the function $f(t)$ thus defined the *basic function* of the continuous operator K.

THE CHARACTERISTIC OPERATOR OF AN INTERPOLATION FORMULA

The *characteristic operator* of a continuously defined interpolation formula will be defined as the continuous operator whose basic function is also the basic function of the interpolation formula. In other words, the characteristic operator G is defined by

$$G=\int_{-\infty}^{\infty} L(t)E^{-t}dt\,, \tag{5}$$

where $L(t)$ is the basic function of the interpolation formula.[6] It is clearly analogous to the graduation operator which the authors have associated with a discrete interpolation formula. Like the basic function, the characteristic operator completely characterizes the interpolation formula, and all the properties of the formula can be deduced from it. Particular interest attaches to the characteristic operator because, for most of the common interpolation formulas, it reduces to a surprisingly simple form in terms of the operators D, δ, μ, and M. For example, the characteristic operator is M^2 for straight-line interpolation, $M^4(1-\frac{1}{6}D^2)$ for plain central third-difference interpolation, $M^3(3M-2\mu)$ for Karup's tangential formula, and $M^4(1-\frac{1}{6}\delta^2)$ for Jenkins' well-known osculatory fifth-difference smoothing interpolation formula. The characteristic operators of a number of continuously defined interpolation formulas are given in Table 1, which follows the main part of the paper.

As the above-mentioned formula of Jenkins is so well known, we digress to point out that the characteristic operator indicates a convenient method of applying the formula in practice, originally suggested by D. C. Fraser in *TFA* XII, 153–54. If we make a preliminary adjustment of the data by deducting $\frac{1}{6}\delta^2 u_n$ from each given value u_n, we can complete the interpolation by applying to the resulting values the four-point formula corresponding to M^4, which (in Everett form) is

$$v_{n+x}=xu_{n+1}+\tfrac{1}{6}x^3\delta^2u_{n+1}+yu_n+\tfrac{1}{6}y^3\delta^2u_n\,,$$

where $y=1-x$.

Perhaps it should be emphasized that the correspondence between an interpolation formula and its characteristic operator is a purely formal one. Operating on a given function with a characteristic operator and using the corresponding interpolation formula to interpolate between given values of the same function are not at all the same thing, and do not give the same result except in special situations. In other words, the characteristic operator *corresponds* to the interpolation formula, but *is not*

[6] In essence, this is equivalent to the characteristic function of an interpolation formula as defined by Schoenberg.

equivalent to it. Nevertheless, because of the fact that the formula and its characteristic operator have the same basic function, each property of the formula is reflected in a specific corresponding property of the operator. The usefulness of the characteristic operator arises from two circumstances: (1) it is usually a much more compact expression than the formula, and (2) several of the properties of interpolation formulas in which one is usually interested happen to correspond to simpler and more immediately obvious properties of the characteristic operator.

As stated earlier, our main purpose is to provide a convenient method for deducing an interpolation formula having previously stipulated properties. To this end, we shall give a set of rules by which it will usually be possible to obtain rather easily the characteristic operator of the formula having the desired properties. The application of these rules is facilitated by some tables which follow the main part of the paper. Certain further tables will enable us to pass readily from the characteristic operator to the corresponding interpolation formula (or vice versa), or, if it is preferred, to compute numerical values of the interpolation coefficients without actually obtaining the formula in algebraic form.

Our rules and tables can also be used in another way. When an interpolation formula is given, its degree and the number of given ordinates it employs are obvious, but fairly extensive analysis may be necessary to determine its order of contact, if this is not known. Sometimes, too, a certain amount of algebra is required to ascertain the number of differences to which the formula is correct. However, if we first obtain the characteristic operator, the former property can be read off merely by inspection, and the latter can be rather quickly deduced.

DEFINITIONS OF SOME SPECIAL TERMS

The rules mentioned in the two preceding paragraphs will employ a few special terms which require definition.

A discrete operator of the form

$$a_0 + a_1\mu\delta + a_2\delta^2 + a_3\mu\delta^3 + \dots$$

will be called a *Stirling operator*, while one of the form

$$a_0\mu + a_1\delta + a_2\mu\delta^2 + a_3\delta^3 + \dots$$

will be called a *Bessel operator*. The analogy with the interpolation formulas bearing these names is obvious.

The *span* of an operator will be defined as the difference between the greatest and smallest exponents of E involved in the expression for the operator. In the case of a discrete operator, this is the difference between the greatest and least exponents of E in the summation (3). For a continuous operator, it is merely the difference, $b - a$, between the limits of integration in equation (4).[7] In the case of a composite operator of the form (6) or (6a) used subsequently, it is necessary to take into account the exponents of E involved in all the terms.

We shall define the *trace* of a continuous operator K, with basic function $f(t)$, as the discrete operator $t(K)$ given by

$$t(K) = \sum_{n=-\infty}^{\infty} f(n) E^{-n} .$$

It will be seen by comparison with equation (1) that the trace of the characteristic operator of an interpolation formula (which, for brevity, we shall refer to as the trace of the interpolation formula) represents the effect of the formula on the given values. Thus, the trace of any formula which always reproduces the given values is 1, while that of Jenkins' smoothing fifth-difference formula is $1 - \frac{1}{36}\delta^4$.

RULES CONNECTING THE PROPERTIES OF AN INTERPOLATION FORMULA WITH THOSE OF ITS CHARACTERISTIC OPERATOR[8]

1. A continuously defined interpolation formula is correct to rth differences if and only if *both* the following conditions (a) and (b) are satisfied:

(a) Its characteristic operator G is of the form $M^{r+1}H$, where H is an operator of the form indicated in Rule 3(a).

(b) The symbolic expansion of H in powers of D alone agrees, up to and including the term (if any) containing D^r, with the similar expansion[9] of M^{-r-1}.

2. If s denotes the number of terms in the linear compound form of a continuously defined interpolation formula correct to rth differences and the span of the operator H is an integer[10] h, then

$$s = h + r + 1 .$$

[7] Here it is assumed that the interval $a \leq t \leq b$ is the smallest interval containing all arguments t corresponding to nonzero values of the basic function $f(t)$. It involves no restriction on the definition of a continuous operator to choose a and b so that this is the case.

[8] Proof of these rules will be found in the Mathematical Appendix at the end of the paper. Throughout the rules, it is tacitly assumed, for the sake of simplicity, that only formulas of finite span are under consideration. Nevertheless, some of the rules still apply, and others apply with certain reservations, to formulas of infinite span.

[9] Table 2, page 446, gives a number of symbolic expansions in powers of D only which will facilitate the application of this rule in most cases.

[10] The span h is always an integer when (as is usually the case) the number of terms s is the same for all arguments (with the possible exception of those differing by an integer

3. (a) A polynomial interpolation formula correct to rth differences is of degree q and the order of contact of successive interpolating arcs is always at least p if and only if H (Rule 1) can be expressed[11] in the form[12]

$$H = J_{-q+r}D^{-q+r} + J_{-q+r+1}D^{-q+r+1} + \ldots + J_{r-p-1}D^{r-p-1}\,, \qquad (6)$$

where the J's are discrete operators and J_{-q+r} is different from zero.[13]

(b) In any polynomial interpolation formula, the degree q, the order of differences r to which the formula is correct, and the minimum order of contact p of successive interpolating arcs satisfy the inequalities:

$$q \geq r\,, \qquad q \geq p + 1\,.$$

(c) Any J's with negative subscripts in equation (6) are necessarily such that the coefficients of negative powers of D vanish in the symbolic expansion of H in powers of D only.

4. (a) A polynomial interpolation formula correct to rth differences (when r is even) is an end-point formula (that is, points of junction of the interpolating arcs occur only at the arguments corresponding to the given values) if and only if the J's in equation (6) can all be expressed as Bessel operators; it is a midpoint formula (that is, points of junction of the interpolating arcs occur only at the arguments midway between the given values) if and only if the J's can all be expressed as Stirling operators.

(b) A polynomial interpolation formula correct to rth differences (when

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from a and b, the extremities of the smallest interval containing all nonzero values of the basic function). If this is not the case, and $h = n + f$, where n is an integer and f a proper fraction, then the number of terms is $n + r + 1$ for some arguments and $n + r + 2$ for others.

[11] This exact expression for H as a *finite* sum of powers of D with ***discrete operators*** as coefficients is not interchangeable with, and should be clearly distinguished from, the (usually ***infinite***) symbolic expansion in powers of D with ***numerical*** coefficients which is referred to in Rules 1(b) and 3(c). Only when H is of span zero are the two expressions identical.

[12] For a general interpolation formula (possibly employing other than polynomial arcs), the order of contact of successive interpolating arcs is always at least p (for $p \leq r$) if and only if H is of the form

$$H = K + J_0 + J_1D + \ldots + J_{r-p-1}D^{r-p-1}\,, \qquad (6a)$$

where K is a continuous (or zero) operator, and the J's are discrete operators. Here $p = r$ implies that $H = K$. In formulas (6) and (6a), $p = 0$ means that the curve of interpolated values is without discontinuities (but may have discontinuities in its first derivative), while $p = -1$ means that discontinuities in the curve itself are permitted.

[13] If the order of differences to which the formula is correct is not specified, we may take $r = -1$ in formula (6) or (6a).

r is odd) is an end-point formula if and only if the J's can all be expressed as Stirling operators; it is a midpoint formula if and only if the J's can all be expressed as Bessel operators.

(c) A polynomial interpolation formula has the points of junction of its interpolating arcs limited to the arguments corresponding to the given values *and* the arguments midway between them if and only if each "J" operator in equation (6) is either a Stirling operator or a Bessel operator, or a sum of both.

5. If a continuously defined interpolation formula is symmetrical, and if all the "J" operators in equation (6) or (6a) are Stirling operators or Bessel operators, or sums of both, then only even powers of δ occur in those J's which are coefficients of even powers of D, and, similarly, only odd powers of δ occur in those J's which are coefficients of odd powers of D.

6. (a) The trace of any continuously defined interpolation formula correct to rth differences can be expressed as a Stirling operator in which the leading term is 1 and the coefficients of powers 1 to r, inclusive, of δ vanish. If the formula is symmetrical, only even powers of δ occur.

(b) If the formula is symmetrical and s (Rule 2) is odd, the span of its trace is at most $s-1$.

(c) If the formula is symmetrical and s (Rule 2) is even, and the curve of interpolated values is free from discontinuities whatever the given values may be (or, in other words, if $p \geq 0$), the span of its trace is at most $s - 2$.

7. If $t(K)$ denotes the trace[14] of the continuous operator K, and j is any positive integer,

$$t(\delta^{2j}K) = \delta^{2j}t(K) ,$$

or

$$t(K) = \delta^{-2j}t(\delta^{2j}K) .$$

8. A continuously defined interpolation formula always reproduces the given values if and only if its trace is 1.

In addition to the rules proper, the following three symbolic identities are frequently needed in the derivation of interpolation formulas:

$$M^kD^k = \delta^k , \tag{7a}$$

$$\delta^kD^{-k} = M^k , \tag{7b}$$

$$\delta^{-k}M^k = D^{-k} , \tag{7c}$$

where k is any positive integer.

[14] The traces of M^l and μM^l for $l = 1, 2, \ldots, 8$ are given in Table 3.

SOME EXAMPLES OF APPLICATION OF THE RULES

Example 1. Find the characteristic operator of a six-point, tangential (*i.e.*, having order of contact 1), fourth-degree, symmetrical, end-point formula, correct to fourth differences.

Solution:

$$G = M^5 H\,, \qquad \text{(Rule 1(a))}$$

$$H = J_0 + J_1 D + J_2 D^2\,, \qquad \text{(Rule 3(a))}$$

J_0, J_1, and J_2 are at most of span 1. (Rule 2)

J_0, J_1, and J_2 are Bessel operators. (Rule 4(a))

J_0 and J_2 contain only even powers of δ, while J_1 contains only odd powers of δ. (Rule 5)

It follows from the last three statements that we may write:

$$J_0 = a\mu\,, \quad J_1 = b\delta\,, \quad J_2 = c\mu\,.$$

Substituting these results in the expression for H gives:

$$H = a\mu + b\delta D + c\mu D^2\,. \qquad (8)$$

Substituting the symbolic expansions of μ and δ in powers of D from Table 2, we have

$$H = a + (\tfrac{1}{8}a + b + c)D^2 + (\tfrac{1}{384}a + \tfrac{1}{24}b + \tfrac{1}{8}c)D^4 + \ldots\,.$$

Again, from Table 2,

$$M^{-5} = 1 - \tfrac{5}{24}D^2 + \tfrac{3}{128}D^4 - \ldots\,.$$

By Rule 1(b),

$$\begin{aligned} a &= 1\,,\\ \tfrac{1}{8}a + b + c &= -\tfrac{5}{24}\,,\\ \tfrac{1}{384}a + \tfrac{1}{24}b + \tfrac{1}{8}c &= \tfrac{3}{128}\,. \end{aligned}$$

Solving these equations, we obtain:

$$a = 1\,, \quad b = -\tfrac{3}{4}\,, \quad c = \tfrac{5}{12}\,.$$

Substituting these values in expression (8) gives

$$G = M^5(\mu - \tfrac{3}{4}\delta D + \tfrac{5}{12}\mu D^2)\,, \qquad (9)$$

which is the characteristic operator of Shovelton's formula.

Example 2. Discuss the properties of Sprague's formula by means of its characteristic operator.

Solution: From Table 1, the characteristic operator is found to be:

$$G = M^5(25M - 24\mu + \tfrac{7}{4}\delta D)\,.$$

Taking $H = 25M - 24\mu + \frac{7}{4}\delta D$, and substituting, from Table 2, the symbolic expansions of M, μ, and δ in powers of D only, we obtain, symbolically,

$$H = 1 - \tfrac{5}{24}D^2 + \tfrac{3}{128}D^4 + \dots .$$

This agrees, as far as the term containing D^4, with the symbolic expansion of M^{-5}, also shown in Table 2. Thus, by Rule 1, the formula is correct to fourth differences.

$$H = 25\delta D^{-1} - 24\mu + \tfrac{7}{4}\delta D . \qquad \text{(formula } (7b))$$

The formula is of the fifth degree and has order of contact 2 (*i.e.*, is osculatory). (Rule 3(a))

It is an end-point formula. (Rule 4(a))

It is symmetrical. (Rule 5)

Since the greatest and least powers of E involved in H are $\frac{1}{2}$ and $-\frac{1}{2}$, the span of H is 1. Therefore, by Rule 2, the formula is a six-point formula.

By Rule 6(a), the trace of the formula is of the form

$$1 + k\delta^6 + \dots .$$

Since the formula is osculatory, the curve of interpolated values is, a fortiori, free from discontinuities whatever the given values may be. Therefore, by Rule 6(c), the span of the trace is at most 4. Therefore, the trace is 1, and, consequently, by Rule 8, the formula always reproduces the given values.

Example 3. Examine the properties of Jenkins' smoothing fifth-difference formula.

Solution:

$$G = M^4(1 - \tfrac{1}{6}\delta^2)\,, \qquad \text{(Table 1)}$$

$$H = 1 - \tfrac{1}{6}D^2 - \dots, \qquad \text{(Table 2)}$$

$$M^{-4} = 1 - \tfrac{1}{6}D^2 + \dots . \qquad \text{(Table 2)}$$

The formula is correct to third differences. (Rule 1)

$$H = J_0 = 1 - \tfrac{1}{6}\delta^2 . \qquad \text{(Rules 1(a) and 3(a))}$$

The formula is of the third degree and is osculatory. (Rule 3(a))

It is a six-point formula (since the span of H is 2). (Rule 2)

It is an end-point formula. (Rule 4(b))

It is symmetrical. (Rule 5)

$t(G)$ is of the form $1 + k\delta^4$. (Rules 6(a) and 6(c))

$t(G) = t(M^4 - \frac{1}{6}\delta^2 M^4) = t(M^4) - \frac{1}{6}\delta^2 t(M^4)$, (Rule 7)

$$k = -\tfrac{1}{6}(\tfrac{1}{6}) = -\tfrac{1}{36}. \qquad \text{(Table 3)}$$

Therefore,

$$t(G) = 1 - \tfrac{1}{36}\delta^4.$$

Example 4. Find the characteristic operator of a five-point, osculatory, symmetrical, midpoint formula of minimum degree, correct to second differences, which always reproduces the given values.

Solution: $G = M^3 H\,,$ (Rule 1(a))

$H = J_{-1}D^{-1} + J_{-2}D^{-2} + \ldots + J_{-q+2}D^{-q+2}\,,$ (Rule 3(a))

$J_{-1}, J_{-2}, \ldots, J_{-q+2}$ are all of span at most 2. (Rule 2)

$J_{-1}, J_{-2}, \ldots, J_{-q+2}$ are all Stirling operators. (Rule 4(a))

J_{-1}, J_{-3}, etc., contain only odd powers of δ, while J_{-2}, J_{-4}, etc., contain only even powers of δ. (Rule 5)

It follows from the last three statements that

$$J_{-1} = a\mu\delta\,, \quad J_{-2} = b + c\delta^2\,, \quad J_{-3} = d\mu\delta\,, \quad J_{-4} = e + f\delta^2\,, \text{ etc.}$$

Using formula (7*b*) and the symbolic expansions in Table 2 gives, symbolically,

$$J_{-1}D^{-1} = a\mu M = a + \tfrac{1}{6}aD^2 + \ldots$$
$$J_{-2}D^{-2} = bD^{-2} + cM^2 = bD^{-2} + c + \tfrac{1}{12}cD^2 + \ldots$$
$$J_{-3}D^{-3} = d\mu MD^{-2} = dD^{-2} + \tfrac{1}{6}d + \tfrac{1}{120}dD^2 + \ldots,$$

etc. Again, from Table 2,

$$M^{-3} = 1 - \tfrac{1}{8}D^2 + \ldots .$$

From these expressions and from Rules 1(b) and 3(c), we obtain the equations:

$$\begin{aligned} b + d + \ldots &= 0\,, \\ a + c + \tfrac{1}{6}d + \ldots &= 1\,, \\ \tfrac{1}{6}a + \tfrac{1}{12}c + \tfrac{1}{120}d + \ldots &= -\tfrac{1}{8}\,. \end{aligned}$$

Now, $q = 3$ would imply that $b = c = d = \ldots = 0$, and these equations could not be satisfied. Therefore q is at least 4.

By Rule 6, $t(G)$ is of the form $1 + k\delta^4$. Now,

$$\begin{aligned} G &= M^3(J_{-1}D^{-1} + J_{-2}D^{-2} + J_{-3}D^{-3} + \ldots) \\ &= a\mu M^4 + bM^3D^{-2} + cM^5 + d\mu M^4D^{-2} + \ldots . \end{aligned}$$

Therefore, by formula (7*c*),

$$G = a\mu M^4 + cM^5 + b\delta^{-2}M^5 + d\mu\delta^{-2}M^6 + \ldots .$$

By Rule 7[15] and Table 3,

$$k = \tfrac{1}{96}a + \tfrac{1}{384}c + \tfrac{1}{7680}d + \ldots ,$$

and, by Rule 8, $k = 0$. If $q = 4$, then $d = \ldots = 0$, and this, together with the three equations previously obtained, gives:

$$\begin{aligned} b &= 0 , \\ a + c &= 1 , \\ \tfrac{1}{6}a + \tfrac{1}{12}c &= -\tfrac{1}{8} , \\ \tfrac{1}{96}a + \tfrac{1}{384}c &= 0 . \end{aligned}$$

The last three equations are found to be inconsistent. If $q = 5$, we have:

$$\begin{aligned} b + d &= 0 , \\ a + c + \tfrac{1}{6}d &= 1 , \\ \tfrac{1}{6}a + \tfrac{1}{12}c + \tfrac{1}{120}d &= -\tfrac{1}{8} , \\ \tfrac{1}{96}a + \tfrac{1}{384}c + \tfrac{1}{7680}d &= 0 . \end{aligned}$$

Solving these equations, we have:

$$a = \tfrac{27}{10} , \quad b = -78 , \quad c = -\tfrac{147}{10} , \quad d = 78 ,$$

whence

$$G = M^3[78D^{-2}(\mu M - 1) - \tfrac{147}{10}M^2 + \tfrac{27}{10}\mu M] , \qquad (10)$$

the characteristic operator of Jenkins' "reproducing" fourth-difference formula.

Example 5. Find the characteristic operator of a two-point, tangential, symmetrical, end-point formula of minimum degree which always reproduces the given values.

Solution:

$G = H = J_{-3}D^{-3} + J_{-4}D^{-4} + \ldots + J_{-q-1}D^{-q-1}$,
(footnote 13, Rule 1(a), and Rule 3(a))

$J_{-3}, J_{-4}, \ldots , J_{-q-1}$ are all of span at most 2. (Rule 2)

$J_{-3}, J_{-4}, \ldots , J_{-q-1}$ are all Stirling operators. (Rule 4(b))

J_{-3}, J_{-5}, etc., contain only odd powers of δ, while J_{-4}, J_{-6}, etc., contain only even powers of δ. (Rule 5)

It follows from the last three statements that

$$J_{-3} = a\mu\delta , \quad J_{-4} = b + c\delta^2 , \quad J_{-5} = d\mu\delta ,$$

[15] See proof of Rule 7 in the appendix for discussion of this application.

etc. Substituting these expressions gives

$$G = a\mu\delta D^{-3} + bD^{-4} + c\delta^2 D^{-4} + d\mu\delta D^{-5} + \ldots$$
$$= a\mu\delta^{-2}M^3 + b\delta^{-4}M^4 + c\delta^{-2}M^4 + d\mu\delta^{-4}M^5 + \ldots \qquad \text{(formula (7}c\text{))}$$
$$t(G) = a\delta^{-2}(1 + \tfrac{1}{4}\delta^2) + b\delta^{-4}(1 + \tfrac{1}{6}\delta^2) + c\delta^{-2}(1 + \tfrac{1}{6}\delta^2)$$
$$+ d\delta^{-4}(1 + \tfrac{1}{3}\delta^2 + \tfrac{1}{48}\delta^4) + \ldots \qquad \text{(Rule 7 and Table 3)}$$
$$= (b + d + \ldots)\delta^{-4} + (a + \tfrac{1}{6}b + c + \tfrac{1}{3}d + \ldots)\delta^{-2}$$
$$+ (\tfrac{1}{4}a + \tfrac{1}{6}c + \tfrac{1}{48}d + \ldots) .$$

Rule 8 then gives the equations:

$$b + d + \ldots = 0 ,$$
$$a + \tfrac{1}{6}b + c + \tfrac{1}{3}d + \ldots = 0 ,$$
$$\tfrac{1}{4}a + \tfrac{1}{6}c + \tfrac{1}{48}d + \ldots = 1 .$$

It is obvious that q is at least 3. Taking $q = 3$ gives:

$$b = 0 ,$$
$$a + \tfrac{1}{6}b + c = 0 ,$$
$$\tfrac{1}{4}a + \tfrac{1}{6}c = 1 .$$

These equations can be solved, giving:

$$a = 12 , \quad b = 0 , \quad c = -12 .$$

Therefore

$$G = 12\mu\delta D^{-3} - 12\delta^2 D^{-4}$$
$$= 12\mu M D^{-2} - 12M^2 D^{-2} , \qquad \text{(formula (7}b\text{))}$$

or

$$G = M[12D^{-2}(\mu - M)] ,$$

which is the characteristic operator of formula (102) of *TASA* XLV, 264.

Even though we did not specify any order of differences to which the formula was to be correct, actually it turns out to be correct to "zeroth" differences—that is, it reproduces a constant function. This formula, of course, is of no practical value, and is included only for illustrative purposes.

Example 6. Find the characteristic operator of a four-point, tangential, symmetrical formula of minimum degree, correct to second differences, points of junction of the interpolating arcs being permitted to occur both at the arguments corresponding to the given values and at those midway between them.

Solution:

$$G = M^3 H , \qquad \text{(Rule 1(a))}$$
$$q \text{ is at least } 2 . \qquad \text{(Rule 3(b))}$$

Let us try $q = 2$. Then

$$H = J_0 . \qquad \text{(Rule 3(a))}$$

J_0 is of span at most 1. (Rule 2)

J_0 is a Stirling operator or a Bessel operator, or a sum of both. (Rule 4(c))

J_0 contains only even powers of δ. (Rule 5)

It follows from the last three statements that

$$H = J_0 = a + b\mu .$$

Therefore, symbolically,

$$H = (a + b) + \tfrac{1}{8}bD^2 + \ldots , \qquad \text{(Table 2)}$$

$$M^{-3} = 1 - \tfrac{1}{8}D^2 + \ldots . \qquad \text{(Table 2)}$$

Rule 1(b) gives the equations:

$$a + b = 1 ,$$

$$\tfrac{1}{8}b = -\tfrac{1}{8} .$$

Solving these equations, we have:

$$a = 2 , \quad b = -1 ,$$

whence

$$G = M^3(2 - \mu) . \qquad (11)$$

While this completes the solution, it may be pointed out that, in view of Rules 7(a) and 7(c), $t(G) = 1$, so that the formula always reproduces the given values.

SOME GENERAL OBSERVATIONS CONCERNING THE RULES

While the details vary in the different cases, a certain general procedure emerges from the examples just given. Rule 1(a) determines the general form of the characteristic operator. Rule 3(a) usually limits the number of terms in H. Rules 2, 4, and 5 limit the individual remaining terms in such a way that frequently the characteristic operator is now fully determined except for certain numerical coefficients. Rule 1(b) then furnishes some equations for determining these coefficients. In some cases, one or more further equations may be supplied by Rules 6, 7, and 8. Formulas (7) provide certain transformations which may be needed in order to apply Rule 7.

It is fairly clear from the rules and the examples that a continuously defined interpolation formula is not completely determined by the kind of properties thus far considered, unless its specifications include the requirement (explicit or implied) that it be a polynomial formula of the lowest degree consistent with the other stipulated properties. It is perhaps partly for this reason that in the earlier work on smooth-junction interpolation so much emphasis was placed on formulas of minimum degree. It is the impression of the present writers that this emphasis may have been somewhat exaggerated. There seems to be no compelling reason for using polynomial arcs in this connection rather than arcs of other curves. If they are employed, sometimes a polynomial of low degree is the most suitable; but there seems to us to be no general reason for considering that one formula must be superior to another merely because it incorporates polynomials of lower degree.[16] A possible alternative to the "minimum degree" criterion will be considered briefly in the next section of this paper.

The rules also bring out certain limitations as to the types of formula that are possible. Rule 3(b) shows that the degree of a polynomial interpolation formula cannot be less than the order of differences to which it is correct, and also that the degree must exceed the order of contact by at least one. The latter fact was previously pointed out by one of the authors (*TASA* XLV, 212), who also stated that a formula of minimum degree for its order of contact can never have the property of reproducing all sets of given values. This is true if one considers only end-point and midpoint formulas. However, a formula which has points of junction of its interpolating arcs both at the arguments corresponding to the given values and at those midway between them can be of minimum degree for its order of contact and also have the reproducing property. This is illustrated by formula (19) below, whose characteristic operator has already been obtained in Example 6.

It follows from Rule 6 that a symmetrical $(r + 1)$-point formula correct to rth differences necessarily has the property of always reproducing the given values if r is even, or if r is odd and the formula has order of contact at least zero. Similarly, a symmetrical $(r + 2)$-point formula correct to rth differences has the reproducing property if r is even and the formula has order of contact at least zero.

In connection with condition (a) of Rule 1, it will be recalled that in the discrete case a similar condition, $G = [m]^{r+1}H$, was obtained. However, in that case, H is a discrete operator—the same kind of operator as G itself. In the continuous case, G is a continuous operator, but H is not a continuous operator unless the formula has order of contact at least r. This

[16] A similar opinion was recently expressed by Dr. Eric Michalup in a paper, "Theorie und Anwendung der 'oskulatorischen' Interpolationsformeln," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, XLVII (1947), 359–407. (See pages 373–74.)

circumstance makes the continuous case somewhat more complicated. The nature of the difference may be illustrated by writing the discrete analogue of the general expression for the characteristic operator of a formula correct to rth differences, namely

$$[m]^{r+1}(J_0 + J_1\delta + J_2\delta^2 + \ldots + J_r\delta^r)\ ,$$

and noting that the expression in parentheses reduces to a single discrete operator. In the continuous case, no such reduction occurs.

It is interesting to observe that the characteristic operator for plain central-difference interpolation to rth differences (which it will be convenient to denote by C_r) is completely determined by Rules 1, 2, and 3(c). For, by Rule 1(a), $C_r = M^{r+1}H$; and the number of terms in the formula when expressed in linear compound form is $r + 1$. Therefore, by Rule 2, H is of span zero, which means that each of the J's in expression (6) reduces to a mere numerical coefficient. By Rule 3(c), the coefficients of negative powers of D vanish. Thus, condition (b) of Rule 1 can be satisfied only if H consists of the terms up to and including D^r in the symbolic expansion of M^{-r-1} in powers of D.

MINIMIZED DERIVATIVE FORMULAS

A reasonable condition to impose on an interpolation formula is to require, in addition to whatever other properties are specifically desired, that it be the smoothest possible formula of its type, according to some stated criterion of smoothness. In connection with discrete interpolation, several writers[17] have considered the problem of determining the smoothest possible formula of a given span correct to a stated order of differences, smoothness being judged by the size of the sum of the squares of some chosen order of differences of the interpolation coefficients. The analogous procedure in connection with continuously defined interpolation formulas would be to minimize the average value of the square of a specified derivative of the basic function. The writers have studied such formulas but have not completed their investigations along this line. In any event, a full discussion of the matter probably would call for an entire paper. However, certain limited observations may be of interest here.

We shall approach the problem by considering the symmetrical discrete interpolation formula of a given span, correct to a stated order of differences, for subdividing the interpolation interval into m equal parts, such that the sum of the squares of a specified order of differences of the sequence of interpolation coefficients is a minimum. We then seek the function defined by this sequence of coefficients as m approaches infinity.

[17] Beers, *RAIA* XXXIII, 245 and XXXIV, 14; *TASA* XLVIII, 53; Greville, *RAIA* XXXIV, 21; *TSA* I, 343; Michalup, *op. cit.;* Vaughan, *JIA* LXXII, 482.

The order of differences whose mean square is to be minimized does not have to be related in any particular way to the order of differences to which the formula is correct. As we are concerned with a discrete formula, the question of continuity of the basic function does not arise, and there seems to be no good reason in this case for imposing any analogous condition, such as the interlocking property considered by Mr. White (*TASA* XLIX, 337). We obtain, however, the rather interesting result that the limiting curve of interpolated values is always continuous, together with its derivatives up to and including the order one less than that of the differences whose mean square is minimized. Thus, a formula which minimizes the mean square of second derivatives must be tangential, and one which minimizes the mean square of third derivatives must be osculatory.

Thus, if we seek a four-point formula, correct to second differences, which minimizes the mean square of second derivatives, we find that Karup's formula is a unique solution. The corresponding formula which minimizes the mean square of third derivatives is the osculatory formula (105) of *TASA* XLV, 264, for which the characteristic operator is $M^3[120D^{-2}(\mu - M) - 9M]$. If we consider six-point formulas correct to third differences, the formula minimizing the mean square of second derivatives is the smoothing, third-degree, tangential formula[18]

$$\begin{aligned} v_{n+x} = {} & xu_{n+1} + \tfrac{1}{6}x(x^2-1)\delta^2 u_{n+1} + \tfrac{1}{108}x^2(5x-12)\delta^4 u_{n+1} \\ & + yu_n + \tfrac{1}{6}y(y^2-1)\delta^2 u_n + \tfrac{1}{108}y^2(5y-12)\delta^4 u_n\,, \end{aligned}$$

where $y = 1 - x$, for which the characteristic operator is $M^4(1 + \frac{5}{18}\delta^2 - \frac{4}{9}\mu\delta D)$. If the mean square of third derivatives is minimized, the result is Vaughan's smoothing, osculatory, fourth-degree formula "C" (*JIA* LXXII, 491),

$$\begin{aligned} v_{n+x} = {} & xu_{n+1} + \tfrac{1}{6}x(x^2-1)\delta^2 u_{n+1} + \tfrac{1}{36}x^3(3x-5)\delta^4 u_{n+1} \\ & + yu_n + \tfrac{1}{6}y(y^2-1)\delta^2 u_n + \tfrac{1}{36}y^3(3y-5)\delta^4 u_n\,. \end{aligned}$$

This formula was originally given as one of three, the other two being denominated "A" and "B." It may be of interest to mention in passing that Jenkins' reproducing fifth-difference formula, formula "A," Jenkins' smoothing fifth-difference formula, formula "B," and formula "C," in that order, form an arithmetic progression, the common difference of the characteristic operators being $M^4(2\mu M - 2 - \frac{1}{3}\delta^2)$, which corresponds to the "Everett" expression

$$\tfrac{1}{72}x^3(3x-4)\delta^4 u_{n+1} + \tfrac{1}{72}y^3(3y-4)\delta^4 u_n\,.$$

[18] This is formula (73) of *TASA* XLV, 261, with $a_{04} = -7/108$.

All five formulas are osculatory, correct to third differences, and of the fourth degree (with the exception of Jenkins' smoothing formula), and are particular cases of formula (84) of *TASA* XLV, 262, being obtained by assigning to the parameter a_{04} in that formula the values 0, $-1/72$, $-1/36$, $-1/24$, and $-1/18$. Formula (111) of *TASA* XLV, 264, is the seventh term of the same progression. As the trace of formula (84) is $1 + a_{04}\delta^4$, this also forms an arithmetic progression for the five formulas.

It will also be noted that, even though no restriction is placed on the form of the basic function, yet, for the minimized derivative formulas considered, it has always turned out to be composed of polynomial arcs. This is due to the particular nature of the minimizing conditions and of the fidelity conditions used. The fidelity condition imposed is correctness to a stated order of differences, or, in other words, the requirement that *polynomials* of a certain degree be reproduced. With such a fidelity condition, the stipulation that the mean square of an order of derivatives be a minimum generally leads to the result that derivatives of some higher order vanish. If, on the contrary, we had required reproduction of all finite Fourier series of a stated order, the basic function would necessarily have involved trigonometric functions.

Though it is not the present object to deal fully with minimizing formulas, it may be said that investigation along the lines sketched has shown that even in the discrete case the successive interpolation coefficients lie on polynomial arcs. In every such case, these arcs may be regarded as constituting the basic function of a certain continuously defined interpolation formula, of which the discrete formula in question is merely a particular application. It has been said in connection with certain published discrete formulas that the mathematical form is indeterminate, but this is true only in the sense that most published minimizing formulas have been ascertained only for the particular case of subdivision in fives, and only the numerical values for that one case were sought. If we ascertain a formula for the general case of subdivision by m, a polynomial of specific degree does emerge as the solution. For example, for a six-point formula correct to fourth differences or a four-point formula correct to second differences, it can be shown that to minimize the mean square of nth differences requires the interpolation formula to be of degree $2n - 1$ (for $n \geq 3$ and $n \geq 2$, in the respective cases).

It has been found that, if we increase the order of finite differences minimized, the effect on the basic function becomes less and less until a definite ascertainable limit is reached. It has also been found that minimizing differences of shorter interval affects the basic function in somewhat the same way as minimizing a higher order of differences. We come to the limiting case of a difference of short interval, of course, in minimizing the mean square of an order of derivatives. For example, Karup's formula (which, as we have seen, minimizes the mean square of *second* derivatives) happens to be very close to the formula of *RAIA* XXXIV, 25, which minimizes the mean square of *third* differences.

As an illustration of some of these points, the general formula (four-point, correct to second differences) to minimize the mean square of second differences at an interval of $1/m$ is, in Everett form:

$$\begin{aligned} v_{n+x} = x u_{n+1} - x(1-x)[(7m^2+m)x+m+7](14m^2+6m \\ +28)^{-1}\delta^2 u_{n+1} + y u_n - y(1-y)[(7m^2+m)y \\ +m+7](14m^2+6m+28)^{-1}\delta^2 u_n\,. \end{aligned} \tag{12}$$

As m approaches ∞, this becomes Karup's formula. When $m = 1$, it is the ordinary third-difference formula.

SPECIAL OPERATORS USED IN CONVERSION BETWEEN INTERPOLATION FORMULAS AND THEIR CHARACTERISTIC OPERATORS

The procedure we shall use in passing from the characteristic operator to the actual formula (and vice versa) is based on the fact that every endpoint formula can be expressed in Everett form, while every midpoint formula can be expressed in Steffensen form, which latter may be illustrated by the case of Jenkins' reproducing fourth-difference formula,

$$\begin{aligned} v_{n-1/2+x} = u_n + \tfrac{1}{2}(x^2-\tfrac{1}{4})\delta u_{n+1/2} + \tfrac{1}{40}x^3(x-\tfrac{1}{2})(13x-18)\delta^3 u_{n+1/2} \\ - \tfrac{1}{2}(y^2-\tfrac{1}{4})\delta u_{n-1/2} - \tfrac{1}{40}y^3(y-\tfrac{1}{2})(13y-18)\delta^3 u_{n-1/2}\,. \end{aligned} \tag{13}$$

It is assumed that this formula is used for interpolation in the interval between arguments $n - \frac{1}{2}$ and $n + \frac{1}{2}$, so that $0 \le x \le 1$. The great majority of formulas we are interested in are of one of these two types. We shall develop special methods for dealing with those which are not.

Any Everett-type formula can be regarded as made up of a number of components of the form

$$k(x^i\delta^{2j}u_{n+1} + y^i\delta^{2j}u_n)\,, \tag{14}$$

and, similarly, any Steffensen-type formula can be considered as made up of components of the form

$$k(x^i\delta^{2j-1}u_{n+1/2} - y^i\delta^{2j-1}u_{n-1/2})\,. \tag{15}$$

If we define Q_i as the characteristic operator of the expression

$$x^i u_{n+1} + y^i u_n\,,$$

and T_i as the characteristic operator of the expression

$$x^i u_{n+1/2} - y^i u_{n-1/2}\,,$$

then the expression (14) corresponds to $k\delta^{2j}Q_i$ and the expression (15) corresponds to $k\delta^{2j-1}T_i$. Even the leading term u_n of the Steffensen form can be written as

$$\delta^{-1}u_{n+1/2} - \delta^{-1}u_{n-1/2}\,,$$

corresponding to the operator $\delta^{-1}T_0$. For example, the characteristic operator of formula (13) can immediately be written down as

$$G = \delta^{-1}T_0 + \delta(\tfrac{1}{2}T_2 - \tfrac{1}{8}T_0) + \delta^3(\tfrac{13}{40}T_5 - \tfrac{49}{80}T_4 + \tfrac{9}{40}T_3)\,. \qquad (16)$$

On the other hand, formula (13) could readily be written down from the expression (16). Thus, an end-point or midpoint interpolation formula is practically interchangeable with the expression for its characteristic operator in terms of the Q's or T's. In this way, we shall find these special operators useful as steppingstones between interpolation formulas and their characteristic operators.

Tables 4 and 5 following the main part of the paper are designed to facilitate conversion of characteristic operators, in both directions, between the expressions in terms of Q's and T's and the usual ones in terms of M, D, δ, and μ. For example, we could substitute for the operators in expression (16) their equivalents from Table 5 and obtain, after some reduction, the expression (10) previously derived in Example 4. Conversely, we could eliminate D from the operator (10) by means of formula (7c) and multiply out, and finally replace each operator of the form M^i or μM^i by the equivalent expression from Table 4. After some simplification, we would obtain the expression (16).[19]

In actual practice, the algebra can usually be somewhat shortened by making use of the fact that most formulas agree, in their first few terms, with the standard Everett or Steffensen formula. Plain central-difference interpolation to rth differences, represented by the characteristic operator C_r, corresponds to Everett's or Steffensen's formula (depending on whether r is odd or even) terminating with $(r-1)$th differences. For example, C_5 corresponds to Everett's formula as far as fourth differences, and C_4 corresponds to Steffensen's formula taken to third differences. Expressions for the C's are given (in slightly different forms) in Tables 1

[19] It may be remarked that the Everett and Steffensen forms have been chosen for this purpose, rather than other possible forms (such as Stirling and Bessel), not only because of their well-known computational advantages, but also because the characteristic operators of their components can be more simply expressed in terms of M, δ, and μ than those arising from the other forms.

and 5. Examples and suggested rules of procedure are given in the next two sections.

In order to deal with the relatively rare case of a mixed end-point-mid-point formula, we shall need two more special types of operators. We shall denote by V_i the characteristic operator of an adjustment term to an Everett-type formula, of the form

$$|x - \tfrac{1}{2}|^i u_p , \tag{17}$$

where u_p denotes the nearer of u_n and u_{n+1}. Similarly, we shall denote by W_i the characteristic operator of an adjustment term to a Steffensen-type formula, of the form

$$(x - \tfrac{1}{2})|x - \tfrac{1}{2}|^{i-1} u_q , \tag{18}$$

where u_q denotes the nearer of $u_{n-1/2}$ and $u_{n+1/2}$. In Part III of Table 5, certain differences of the V's and W's are expressed in terms of M, D, and μ.

EXAMPLES OF CONVERSION BETWEEN INTERPOLATION FORMULAS AND THEIR CHARACTERISTIC OPERATORS

We shall give some examples of conversion between interpolation formulas and their characteristic operators before suggesting rules of procedure to be followed, in the belief that the latter will be more comprehensible after the examples have been read.

Example 7. Obtain the characteristic operator of formula (13).

Solution: The terms of this formula as far as first differences agree with the corresponding terms of the standard Steffensen formula. The latter formula taken to first differences is correct to second differences. Therefore, the characteristic operator for these terms is C_2.

By the definition of the T's the characteristic operator for the third-difference terms is

$$\delta^3(\tfrac{13}{40}T_5 - \tfrac{49}{80}T_4 + \tfrac{9}{40}T_3) .$$

Combining these two results, we have:

$$G = C_2 + \tfrac{13}{40}\delta^3 T_5 - \tfrac{49}{80}\delta^3 T_4 + \tfrac{9}{40}\delta^3 T_3 .$$

Substitution from Table 5 of the equivalents for the various operators gives

$$\begin{aligned} G &= M^3 - \tfrac{1}{8}\delta^2 M + \tfrac{13}{40}\delta^{-2}(240\mu M^6 - 240M^5 - 40\delta^2 M^3 - 2\delta^4 M) \\ &\quad - \tfrac{49}{80}(24M^5 - 24M^3 - 2\delta^2 M) + \tfrac{9}{40}(12\mu M^4 - 12M^3 - 2\delta^2 M) \\ &= 78\mu\delta^{-2}M^6 - 78\delta^{-2}M^5 - \tfrac{147}{10}M^5 + \tfrac{27}{10}\mu M^4 . \end{aligned}$$

Using formula (7c), we obtain at once expression (10).

Example 8. Deduce Shovelton's formula from its characteristic operator.

Solution: As this end-point formula is correct to fourth differences, it will agree with the standard Everett formula as far as the second-difference terms. (If it agreed as far as the fourth-difference terms, it would be correct to fifth differences.) This much of the standard formula is correct to third differences, and the corresponding characteristic operator is C_3. Subtracting from the expression (9) the expression for C_3 from Table 1 and applying formula (7*a*) gives

$$\begin{aligned} G - C_3 &= M^4(\mu M - 1 - \tfrac{3}{4}\delta^2 + \tfrac{5}{12}\mu\delta D + \tfrac{1}{6}D^2) \\ &= \mu M^5 - M^4 - \tfrac{3}{4}\delta^2 M^4 + \tfrac{5}{12}\mu\delta^2 M^3 + \tfrac{1}{6}\delta^2 M^2 . \end{aligned}$$

Substituting the equivalent expressions from Table 4, we have

$$G - C_3 = \delta^4(\tfrac{1}{48}Q_4 - \tfrac{1}{8}Q_3 + \tfrac{5}{48}Q_2) = \tfrac{1}{48}\delta^4(Q_4 - 6Q_3 + 5Q_2) .$$

This corresponds to the Everett terms

$$\tfrac{1}{48}x^2(x-1)(x-5)\delta^4 u_{n+1} + \tfrac{1}{48}y^2(y-1)(y-5)\delta^4 u_n .$$

Adding these to the standard terms gives the Everett form of Shovelton's formula.

Example 9. Obtain the characteristic operator of the formula:

$$v_{n+x} = xu_{n+1} - \tfrac{1}{4}x^2\delta^2 u_{n+1} + yu_n - \tfrac{1}{4}y^2\delta^2 u_n + (x - \tfrac{1}{2})^2\delta^2 u_p , \quad (19)$$

where u_p denotes the nearer of u_n and u_{n+1}.

Solution: The first-degree terms agree with the standard Everett formula, and are correct to first differences. Thus, by the definitions of the Q's and V's, the characteristic operator is

$$G = C_1 - \tfrac{1}{4}\delta^2 Q_2 + \delta^2 V_2 .$$

Substituting the expressions for these operators from Table 5, we obtain the expression (11) of Example 6.

Example 10. Obtain formula (19) in the form of a midpoint formula with an adjustment term.

Solution: Examining in the light of Rule 4(a) the operator (11) of Example 6, which corresponds to this formula, we observe that the term $2M^3$ corresponds to a "midpoint" expression, while $-\mu M^3$ represents an "end-point" expression. We should like, therefore, to find an adjustment term whose characteristic operator consists of $-\mu M^3$ plus terms representing midpoint expressions. The formula (19) could then be written as a midpoint formula plus this adjustment term. With this in mind, we look in the part of Table 5 which gives expressions for certain differences of the W's, since each of these expressions consists of a single end-point term combined with several midpoint terms. We find that the expression for $\delta^3 W_2$ contains the term $4\mu M^3$. We write, therefore,

$$G + \tfrac{1}{4}\delta^3 W_2 = M^3(1 - \tfrac{1}{8}D^2)\,.$$

Since formula (19) is correct to second differences, we now subtract the expression for C_2 from Table 1 and obtain

$$G + \tfrac{1}{4}\delta^3 W_2 - C_2 = 0\,,$$

or,

$$G = C_2 - \tfrac{1}{4}\delta^3 W_2\,.$$

In view of the definitions of C_2 and W_2, formula (19) can, therefore, be written as

$$\begin{aligned} v_{n-1/2+x} = u_n &+ \tfrac{1}{2}(x^2 - \tfrac{1}{4})\delta u_{n+1/2} - \tfrac{1}{2}(y^2 - \tfrac{1}{4})\delta u_{n-1/2} \\ &- \tfrac{1}{4}(x - \tfrac{1}{2})\,|x - \tfrac{1}{2}|\,\delta^3 u_q\,. \end{aligned}$$

Example 11. Obtain, in the most convenient form, the interpolation formula whose characteristic operator is $M^4(\frac{11}{3} - \frac{8}{3}\mu + \frac{1}{6}\delta^2)$.

Solution: With the aid of Rule 1 and Table 2, it is found that the formula is correct to third differences. By Rule 4(b), this operator contains two end-point terms and one midpoint term. This suggests that it may be easier to express the corresponding interpolation formula as an end-point formula plus an adjustment term. We therefore look in Table 5 among the expressions for certain differences of the V's, each of which consists of one midpoint term together with several end-point terms. In this case, we see that we can eliminate the term $-\frac{8}{3}\mu M^4$ by adding $\frac{2}{9}\delta^4 V_3$. Therefore,

$$G + \tfrac{2}{9}\delta^4 V_3 = M^4(1 + \tfrac{1}{6}\delta^2 - \tfrac{1}{3}D^2)\,.$$

Now subtracting the expression for C_3 from Table 1, we obtain

$$G + \tfrac{2}{9}\delta^4 V_3 - C_3 = \tfrac{1}{6}M^4(\delta^2 - D^2)\,.$$

Using formula (7*a*), this reduces to

$$\tfrac{1}{6}\delta^2(M^4 - M^2)\,,$$

and substituting the equivalent expressions from Table 4 gives

$$G + \tfrac{2}{9}\delta^4 V_3 - C_3 = \tfrac{1}{36}\delta^4 Q_3\,,$$

or

$$G = C_3 + \tfrac{1}{36}\delta^4 Q_3 - \tfrac{2}{9}\delta^4 V_3 .$$

By the definitions of C_3, Q_3, and V_3, the formula is, therefore,

$$\begin{aligned} v_{n+x} = {} & x u_{n+1} + \tfrac{1}{6}x(x^2-1)\delta^2 u_{n+1} + \tfrac{1}{36}x^3\delta^4 u_{n+1} + y u_n \\ & + \tfrac{1}{6}y(y^2-1)\delta^2 u_n + \tfrac{1}{36}y^3\delta^4 u_n - \tfrac{2}{9}\left|x-\tfrac{1}{2}\right|^3 \delta^4 u_p . \end{aligned} \tag{20}$$

SUGGESTED PROCEDURE IN CONVERSION BETWEEN INTERPOLATION FORMULAS AND THEIR CHARACTERISTIC OPERATORS

I. *Conversion from Operator to Formula:*

1. (a) If the formula is an end-point formula correct to rth differences, subtract from the given operator the expression in Table 1 for C_r if r is odd, or C_{r-1} if r is even.

(b) If the formula is a midpoint formula correct to rth differences, subtract from the given operator the expression in Table 1 for C_r if r is even, or C_{r-1} if r is odd.

(c) If the formula is a mixed end-point-midpoint formula, pass at once to Step 5 below.

2. Multiply out the remainder, and eliminate D's by means of formulas (7*a*) and (7*c*).

3. In each term, substitute the equivalent expression from Table 4, and simplify.

4. Write the Everett or Steffensen expression corresponding to the operator subtracted in Step 1(a) or 1(b) and add further terms corresponding to the operator resulting from Step 3, in accordance with the definitions of the Q's and T's.

5. (Applies only to mixed end-point-midpoint formulas.)

(a) Ascertain which are the end-point and midpoint terms in the given operator. End-point terms contain a Stirling operator combined with an even power of M or a Bessel operator combined with an odd power of M; midpoint terms contain a Stirling operator combined with an odd power of M or a Bessel operator combined with an even power of M.[20]

(b) Eliminate the midpoint terms by adding or subtracting appropriate multiples of one or more of the differences of the V operators in Part III of Table 5; or eliminate the end-point terms by subtracting appropriate multiples of one or more of the differences of the W operators found in the same place.

[20] See proof of Rule 4 in the appendix.

(c) Follow Steps 1–4 to find the formula corresponding to the remainder after Step (b), and add the adjustment term or terms corresponding to the operator or operators subtracted in Step (b).

II. *Conversion from Formula to Operator:* (It is assumed that the formula is given in Everett or Steffensen form, possibly with an adjustment term of the form (17) or (18).)

1. If the formula agrees, up to and including the rth-difference terms, with the standard Everett or Steffensen formula, the characteristic operator of the agreeing terms is C_{r+1}.
2. Write the characteristic operator of the remaining terms of the formula in terms of the Q's, T's, V's, and W's by means of the definitions of these operators, and add it to C_{r+1}.
3. In each term of the resulting operator substitute the equivalent expression from Table 5, and simplify.
4. If desired, use formulas (7*a*) and/or (7*c*) to express the result in factored form.

COMPUTATION OF INTERPOLATION COEFFICIENTS FROM THE CHARACTERISTIC OPERATOR

If a formula whose characteristic operator has been deduced by means of the rules previously given is to be used primarily for numerical calculations, it may be preferred to deduce the interpolation coefficients directly from the characteristic operator, without actually obtaining the formula in algebraic form. These coefficients are, as previously explained, appropriate values of the basic function. In many instances, these values can be easily computed with the help of Table 6, which gives, for arguments at intervals of 0.1, values of the basic functions of a number of operators of the form $\delta^{2k}M^l$ or $\mu\delta^{2k}M^l$, which occur frequently as terms in the characteristic operators of known interpolation formulas. Upon multiplying out and applying formula (7*a*), many characteristic operators can be expressed entirely in terms of the operators which appear in the headings of Table 6. Values of the basic function of the total operator are obtained, of course, by adding together the corresponding values of the basic functions of the individual terms. An example of the procedure is given below.

As all the basic functions tabulated in Table 6 are composed of polynomial arcs, values for other arguments than those shown can be computed exactly by ordinary finite-difference interpolation to the appropriate order of differences. In each case, the degree of the polynomial arcs is one less than the exponent[21] of M. In making such an interpolation, care

[21] This is a consequence of footnote 13, formula (7*b*), and Rule 3(a).

must be exercised, of course, to use tabular values all of which lie on the same arc. In this connection, it may be pointed out that the basic functions of "end-point" terms have their points of junction at integral arguments, while those of "midpoint" terms have them at arguments which are odd multiples of $\frac{1}{2}$. As previously stated, end-point terms are those of the form $\delta^{2k}M^{2l}$ or $\mu\delta^{2k}M^{2l-1}$, while midpoint terms are of the form[22] $\delta^{2k}M^{2l-1}$ or $\mu\delta^{2k}M^{2l}$.

A general method for determining other basic functions directly from their characteristic operators is given in the appendix, page 474.

Example 12. Given $u_0 = 43$, $u_1 = 48$, $u_2 = 51$, $u_3 = 49$, $u_4 = 50$, $u_5 = 54$, compute $u_{2.3}$ by Shovelton's formula, using only Table 6 and the characteristic operator (9).

Solution: Denoting the basic function of the formula by $L(x)$, we have, by equation (1),

$$v_{2.3} = L(2.3)u_0 + L(1.3)u_1 + L(0.3)u_2 + L(-0.7)u_3 + L(-1.7)u_4 + L(-2.7)u_5 \,.$$

Multiplying out the expression (9) and using formula (7*a*) gives

$$G = \mu M^5 - \tfrac{3}{4}\delta^2 M^4 + \tfrac{5}{12}\mu\delta^2 M^3 \,.$$

The computation of the basic function for the six arguments required is shown in the accompanying table. Substituting the given values and the

Argument:	2.3	1.3	0.3	−0.7	−1.7	−2.7
Basic function of:						
μM^5	0.0050	0.1540	0.4362	0.3458	0.0588	0.0002
$-\frac{3}{4}\delta^2 M^4$	−0.0429	−0.3569	0.5813	0.0763	−0.2544	−0.0034
$\frac{5}{12}\mu\delta^2 M^3$	0.0510	0.0969	−0.1896	−0.1062	0.1385	0.0094
G	0.0131	−0.1060	0.8279	0.3159	−0.0571	0.0062

coefficients from the table in the above expression for $v_{2.3}$ gives 50.66 as the interpolated value.

[22] For proof of both statements, see the proof of Rule 4 in the appendix.

TABLE 1

CHARACTERISTIC OPERATORS OF CERTAIN PUBLISHED INTERPOLATION FORMULAS

Name or Originator of Formula*	Where Published	Year of Publication	Characteristic Operator
I. Reproducing Formulas			
Plain central-difference interpolation:			
to 1st differences			$C_1 = M^2$
to 2nd differences			$C_2 = M^3(1 - \frac{1}{8}D^2)$
to 3rd differences			$C_3 = M^4(1 - \frac{1}{6}D^2)$
to 4th differences			$C_4 = M^5(1 - \frac{5}{24}D^2 + \frac{3}{128}D^4)$
to 5th differences			$C_5 = M^6(1 - \frac{1}{4}D^2 + \frac{1}{30}D^4)$
to 6th differences			$C_6 = M^7(1 - \frac{7}{24}D^2 + \frac{259}{5760}D^4 - \frac{5}{1024}D^6)$
to 7th differences			$C_7 = M^8(1 - \frac{1}{3}D^2 + \frac{7}{120}D^4 - \frac{1}{140}D^6)$
Sprague	*JIA* XXII, 280	1880	$M^5(25M - 24\mu + \frac{7}{4}\delta D)$
Karup	*TICA* 2nd, 83	1898	$M^3(3M - 2\mu)$
Henderson	*TASA* IX, 217	1906	$M^4(1 - \frac{1}{2}\delta^2 + \frac{1}{3}\mu\delta D)$
Curve of Sines	See footnote†	1907	$M^5(P - \frac{1}{2}\pi^2\mu + \frac{1}{8}\pi^2\delta D)$†
Buchanan	*JIA* XLII, 374	1908	$M^4(30M^2 - 28\mu M - 1 + 2\delta^2)$
Shovelton	*JIA* XLVII, 287	1913	$M^5(\mu - \frac{3}{4}\delta D + \frac{5}{12}\mu D^2)$
Henderson	*TASA* XXII, 191	1921	$M^4(1 - \frac{1}{6}\delta^2 + \frac{1}{36}\delta^2 D^2)$
Reilly ($h = 2$, $k = 3$)	*RAIA* XIII, 21	1924	$M^5[2940D^{-2}(\mu - M) - 290M + 46\mu]$

* Notations in parentheses following the name of an author employ the author's own symbolism to indicate which formula is intended where several have been published on the same page, as in a table of formulas.

† Published in 1907 by Dr. John Tatham in the *Supplement to the Sixty-Fifth Annual Report of the Registrar-General of Births, Deaths, and Marriages in England and Wales, 1891–1900, Part I*. However, the formula was also used in the construction of life tables published in 1895 in the *Supplement to the Fifty-Fifth Annual Report of the Registrar-General of Births, Deaths, and Marriages in England and Wales, 1881–1890, Part I*. P is a continuous operator of span 1 with the basic function

$$\frac{1}{48}\,\frac{d^5}{dt^5}\left[(t^2 - \tfrac{1}{4})(t^2 - \tfrac{9}{4})\sin \pi t\right]$$

for $-\frac{1}{2} < t < \frac{1}{2}$.

TABLE 1—*Continued*

Name or Originator of Formula*	Where Published	Year of Publication	Characteristic Operator
Reilly ($h = 2$, $k = 4$)	*RAIA* XIV, 19	1925	$M^5\{400M - 1260D^{-2}[13\mu - 59M + 552D^{-2}(\mu - M)]\}$
Jenkins, 5th difference	*RAIA* XV, 89	1926	$M^4(-4\mu M + 5 + \frac{1}{2}\delta^2)$
Jenkins, 4th difference	*TASA* XXXI, 24	1930	$M^3[78D^{-2}(\mu M - 1) - \frac{147}{10}M^2 + \frac{27}{10}\mu M]$
Jenkins, 2nd difference	*TASA* XXXI, 31	1930	$M[12D^{-2}(\mu M - 1) - M^2]$
Greville (67, $a_{13} = -\frac{1}{4}$)	*TASA* XLV, 261	1944	$M^3(-\frac{3}{2}\mu M + \frac{5}{2} + \frac{1}{8}\delta^2)$
Greville (102)	*TASA* XLV, 264	1944	$M[12D^{-2}(\mu - M)]$
Greville (103)	*TASA* XLV, 264	1944	$M\{60D^{-2}[M - 12D^{-2}(\mu - M)]\}$
Greville (105)	*TASA* XLV, 264	1944	$M^3[120D^{-2}(\mu - M) - 9M]$
Michalup (MMT)	*MVSV*‡ XLVII, 383	1947	$M^5(\frac{725}{99}M - \frac{626}{99}\mu - \frac{7}{396}\delta D + \frac{175}{594}\mu D^2)$
Michalup (MMA)	*MVSV*‡ XLVII, 384	1947	$M^5(\frac{75}{4}M - \frac{71}{4}\mu + \frac{6}{5}\delta D - \frac{7}{240}\mu D^2 + \frac{1}{160}\delta D^3)$
Michalup (MMC)	*MVSV*‡ XLVII, 384	1947	$M^5(\frac{175}{8}M - \frac{167}{8}\mu + \frac{61}{40}\delta D - \frac{17}{480}\mu D^2 + \frac{7}{960}\delta D^3)$
Vaughan (12)	*TSA* VI, 435	1954	$M^3(14m^2 + 6m + 28)^{-1}[6m(7m + 1)M - 28(m^2 - 1)\mu - (m + 7)\delta D]$
Vaughan (19)	*TSA* VI, 438	1954	$M^3(2 - \mu)$
Greville (20)	*TSA* VI, 440	1954	$M^4(\frac{11}{3} - \frac{8}{3}\mu + \frac{1}{6}\delta^2)$
	II. Smoothing Formulas		
Jenkins, 3rd difference	*TASA* XXVIII, 200	1927	μM^3
Jenkins, 5th difference	*TASA* XXVIII, 202	1927	$M^4(1 - \frac{1}{6}\delta^2)$

* Notations in parentheses following the name of an author employ the author's own symbolism to indicate which formula is intended where several have been published on the same page, as in a table of formulas.

‡ *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker.*

TABLE 1—*Continued*

Name or Originator of Formula*	Where Published	Year of Publication	Characteristic Operator
Jenkins, 4th difference	*TASA* XXXI, 12	1930	$M^4(\frac{7}{2}M - \frac{5}{2}\mu)$
Jenkins, 2nd difference	*TASA* XXXI, 30	1930	M^3
Reid and Dow	*TFA* XIV, 189	1933	$M^4(1 - \frac{1}{6}\delta^2 + b\delta^4)$
Greville (69)	*TASA* XLV, 261	1944	$M^4(\mu - \frac{7}{24}\delta D)$
Greville (70)	*TASA* XLV, 261	1944	$M^3(1 - \frac{1}{8}\delta^2)$
Greville (73, $a_{04} = -\frac{7}{108}$)	*TASA* XLV, 261	1944	$M^4(1 + \frac{5}{18}\delta^2 - \frac{4}{9}\mu\delta D)$
Greville (74)	*TASA* XLV, 261	1944	$M^3(\mu - \frac{1}{4}\mu\delta^2)$
Greville (84)	*TASA* XLV, 262	1944	$M^4[-4\mu M + 5 + \frac{1}{2}\delta^2 - 72a_{04}(2\mu M - 2 - \frac{1}{3}\delta^2)]$
Greville (104)	*TASA* XLV, 264	1944	$M^2[12D^{-2}(\mu - M)]$
Greville (106)	*TASA* XLV, 264	1944	μM^4
Greville (107)	*TASA* XLV, 264	1944	$M^3(1 + \frac{3}{8}\delta^2)$
Greville (108)	*TASA* XLV, 264	1944	$M^3(\frac{13}{2}\mu M - \frac{11}{2} - \frac{29}{24}\delta^2)$
Greville (109)	*TASA* XLV, 264	1944	$M^3(\mu + \frac{2}{3}\mu\delta^2)$
Greville (110)	*TASA* XLV, 264	1944	$M^4(1 + \frac{1}{2}\delta^2 - \frac{2}{3}\mu\delta D)$
Greville (111)	*TASA* XLV, 264	1944	$M^4(8\mu M - 7 - \frac{3}{2}\delta^2)$
Vaughan (A)	*JIA* LXXII, 491	1946	$M^4(-2\mu M + 3 + \frac{1}{6}\delta^2)$
Vaughan (B)	*JIA* LXXII, 491	1946	$M^4(2\mu M - 1 - \frac{1}{2}\delta^2)$
Vaughan (C)	*JIA* LXXII, 491	1946	$M^4(4\mu M - 3 - \frac{5}{6}\delta^2)$
Michalup (M3)	*MVSV*‡ XLVII, 377	1947	$M^5(4M - 3\mu)$
Michalup (M4)	*MVSV*‡ XLVII, 379	1947	$M^4[45D^{-2}(\mu M - 1) - \frac{13}{2}M^2]$
Michalup (M2)	*MVSV*‡ XLVII, 380	1947	$M^5(\mu - \frac{1}{3}\delta D)$
Michalup (M4A)	*MVSV*‡ XLVII, 381	1947	$M^6(1 - \frac{1}{4}D^2)$
Michalup (3M4)	*MVSV*‡ XLVII, 402	1947	$M^3[12D^{-2}(\mu - M)]$

* Notations in parentheses following the name of an author employ the author's own symbolism to indicate which formula is intended where several have been published on the same page, as in a table of formulas.

‡ *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker.*

TABLE 2

SYMBOLIC EXPANSIONS OF CERTAIN OPERATORS* IN POWERS OF D

$M = 1 + \frac{1}{24}D^2 + \frac{1}{1920}D^4 + \frac{1}{322,560}D^6 + \ldots$	$M^{-1} = 1 - \ldots$
$\mu M = 1 + \frac{1}{6}D^2 + \frac{1}{120}D^4 + \frac{1}{5040}D^6 + \ldots$	$M^{-2} = 1 - \ldots$
$M^2 = 1 + \frac{1}{12}D^2 + \frac{1}{360}D^4 + \frac{1}{20,160}D^6 + \ldots$	$M^{-3} = 1 - \frac{1}{8}D^2 + \ldots$
$\mu = 1 + \frac{1}{8}D^2 + \frac{1}{384}D^4 + \frac{1}{46,080}D^6 + \ldots$	$M^{-4} = 1 - \frac{1}{6}D^2 + \ldots$
$\delta = D + \frac{1}{24}D^3 + \frac{1}{1920}D^5 + \frac{1}{322,560}D^7 + \ldots$	$M^{-5} = 1 - \frac{5}{24}D^2 + \frac{3}{128}D^4 - \ldots$
$\mu\delta = D + \frac{1}{6}D^3 + \frac{1}{120}D^5 + \frac{1}{5040}D^7 + \ldots$	$M^{-6} = 1 - \frac{1}{4}D^2 + \frac{1}{30}D^4 - \ldots$
$\delta^2 = D^2 + \frac{1}{12}D^4 + \frac{1}{360}D^6 + \ldots$	$M^{-7} = 1 - \frac{7}{24}D^2 + \frac{259}{5760}D^4 - \frac{5}{1024}D^6 + \ldots$
$\mu\delta^2 = D^2 + \frac{5}{24}D^4 + \frac{91}{5760}D^6 + \ldots$	$M^{-8} = 1 - \frac{1}{3}D^2 + \frac{7}{120}D^4 - \frac{1}{140}D^6 + \ldots$

* Negative powers of M are expanded as far as D^{r-1} (the exponent of M being $-r$). Other operators are expanded as far as D^7.

TABLE 3

TRACES OF POWERS AND MEAN POWERS OF M

Operator	Trace	Operator	Trace
M	1	μM	indeterminate
M^2	1	μM^2	$1 + \frac{1}{4}\delta^2$
M^3	$1 + \frac{1}{8}\delta^2$	μM^3	$1 + \frac{1}{4}\delta^2$
M^4	$1 + \frac{1}{6}\delta^2$	μM^4	$1 + \frac{7}{24}\delta^2 + \frac{1}{96}\delta^4$
M^5	$1 + \frac{5}{24}\delta^2 + \frac{1}{384}\delta^4$	μM^5	$1 + \frac{1}{3}\delta^2 + \frac{1}{48}\delta^4$
M^6	$1 + \frac{1}{4}\delta^2 + \frac{1}{120}\delta^4$	μM^6	$1 + \frac{3}{8}\delta^2 + \frac{61}{1920}\delta^4 + \frac{1}{7680}\delta^6$
M^7	$1 + \frac{7}{24}\delta^2 + \frac{91}{5760}\delta^4 + \frac{1}{46,080}\delta^6$	μM^7	$1 + \frac{5}{12}\delta^2 + \frac{2}{45}\delta^4 + \frac{1}{1440}\delta^6$
M^8	$1 + \frac{1}{3}\delta^2 + \frac{1}{40}\delta^4 + \frac{1}{5040}\delta^6$	μM^8	$1 + \frac{11}{24}\delta^2 + \frac{113}{1920}\delta^4 + \frac{547}{322,560}\delta^6 + \frac{1}{1,290,240}\delta^8$

TABLE 4

POWERS AND MEAN POWERS OF M IN TERMS OF SPECIAL OPERATORS

Part I. End-point Formulas

$$\mu M = \tfrac{1}{2}Q_0$$

$$M^2 = Q_1$$

$$\mu M^3 = Q_1 + \tfrac{1}{4}\delta^2 Q_2$$

$$M^4 = Q_1 + \tfrac{1}{6}\delta^2 Q_3$$

$$\mu M^5 = Q_1 + \delta^2(\tfrac{1}{6}Q_1 + \tfrac{1}{6}Q_3) + \tfrac{1}{48}\delta^4 Q_4$$

$$M^6 = Q_1 + \delta^2(\tfrac{1}{12}Q_1 + \tfrac{1}{6}Q_3) + \tfrac{1}{120}\delta^4 Q_5$$

$$\mu M^7 = Q_1 + \delta^2(\tfrac{1}{4}Q_1 + \tfrac{1}{6}Q_3) + \delta^4(\tfrac{1}{120}Q_1 + \tfrac{1}{36}Q_3 + \tfrac{1}{120}Q_5) + \tfrac{1}{1440}\delta^6 Q_6$$

$$M^8 = Q_1 + \delta^2(\tfrac{1}{6}Q_1 + \tfrac{1}{6}Q_3) + \delta^4(\tfrac{1}{360}Q_1 + \tfrac{1}{72}Q_3 + \tfrac{1}{120}Q_5) + \tfrac{1}{5040}\delta^6 Q_7$$

$$\mu M^9 = Q_1 + \delta^2(\tfrac{1}{3}Q_1 + \tfrac{1}{6}Q_3) + \delta^4(\tfrac{1}{40}Q_1 + \tfrac{1}{24}Q_3 + \tfrac{1}{120}Q_5) + \delta^6(\tfrac{1}{5040}Q_1 + \tfrac{1}{720}Q_3 + \tfrac{1}{720}Q_5 + \tfrac{1}{5040}Q_7) + \tfrac{1}{80,640}\delta^8 Q_8$$

$$M^{10} = Q_1 + \delta^2(\tfrac{1}{4}Q_1 + \tfrac{1}{6}Q_3) + \delta^4(\tfrac{1}{80}Q_1 + \tfrac{1}{36}Q_3 + \tfrac{1}{120}Q_5) + \delta^6(\tfrac{1}{20,160}Q_1 + \tfrac{1}{2160}Q_3 + \tfrac{1}{1440}Q_5 + \tfrac{1}{5040}Q_7) + \tfrac{1}{362,880}\delta^8 Q_9$$

Part II. Midpoint Formulas

$$M = \delta^{-1}T_0$$

$$\mu M^2 = \delta^{-1}T_0 + \tfrac{1}{2}\delta T_1$$

$$M^3 = \delta^{-1}T_0 + \tfrac{1}{2}\delta T_2$$

$$\mu M^4 = \delta^{-1}T_0 + \delta(\tfrac{1}{6}T_0 + \tfrac{1}{2}T_2) + \tfrac{1}{12}\delta^3 T_3$$

$$M^5 = \delta^{-1}T_0 + \delta(\tfrac{1}{12}T_0 + \tfrac{1}{2}T_2) + \tfrac{1}{24}\delta^3 T_4$$

$$\mu M^6 = \delta^{-1}T_0 + \delta(\tfrac{1}{4}T_0 + \tfrac{1}{2}T_2) + \delta^3(\tfrac{1}{120}T_0 + \tfrac{1}{12}T_2 + \tfrac{1}{24}T_4) + \tfrac{1}{240}\delta^5 T_5$$

$$M^7 = \delta^{-1}T_0 + \delta(\tfrac{1}{6}T_0 + \tfrac{1}{2}T_2) + \delta^3(\tfrac{1}{360}T_0 + \tfrac{1}{24}T_2 + \tfrac{1}{24}T_4) + \tfrac{1}{720}\delta^5 T_6$$

$$\mu M^8 = \delta^{-1}T_0 + \delta(\tfrac{1}{3}T_0 + \tfrac{1}{2}T_2) + \delta^3(\tfrac{1}{40}T_0 + \tfrac{1}{8}T_2 + \tfrac{1}{24}T_4) + \delta^5(\tfrac{1}{5040}T_0 + \tfrac{1}{240}T_2 + \tfrac{1}{144}T_4 + \tfrac{1}{720}T_6) + \tfrac{1}{10,080}\delta^7 T_7$$

$$M^9 = \delta^{-1}T_0 + \delta(\tfrac{1}{4}T_0 + \tfrac{1}{2}T_2) + \delta^3(\tfrac{1}{80}T_0 + \tfrac{1}{12}T_2 + \tfrac{1}{24}T_4) + \delta^5(\tfrac{1}{20,160}T_0 + \tfrac{1}{720}T_2 + \tfrac{1}{288}T_4 + \tfrac{1}{720}T_6) + \tfrac{1}{40,320}\delta^7 T_8$$

$$\mu M^{10} = \delta^{-1}T_0 + \delta(\tfrac{5}{12}T_0 + \tfrac{1}{2}T_2) + \delta^3(\tfrac{7}{144}T_0 + \tfrac{1}{6}T_2 + \tfrac{1}{24}T_4) + \delta^5(\tfrac{17}{12,096}T_0 + \tfrac{1}{80}T_2 + \tfrac{1}{96}T_4 + \tfrac{1}{720}T_6) + \delta^7(\tfrac{1}{362,880}T_0 + \tfrac{1}{10,080}T_2 + \tfrac{1}{2880}T_4 + \tfrac{1}{4320}T_6 + \tfrac{1}{40,320}T_8) + \tfrac{1}{725,760}\delta^9 T_9$$

TABLE 5

SPECIAL OPERATORS IN TERMS OF M, δ, μ, AND D

Part I. End-point Formulas

A. Standard Formulas

$$C_1 = M^2$$
$$C_3 = M^4 - \tfrac{1}{6}\delta^2 M^2$$
$$C_5 = M^6 - \tfrac{1}{4}\delta^2 M^4 + \tfrac{1}{30}\delta^4 M^2$$
$$C_7 = M^8 - \tfrac{1}{3}\delta^2 M^6 + \tfrac{7}{120}\delta^4 M^4 - \tfrac{1}{140}\delta^6 M^2$$

B. Additional Terms

$$Q_0 = 2\mu M$$
$$Q_1 = M^2$$
$$\delta^2 Q_2 = 4\mu M^3 - 4M^2$$
$$\delta^2 Q_3 = 6M^4 - 6M^2$$
$$\delta^4 Q_4 = 48\mu M^5 - 48M^4 - 8\delta^2 M^2$$
$$\delta^4 Q_5 = 120M^6 - 120M^4 - 10\delta^2 M^2$$
$$\delta^6 Q_6 = 1440\mu M^7 - 1440M^6 - 240\delta^2 M^4 - 12\delta^4 M^2$$
$$\delta^6 Q_7 = 5040M^8 - 5040M^6 - 420\delta^2 M^4 - 14\delta^4 M^2$$
$$\delta^8 Q_8 = 80{,}640\mu M^9 - 80{,}640M^8 - 13{,}440\delta^2 M^6 - 672\delta^4 M^4 - 16\delta^6 M^2$$
$$\delta^8 Q_9 = 362{,}880M^{10} - 362{,}880M^8 - 30{,}240\delta^2 M^6 - 1008\delta^4 M^4 - 18\delta^6 M^2$$

Part II. Midpoint Formulas

A. Standard Formulas

$$C_0 = M$$
$$C_2 = M^3 - \tfrac{1}{8}\delta^2 M$$
$$C_4 = M^5 - \tfrac{5}{24}\delta^2 M^3 + \tfrac{3}{128}\delta^4 M$$
$$C_6 = M^7 - \tfrac{7}{24}\delta^2 M^5 + \tfrac{259}{5760}\delta^4 M^3 - \tfrac{5}{1024}\delta^6 M$$

B. Additional Terms

$$T_0 = \delta M$$
$$\delta T_1 = 2\mu M^2 - 2M$$
$$\delta T_2 = 2M^3 - 2M$$
$$\delta^3 T_3 = 12\mu M^4 - 12M^3 - 2\delta^2 M$$
$$\delta^3 T_4 = 24M^5 - 24M^3 - 2\delta^2 M$$
$$\delta^5 T_5 = 240\mu M^6 - 240M^5 - 40\delta^2 M^3 - 2\delta^4 M$$

TABLE 5—*Continued*

Part II. Midpoint Formulas—*Continued*

$$\delta^6 T_6 = 720M^7 - 720M^5 - 60\delta^2 M^3 - 2\delta^4 M$$

$$\delta^7 T_7 = 10{,}080\mu M^8 - 10{,}080M^7 - 1680\delta^2 M^5 - 84\delta^4 M^3 - 2\delta^6 M$$

$$\delta^7 T_8 = 40{,}320M^9 - 40{,}320M^7 - 3360\delta^2 M^5 - 112\delta^4 M^3 - 2\delta^6 M$$

$$\delta^9 T_9 = 725{,}760\mu M^{10} - 725{,}760M^9 - 120{,}960\delta^2 M^7 - 6048\delta^4 M^5 - 144\delta^6 M^3 - 2\delta^8 M$$

Part III. Mixed End-point-Midpoint Formulas

$$V_0 = M$$

$$\delta^2 V_1 = M^2(2\mu - 2)$$

$$\delta^2 V_2 = M^2(2M - 2)$$

$$\delta^4 V_3 = M^4(12\mu - 12 - \tfrac{3}{2}D^2)$$

$$\delta^4 V_4 = M^4(24M - 24 - D^2)$$

$$\delta^6 V_5 = M^6(240\mu - 240 - 30D^2 - \tfrac{5}{8}D^4)$$

$$\delta^6 V_6 = M^6(720M - 720 - 30D^2 - \tfrac{3}{8}D^4)$$

$$\delta^8 V_7 = M^8(10{,}080\mu - 10{,}080 - 1260D^2 - \tfrac{105}{4}D^4 - \tfrac{7}{32}D^6)$$

$$\delta^8 V_8 = M^8(40{,}320M - 40{,}320 - 1680D^2 - 21D^4 - \tfrac{1}{8}D^6)$$

$$\delta W_0 = M(2\mu - 2)$$

$$\delta W_1 = M(M - 1)$$

$$\delta^3 W_2 = M^3(4\mu - 4 - \tfrac{1}{2}D^2)$$

$$\delta^3 W_3 = M^3(6M - 6 - \tfrac{1}{4}D^2)$$

$$\delta^5 W_4 = M^5(48\mu - 48 - 6D^2 - \tfrac{1}{8}D^4)$$

$$\delta^5 W_5 = M^5(120M - 120 - 5D^2 - \tfrac{1}{16}D^4)$$

$$\delta^7 W_6 = M^7(1440\mu - 1440 - 180D^2 - \tfrac{15}{4}D^4 - \tfrac{1}{32}D^6)$$

$$\delta^7 W_7 = M^7(5040M - 5040 - 210D^2 - \tfrac{21}{8}D^4 - \tfrac{1}{64}D^6)$$

$$\delta^9 W_8 = M^9(80{,}640\mu - 80{,}640 - 10{,}080D^2 - 210D^4 - \tfrac{7}{4}D^6 - \tfrac{1}{128}D^8)$$

TABLE 6

VALUES OF THE BASIC FUNCTION $L(x)$ OF CERTAIN CONTINUOUS OPERATORS*

x	Operator					
	M^6	μM^5	M^4	$\delta^2 M^4$	$\delta^4 M^4$	μM^3
0.0	.5500 00000	.4583 33333	.66666	−1.00000	2.66666	.5000
.1	.5450 24166	.4558 37500	.65716	− .97166	2.57250	.4975
.2	.5303 73333	.4484 00000	.63066	− .89333	2.31333	.4900
.3	.5068 22500	.4361 70833	.59016	− .77500	1.92416	.4775
.4	.4755 46666	.4194 00000	.53866	− .62666	1.44000	.4600
.5	.4380 20833	.3984 37500	.47916	− .45833	.89583	.4375
.6	.3959 20000	.3737 33333	.41466	− .28000	.32666	.4100
.7	.3510 19166	.3458 37500	.34816	− .10166	− .23250	.3775
.8	.3050 93333	.3154 00000	.28266	.06666	− .74666	.3400
.9	.2598 17500	.2831 70833	.22116	.21500	−1.18083	.2975
1.0	.2166 66666	.2500 00000	.16666	.33333	−1.50000	.2500
1.1	.1768 17083	.2168 27083	.12150	.41416	−1.67850	.2025
1.2	.1410 80000	.1845 66666	.08533	.46000	−1.72800	.1600
1.3	.1099 17916	.1539 93750	.05716	.47583	−1.66950	.1225
1.4	.0834 93333	.1257 33333	.03600	.46666	−1.52400	.0900
1.5	.0617 18750	.1002 60416	.02083	.43750	−1.31250	.0625
1.6	.0443 06666	.0779 00000	.01066	.39333	−1.05600	.0400
1.7	.0308 19583	.0588 27083	.00450	.33916	− .77550	.0225
1.8	.0207 20000	.0430 66666	.00133	.28000	− .49200	.0100
1.9	.0134 20416	.0304 93750	.00016	.22083	− .22650	.0025
2.0	.0083 33333	.0208 33333	.00000	.16666	.00000	.0000
2.1	.0049 20750	.0136 68750		.12150	.17116	
2.2	.0027 30666	.0085 33333		.08533	.28933	
2.3	.0014 00583	.0050 02083		.05716	.36150	
2.4	.0006 48000	.0027 00000		.03600	.39466	
2.5	.0002 60416	.0013 02083		.02083	.39583	
2.6	.0000 85333	.0005 33333		.01066	.37200	
2.7	.0000 20250	.0001 68750		.00450	.33016	
2.8	.0000 02666	.0000 33333		.00133	.27733	
2.9	.0000 00083	.0000 02083		.00016	.22050	
3.0	.0000 00000	.0000 00000		.00000	.16666	
3.1					.12150	
3.2					.08533	
3.3					.05716	
3.4					.03600	
3.5					.02083	
3.6					.01066	
3.7					.00450	
3.8					.00133	
3.9					.00016	
4.0					.00000	

* An italic final digit denotes infinite repetition of this digit. Only positive values of x are shown, since $L(-x) = L(x)$.

TABLE 6—*Continued*

x	Operator					
	$\mu\delta^3 M^3$	$\delta^4 M^2$	M^5	μM^4	M^3	$\delta^2 M^3$
0.0	−.5000	6.0	.5989 5833	.479 166	.750	−1.250
.1	−.4950	5.0	.5927 3333	.476 666	.740	−1.220
.2	−.4800	4.0	.5743 5833	.469 166	.710	−1.130
.3	−.4550	3.0	.5447 3333	.456 666	.660	− .980
.4	−.4200	2.0	.5053 5833	.439 166	.590	− .770
.5	−.3750	1.0	.4583 3333	.416 666	.500	− .500
.6	−.3200	0.0	.4063 1666	.389 333	.405	− .220
.7	−.2550	−1.0	.3520 6666	.358 000	.320	.020
.8	−.1800	−2.0	.2979 8333	.323 666	.245	.220
.9	−.0950	−3.0	.2460 6666	.287 333	.180	.380
1.0	.0000	−4.0	.1979 1666	.250 000	.125	.500
1.1	.0925	−3.5	.1547 3333	.212 666	.080	.580
1.2	.1700	−3.0	.1173 1666	.176 333	.045	.620
1.3	.2325	−2.5	.0860 6666	.142 000	.020	.620
1.4	.2800	−2.0	.0609 8333	.110 666	.005	.580
1.5	.3125	−1.5	.0416 6666	.083 333	.000	.500
1.6	.3300	−1.0	.0273 3750	.060 750		.405
1.7	.3325	− .5	.0170 6666	.042 666		.320
1.8	.3200	.0	.0100 0416	.028 583		.245
1.9	.2925	.5	.0054 0000	.018 000		.180
2.0	.2500	1.0	.0026 0416	.010 416		.125
2.1	.2025	.9	.0010 6666	.005 333		.080
2.2	.1600	.8	.0003 3750	.002 250		.045
2.3	.1225	.7	.0000 6666	.000 666		.020
2.4	.0900	.6	.0000 0416	.000 083		.005
2.5	.0625	.5	.0000 0000	.000 000		.000
2.6	.0400	.4				
2.7	.0225	.3				
2.8	.0100	.2				
2.9	.0025	.1				
3.0	.0000	.0				

MATHEMATICAL APPENDIX

DISTRIBUTION OPERATORS

The term "distribution," as used by Schwartz for the case of a single variable, means, in essence, an operator of the form

$$K + J_0 + J_1D + J_2D^2 + \ldots + J_rD^r , \tag{21}$$

where K is a continuous operator and the J's are discrete operators. We shall therefore call such an operator a *distribution operator.* Such an operator has already been encountered in equation (6*a*) of footnote 12. It is clear that the continuous operator, the discrete operator, and the differential operator D^i are all particular cases of the distribution operator. We shall say that a distribution operator H of the form (21) is *of finite range* if all arguments x corresponding to nonzero values of the basic function $f(x)$ of K, and all arguments x whose negatives appear as exponents of E when the "J" operators are expressed in the form (3) with nonzero coefficients, are contained in a finite interval $a \leq x \leq b$. We shall denote this interval by (a, b). The smallest interval having this property will be called the *range* of H. If (a, b) is the range of H, this implies that either (1) E^{-a} occurs in the expression for one or more of the J's or (2) any interval, however small, extending to the right of the argument a contains arguments for which $f(x)$ is different from zero. Of course, both statements may be true at the same time. Similar remarks apply to the upper end of the interval. Evidently, the span of H, as defined in the main part of the paper, is $b - a$. In this appendix, we shall not limit our attention exclusively to distribution operators of finite range.

The basic function of the continuous term K of a distribution operator may have discontinuities. This discussion will be limited, however, to cases in which this function is piecewise continuous, and has piecewise continuous derivatives of all orders. We call a function *piecewise continuous* when it has, in any finite interval, at most a finite number of discontinuities, and, further, all its discontinuities are "jump" discontinuities. We say that a function $f(x)$ has a *jump discontinuity* at $x = a$ if $f(x)$ approaches a finite limit when x approaches a from above, and also when x approaches a from below, but the two limits are different. In such a case, we shall find it convenient to denote the respective limits by $f(a + 0)$ and $f(a - 0)$. We shall call the difference, $f(a + 0) - f(a - 0)$, the *jump* of the function $f(x)$ at the point of discontinuity $x = a$. We shall call a function *piecewise analytic* if it is piecewise continuous and has piecewise continuous derivatives of all orders. The curves of interpolated values produced by polynomial interpolation formulas applied centrally in each interpolation interval over a number of intervals, and the basic functions of such formulas, are examples of piecewise analytic functions. Throughout this appendix, the basic functions of all continuous operators will be assumed, without explicit statement to that effect, to be piecewise analytic.

A jump discontinuity in the basic function $L(x)$ of an interpolation formula means that the interpolated value at corresponding points has dual values. For example, in plain central-difference interpolation to second differences, $v_{n+1/2}$ can be calculated either from u_{n-1}, u_n, and u_{n+1} or from u_n, u_{n+1}, and u_{n+2}; and, unless the given values lie exactly on a polynomial curve of the second degree, the two values will differ. In the same way, a discontinuity in the $L(x)$ function for an integral argument would leave us with dual values at the given points. In interpolation formulas of practical value, $L(x)$ is continuous, and comes to zero at the ends. However, all polynomial interpolation formulas have discontinuities in some of the derivatives of the basic function.

We shall also follow Schwartz in studying distribution operators through their effect on a test-function. By a *test-function* we shall mean any function $\phi(x)$ having the following two properties:

(1) There exist two finite real numbers α and β such that $\phi(x) = 0$ for $x \leq \alpha$ and for $x \geq \beta$.

(2) $\phi(x)$ possesses at every point derivatives of all orders.

In our investigations the test-function will serve only a purely formal purpose, and we shall not have occasion to make any actual calculations with particular test-functions. However, inasmuch as functions having the properties stated are somewhat unusual, it may be well to show that they actually exist. An example given by Schwartz is the function defined by

$$\phi(x) = \begin{cases} 0 & \text{for } |x| \geq h\,, \\ e^{-h^2/(h^2-x^2)} & \text{for } |x| < h\,. \end{cases}$$

The convenience of the test-function arises from the fact that *any* distribution operator can be applied to such a function. It should perhaps be pointed out, however, that most distribution operators can also be applied (and we shall apply them) to a great many functions other than test-functions. When a distribution operator of finite range is applied to a test-function, it is fairly clear that the resulting function is also a test-function. Applying to a test-function a distribution operator of infinite range gives a function which has property (2), but, in general, does not have property (1) and thus fails to qualify as a test-function.

It is clear that a distribution operator is a linear operator. In other words, we always have

$$H[k\phi(x)] = kH\phi(x)\,, \tag{22a}$$

$$H[\phi(x) + \psi(x)] = H\phi(x) + H\psi(x)\,, \tag{22b}$$

where H is any distribution operator, k is a constant, and $\phi(x)$ and $\psi(x)$ are functions to which H can be applied.

PRODUCTS OF DISTRIBUTION OPERATORS

If H_1 is any distribution operator, and H_2 is another distribution operator such that, for any test-function $\phi(x)$, the function $f(x) = H_2[H_1\phi(x)]$ is always defined, we shall call the entire operation leading from $\phi(x)$ to $f(x)$ the *product*[23] of H_2 and H_1, and shall write

$$f(x) = H_3\phi(x) = H_2[H_1\phi(x)]\,,$$

or, symbolically,

$$H_3 = H_2H_1\,.$$

If at least one of the operators H_1 and H_2 is of finite range, their product is always defined. It may not be defined if both are of infinite range.

We are particularly interested in the form of the product DK, where K is a continuous operator whose basic function $f(t)$ has jump discontinuities. If $\phi(x)$ is a test-function, we have

$$K\phi(x) = \int_{-\infty}^{\infty} f(t)\,\phi(x-t)\,dt = \int_{-\infty}^{\infty} f(x-s)\,\phi(s)\,ds\,, \tag{23}$$

writing $s = x - t$. Let $\ldots, t_{-1}, t_0, t_1, \ldots$ denote the points of discontinuity of $f(t)$, arranged in increasing order of magnitude. Then,

$$K\phi(x) = \Sigma_i \int_{x-t_{i+1}}^{x-t_i} f(x-s)\,\phi(s)\,ds\,.$$

Applying the usual rule for differentiation of an integral with variable limits of integration, we have

$$\begin{aligned} DK\phi(x) &= \Sigma_i \int_{x-t_{i+1}}^{x-t_i} f'(x-s)\,\phi(s)\,ds \\ &\quad + \Sigma_i\,[f(t_i+0)\,\phi(x-t_i) - f(t_{i+1}-0)\,\phi(x-t_{i+1})] \\ &= \int_{-\infty}^{\infty} f'(x-s)\phi(s)\,ds + \Sigma_i[\,f(t_i+0) - f(t_i-0)\,]\phi(x-t_i)\,. \end{aligned}$$

Symbolically, we may write, therefore,

$$DK = K' + \Sigma_i\, b_i\, E^{-t_i}\,, \tag{24}$$

[23] Schwartz also considers other types of product. He calls this kind "*produit de composition.*"

where K' denotes the continuous operator whose basic function is $f'(t)$ and b_i denotes the jump of $f(t)$ at the argument t_i.

Generalizing this result, let $\ldots, t_{-1}, t_0, t_1, \ldots$ denote the arguments at which there is a discontinuity in $f(t)$ or in any of its first r derivatives, and let $b_i^{(k)}$ denote the jump of the kth derivative $f^{(k)}(t)$ at the argument t_i. Then, by repeated application of formula (24) we obtain[24]

$$D^rK = K^{(r)} + J_{-r} + J_{-r+1}D + \ldots + J_{-1}D^{r-1}, \qquad (25)$$

where

$$J_{-k} = \Sigma_i \, b_i^{(k-1)}E^{-t_i}, \qquad (26)$$

and $K^{(r)}$ denotes the continuous operator whose basic function is $f^{(r)}(t)$.

In general, multiplication of distribution operators is commutative, associative, and distributive. In other words, we have:

$$H_1H_2 = H_2H_1,$$
$$H_1(H_2H_3) = (H_1H_2)H_3,$$
$$H_1(H_2 + H_3) = H_1H_2 + H_1H_3.$$

The distributive property is essentially a matter of definition, and follows from the relation (22*b*) if we define $(H_1 + H_2)\phi(x)$ to mean $H_1\phi(x) + H_2\phi(x)$. It can be shown that the other two properties always hold if not more than one operator in any given product is of infinite range,[25] as will be the case in all applications to be made in this appendix. If H_1 is a distribution operator of finite or infinite range and $\phi(x)$ is a test-function, $F(x) = H_1\phi(x)$ is of the form

$$\int_{-\infty}^{\infty} f(t)\,\phi(x-t)\,dt + \sum_{j=0}^{r} \Sigma_i a_{ij}\phi^{(j)}(x - x_i), \qquad (27)$$

where $f(t)$ is the basic function of the continuous term of H_1. In the following deductions, it should be borne in mind that the limits of integration and the summation with respect to i in expression (27) are, in reality, finite, because of property (1) of the test-function $\phi(x)$. By actual differentiation we find that

$$F^{(k)}(x) = H_1\phi^{(k)}(x),$$

or, symbolically, $D^kH_1 = H_1D^k$. Also, it is easily verified that $E^hF(x) = F(x + h) = H_1\phi(x + h)$, so that $E^hH_1 = H_1E^h$. Thus, if J is any discrete operator of finite range, $JH_1 = H_1J$, making use of the distributive property and relation (22*a*). It follows that any operator of the form JD^k, where J is of finite range, is commutative with H_1 (since $D^k\phi(x)$ is

[24] Equations (24) and (25) are due to Schwartz, *op. cit.*, Tome I, pp. 37–38.

[25] This is proved by Schwartz, *op. cit.*, Tome II, p. 14.

also a test-function). Finally, if K is a continuous operator of finite range, with basic function $g(s)$, it can be shown by integration with respect to s that $KF(x) = KH_1\phi(x) = H_1K\phi(x)$. In view of the distributive property, it follows that any distribution operator H_2 of finite range is commutative with H_1.

It might have been supposed that a more general type of operator than the form (21) would be obtained by taking as coefficients of the various powers of D sums of discrete and continuous operators. Formula (25), together with the commutative property, shows that this is not the case, and that such an operator would reduce to a distribution operator. Moreover, it is not difficult to see that the product of two continuous operators, or of a continuous and a discrete operator, when defined, is a continuous operator; while the product of two discrete operators, when defined, is a discrete operator. From these statements and equation (25) it follows that the product of any two distribution operators, when defined, is a distribution operator. If both are of finite range, it is clear that the product is of finite range.

We are now in a position to prove the associative property. By the definition of the product, $(H_2H_3)\phi(x) = H_2H_3\phi(x)$, and therefore $H_1(H_2H_3)\phi(x) = H_1H_2H_3\phi(x)$. Similarly, $(H_1H_2)H_3\phi(x) = H_1H_2H_3\phi(x)$, provided H_3 is of finite range, so that $H_3\phi(x)$ is a test-function. On the other hand, if H_3 is of infinite range, (H_1H_2) is of finite range (since, by hypothesis, not more than one of the three operators is of infinite range), and we have, by the commutative property, $(H_1H_2)H_3\phi(x) = H_3(H_1H_2)\phi(x) = H_3H_1H_2\phi(x) = H_1H_3H_2\phi(x) = H_1H_2H_3\phi(x)$, since $H_2\phi(x)$ is a test-function.

It will be convenient to refer to the product D^rH (where H is any distribution operator) as the rth *derivative*[26] of H. If H is of finite range, it can be shown that the range of D^rH is the same as that of H. We shall consider first the case $r = 1$. If H is of the form (21), we have

$$DH = DK + J_0D + J_1D^2 + \dots + J_rD^{r+1}.$$

It is clearly sufficient to show that DK has the same range as K. If the range of K is (a, b), it is evident from equation (24) that the range of DK is contained in (a, b). Moreover, $f'(t)$, the derivative of the basic function $f(t)$ of K, cannot be identically zero in an interval extending to the right of the argument a unless a discontinuity in $f(t)$ occurs at $t = a$, for otherwise $f(t)$ would have to be identically zero in the same interval, contrary to the definition of the range. Similar considerations apply to the

[26] This is essentially equivalent to Schwartz' definition of the rth derivative of a distribution for the case of one variable.

upper end of the interval. The extension to a derivative of any order is immediate. As an example, we note that the range of M^r is $(-r/2, r/2)$, since this is the range of $\delta^r = D^r M^r$.

ORDER OF A DISTRIBUTION OPERATOR

A distribution operator of the form (21), in which J_r is different from zero, will be called an *operator of order r*.[27] In particular, a discrete operator, or a sum of a continuous and a discrete operator, is of order zero. It is clear that the derivative of an operator of order r is of order $r + 1$. It will be convenient also to define an operator of negative order $-r$ as one whose rth derivative is of order zero. Such an operator would be a continuous operator whose basic function and its first $r - 2$ derivatives are everywhere continuous, but whose $(r - 1)$th derivative has discontinuities. An operator of order -1 is a continuous operator whose basic function has discontinuities. With this understanding, it is evident that, whatever may be the order of a distribution operator, multiplication by D^r always increases the order by r. This statement is somewhat generalized in the following theorem:

Theorem 1. If H_1 and H_2 are distribution operators of order r_1 and r_2, respectively, the product H_1H_2, if it is defined, is of order $r_1 + r_2$.

This is obvious except when r_1 and r_2 are both negative. In that case, we have

$$D^{-(r_1+r_2)}H_1H_2 = (D^{-r_1}H_1)(D^{-r_2}H_2)\,,$$

which is the product of two operators of order zero and therefore is of order zero. Hence, by the definition of negative order, H_1H_2 is of order $r_1 + r_2$.

The following theorem will be useful later.

Theorem 2. If K is a continuous operator with basic function $f(t)$ and H is a distribution operator such that HK is a continuous operator, and at least one of the operators H and K is of finite range, then the basic function of HK is $Hf(t)$.

If H is of positive order r, it follows from Theorem 1 and from the fact that HK is a continuous operator (and therefore of order not exceeding -1) that K is of order not exceeding $-r - 1$. In other words, $f(t)$ has no discontinuities in its derivatives of order $0, 1, \ldots, r - 1$. It follows from equations (25) and (26) that $D^iK = K^{(i)}$ for $i = 1, 2, \ldots, r$. Recalling from equation (23) that

$$K\phi(x) = \int_{-\infty}^{\infty} f(x - s)\,\phi(s)\,ds\,,$$

[27] The notion of the order of a distribution operator is due to Schwartz, *op. cit.*, Tome I, p. 25.

where $\phi(x)$ is any test-function, it is evident that we can obtain $D^i K\phi(x) = K^{(i)}\phi(x)$ (for $i = 1, 2, \ldots, r$) by merely replacing $f(x - s)$ in the integrand by its ith derivative. If H is of zero or negative order, of course no differentiation is involved. Moreover, if the range of nonzero values of $\phi(s)$ is (α, β), the limits of integration can be changed to α and β. If either K or H is of finite range, it follows that all integrations and summations involved in the calculation of $HK\phi(x)$ for a particular value of x are, in reality, over a finite interval: therefore the order of operations can be reversed. In the case of a summation, this is a consequence of the elementary fact that the integral of a sum of functions is the sum of their integrals. In the case of an integration, it follows from the piecewise continuity of all functions involved in the integrand. Consequently, we can obtain $HK\phi(x)$ by operating under the integral sign with H on $f(x - s)$. This gives

$$HK\phi(x) = \int_{-\infty}^{\infty} [Hf(x-s)]\,\phi(s)\,ds\,,$$

which implies, in view of equation (23), that $Hf(t)$ is the basic function of HK.

POLYNOMIAL OPERATORS

A continuous operator whose basic function is a polynomial of degree q will be called a *polynomial operator of degree q*. Clearly, such an operator is of infinite range. The continuous operator whose basic function is identically zero, which reduces every test-function to zero, will be considered a special case of a polynomial operator, and will be called the *zero operator*. It will be represented by the ordinary zero symbol, as the context will always show whether the zero operator or the number zero is intended.

If H is a distribution operator such that $D^q H = 0$, where q is a positive integer, it is easily seen that H must be a polynomial operator of degree at most $q - 1$. First, we point out that H is necessarily a continuous operator, for otherwise $D^q H$ would not be a continuous operator. Let $f(t)$ denote the basic function of H. Then, by Theorem 2, $D^q f(t) = 0$. This shows that $f(t)$ is a polynomial of degree at most $q - 1$. It follows that two distribution operators H_1 and H_2 which have the same rth derivative differ by a polynomial operator of degree $q - 1$ or less, for we have $D^q(H_1 - H_2) = 0$. Thus, a distribution operator H of finite range is completely determined by its rth derivative.

It is easily seen that the product of a polynomial operator P of degree q with any distribution operator H of finite range is another polynomial operator of degree at most q. The product certainly exists since one factor is of finite range. Moreover,

$$D^{q+1}HP = HD^{q+1}P = 0\,,$$

showing that HP is a polynomial operator of degree at most q.

For brevity, we shall refer to a discrete operator which is the trace of a polynomial operator of degree q as a *polynomial trace of degree q*. We shall also say that one distribution operator *annihilates* another if their product is the zero operator. We shall need the following theorem.

Theorem 3. A distribution operator H of finite range annihilates every polynomial trace of degree q or less if and only if it is of the form $\delta^{q+1}H'$, where H' is a distribution operator of finite range.

If $t(P)$ is a polynomial trace of degree q or less, evidently

$$\delta^{q+1}H't(P) = H'\delta^{q+1}t(P) = 0 .$$

To prove the converse, we shall consider first an operator R, which is either a continuous or discrete operator of finite range and annihilates every polynomial trace of degree q or less. If R is continuous, let $f(x)$ denote its basic function. If it is discrete and of the form $\Sigma_i a_i E^{-x_i}$, we define a "basic function" $f(x)$ equal to a_i when $x = x_i$ and equal to zero for all values of x not equal to any x_i. We now define

$$f_1(x) = \sum_{n=0}^{\infty} f(x - n - \tfrac{1}{2}) ,$$

where the infinite upper limit is used for algebraic convenience. Evidently

$$\delta f_1(x) = f(x) . \tag{28}$$

If (a, b) is the range of R, we see that $f_1(x) = 0$ for $x < a + \frac{1}{2}$. For $x > b - \frac{1}{2}$, we have

$$f_1(x) = \sum_{n=-\infty}^{\infty} f(x - n - \tfrac{1}{2})$$

Now, let P_0 denote the polynomial operator of degree zero whose basic function is identically 1, and let $f_0(x)$ denote the basic function (in quotes, if R is discrete) of the product $Rt(P_0)$. Then, by Theorem 2 if R is continuous, or by symbolic multiplication if R is discrete, we have

$$f_0(x) = \sum_{n=-\infty}^{\infty} f(x - n) .$$

Thus, $f_1(x) = f_0(x - \frac{1}{2})$ for $x > b - \frac{1}{2}$. But, since $q \geq 0$, $t(P_0)$ is annihilated by R, and therefore $f_0(x)$ is identically zero. Hence $f_1(x) = 0$ for $x > b - \frac{1}{2}$.

Moreover, equation (28) shows that in the continuous case $f_1(x)$ is piecewise analytic, for, if it were not, either by reason of having other than jump discontinuities or an infinite number of discontinuities in a finite interval, such irregularities necessarily would occur also in $f(x)$.

Now let R_1 denote the operator (of finite range) whose basic function (in quotes, if R is discrete) is $f_1(x)$. Then, by Theorem 2 in the continuous case, or by symbolic multiplication in the discrete case,

$$\delta R_1 = R .$$

If $t(P)$ is any polynomial trace of degree q or less,

$$Rt(P) = 0 = R_1 \delta t(P) .$$

But, since $\delta t(P) = E^{-1/2} t(E^{1/2} \delta P)$, and since any polynomial is the first difference of a polynomial of the next higher degree, we conclude that R_1 annihilates every polynomial trace of degree $q - 1$ or less.

By repeated application of this result we eventually find that

$$R = \delta^{q+1} R' ,$$

where R' is a continuous or discrete operator according as R is continuous or discrete, and is of finite range.

Finally, if H is a distribution operator of the form (21), a polynomial trace is annihilated by H only if it is annihilated separately by the continuous term K and by each discrete "J" operator. Therefore, if H is of finite range, it is of the form $\delta^{q+1} H'$, where H' is a distribution operator of finite range.

APPLICATION TO INTERPOLATION FORMULAS

If $u_a, u_{a+1}, \ldots, u_{a+N}$ are a set of given values on which interpolation is to be performed, we shall call the discrete operator

$$U = \sum_{n=a}^{a+N} u_n E^{-n}$$

the *characteristic operator of the given values.* Formula (1) defines a piecewise analytic function v_x if we adopt the convention[28] that $u_n = 0$ for $n < a$ and for $n > a + N$. This function v_x may be considered as the basic function of a continuous operator K, which we shall call the *characteristic operator of the interpolated values.* If G denotes the characteristic operator of the interpolation formula, it follows from equations (1) and (5) and Theorem 2 that

$$GU = K , \tag{29}$$

[28] This, of course, is not a practical assumption, and is made only for algebraic convenience. In this paper, our interest is only in the main part of the v_x curve, which is unaffected by this assumption.

or, in other words, the characteristic operator of the given values multiplied by the characteristic operator of the interpolation formula gives the characteristic operator of the interpolated values.[29]

MAINTENANCE OF DEGREE

It follows from equation (29) that a continuously defined interpolation formula is correct to rth differences if and only if its characteristic operator G satisfies the equation

$$Gt(P) = P\,, \tag{30}$$

for every polynomial operator P of degree r or less. It will be convenient to consider the somewhat weaker requirement to the effect that, for every such polynomial operator P,

$$Gt(P) = Q\,,$$

where Q is a polynomial operator, but not necessarily identical with P. A continuous operator which satisfies this weaker condition will be said to maintain[30] the degree r. In justification of this terminology, we point out that

$$\delta^{r+1}Q = \delta^{r+1}Gt(P) = G\delta^{r+1}t(P) = 0\,,$$

showing that if Q is a polynomial operator, its degree does not exceed that of P.

Therefore, if G maintains the degree r and P is a polynomial operator of degree r or less,

$$D^{r+1}Gt(P) = D^{r+1}Q = 0\,,$$

which shows that the operator $D^{r+1}G$ annihilates all polynomial traces of degree r or less. If G is of finite range, we have, therefore, by Theorem 3,

$$D^{r+1}G = \delta^{r+1}H\,,$$

[29] These definitions and this relationship are, of course, strictly analogous to those relating to the characteristic functions employed by Schoenberg in connection with smoothing formulas.

[30] This concept is due to Schoenberg, who uses, however, the word "preserve" rather than "maintain." One reason for using a slightly different terminology is the fact that our concept differs in certain details from his. His definition requires that Q be of the same degree as P, and that it differ from P by a polynomial operator of lower degree. This implies, as he points out, that the operator G preserves the leading term of a polynomial of degree r or less. We wish to include certain operators (such as $\delta^k M^l$) which actually reduce the degree. Moreover, our definition has the advantage, from our standpoint, that any linear combination of operators which maintain the degree r is an operator which maintains the degree r. Operators which preserve the degree, according to Schoenberg's definition, do not have this property.

where H is a distribution operator of finite range. Since G is a continuous operator, and therefore of order not exceeding -1, while δ^{r+1} is of order zero, it follows from Theorem 1 that H is of order not exceeding r.

Now, $M^{r+1}H$ is also of finite range, and

$$D^{r+1}M^{r+1}H = \delta^{r+1}H = D^{r+1}G \,.$$

Since a distribution operator of finite range is completely determined by its $(r+1)$th derivative, we have, therefore, $G = M^{r+1}H$. On the other hand, if $G = M^{r+1}H$ and P is a polynomial operator of degree r or less,

$$D^{r+1}Gt(P) = H\delta^{r+1}t(P) = 0 \,,$$

and G maintains the degree r. We have shown, therefore, that a continuous operator G of finite range maintains the degree r if and only if condition (a) of Rule 1 is satisfied.

In connection with discrete interpolation, it has been shown (*TSA* I, 349) that an operator which maintains the degree r, when applied to values of a polynomial of degree r or less, gives the same result whether applied as an interpolation formula or as a graduation formula. The continuous analogue of this observation is provided by the following theorem.

Theorem 4. A continuous operator G of finite range maintains the degree r if and only if $Gt(P) = GP$ for every polynomial operator P of degree r or less.

If G is of finite range, it has already been shown that GP is a polynomial operator. Thus, $Gt(P)$, if equal to GP, is a polynomial operator, and therefore G maintains the degree r.

To prove the converse, let us suppose that G maintains the degree r. This implies that $Gt(P) = Q$, a polynomial operator of degree r or less. Now, the basic function of Q, being a polynomial, is completely determined by its values for integral arguments. However, a little reflection will convince the reader that, in view of Theorem 2, these are identical with the values for the same integral arguments of the basic function of $Pt(G)$, which evidently is also a polynomial operator of degree r or less. Therefore,

$$Gt(P) = Q = Pt(G) \,. \tag{31}$$

Moreover, replacing P by E^hP, which is a polynomial operator of the same degree,

$$Gt(E^hP) = E^hPt(G) = E^hQ \,,$$

or

$$GE^{-h}t(E^hP) = Q \,.$$

Let $f(x)$, $p(x)$, and $g(x)$ denote the basic functions of G, P, and Q, respectively. Then, by Theorem 2,

$$E^{-h}t(E^hP)f(x) = g(x) \,,$$

or

$$\sum_{n=-\infty}^{\infty} p(n+h)\, f(x-n-h) = g(x)\,.$$

Integrating both sides with respect to h between the limits 0 and 1 and rearranging the left member gives

$$\int_{-\infty}^{\infty} p(y)\, f(x-y)\, dy = g(x)\,,$$

or, in other words,

$$Pf(x) = g(x)\,.$$

It follows from Theorem 2 that $g(x)$ is the basic function of GP. In other words,

$$GP = Q = Gt(P)\,,$$

as required by the theorem.

We shall say that a distribution operator H is *correct to rth differences* if, for every polynomial operator P of degree r or less, $HP = P$. It follows from this definition, equation (30), and Theorem 4 that a continuously defined interpolation formula of finite span is correct to rth differences if and only if its characteristic operator G maintains the degree r and is also correct to rth differences.

SYMBOLIC EXPANSIONS IN POWERS OF D

Using the symbolic expansion

$$E^{-t} = e^{-tD} = 1 - tD + \tfrac{1}{2}t^2D^2 - \ldots\,, \tag{32}$$

any continuous or discrete operator of finite range, and therefore any distribution operator of finite range, can be expanded symbolically in a series of powers of D with numerical coefficients. In the case of a continuous operator K with basic function $f(t)$, the coefficient of D^h (for $h \geq 0$) in its symbolic expansion is

$$\frac{(-1)^h}{h!}\int_{-\infty}^{\infty} t^h f(t)\, dt\,. \tag{33}$$

In the symbolic expansion of the discrete operator

$$J = \Sigma_i c_i E^{-t_i}\,, \tag{34}$$

the coefficient of D^h is

$$\frac{(-1)^h}{h!}\, \Sigma_i c_i t_i^h\,. \tag{35}$$

The first few terms of the symbolic expansions in powers of D of a number of continuous and discrete operators are given in Table 2.

These expansions can be considered valid only when applied to polynomials, since, for other functions, the resulting infinite series usually do not converge. In the case of polynomials, however, the expansion (32) is clearly valid, since it amounts to expanding the polynomial in a Taylor series, which, in this case, terminates after a finite number of terms. Therefore, the symbolic expansion in powers of D of any distribution operator of finite range, when applied to a polynomial of any degree, gives the same result as the operator itself. It is evident, therefore, that a distribution operator of finite range is correct to rth differences if and only if its symbolic expansion is of the form $1 + kD^{r+1} + \ldots$. The following two theorems will be needed.

Theorem 5. If H_1 and H_2 are distribution operators of finite range, the symbolic expansion of their product in powers of D is identical with the product of their symbolic expansions.

For, if H_1 and H_2 are applied in succession to any polynomial, the result is the same as that of applying their symbolic expansions in succession (since the result of the first application is again a polynomial), which, in turn, is clearly the same as would be obtained by applying the product of their symbolic expansions. This product, then, must be identical with the symbolic expansion of the product of the operators, since two different symbolic expansions could not give the same result for all polynomials.

Theorem 6. A distribution operator H of finite range can be expressed in the form $D^q H_q$, where H_q is a distribution operator of finite range, if and only if the symbolic expansion of H lacks powers of D less than q. The operator H_q is unique.

If it is given that $H = D^q H_q$, where H_q is of finite range, it follows at once from Theorem 5 that the symbolic expansion of H lacks powers less than q.

To prove the converse, we shall consider first the case $q = 1$. Let

$$H = K + J_0 + J_1 D + \ldots + J_r D^r , \tag{36}$$

where K has the basic function $f(t)$ and J_0 is of the form (34), and suppose that the symbolic expansion of H in powers of D lacks powers less than q. Now, let

$$H_1 = K_1 + J_1 + J_2 D + \ldots + J_r D^{r-1} ,$$

where the basic function of K_1 is

$$f_1(t) = \int_{-\infty}^{t} f(s)\, ds + \sum_{t_i < t} c_i .$$

It is clear that $f_1(t)$ has a jump discontinuity of jump c_i at each argument t_i and has, for all other arguments, the derivative $f(t)$. It follows from equation (24) that $DH_1 = H$.

We shall now show that H_1 is of finite range. It is clearly sufficient to show that K_1 is of finite range. Let (a, b) be the range of H. Evidently, $f_1(t) = 0$ for $t < a$. For $t > b$, $f_1(t)$ is constant and equal to

$$\int_{-\infty}^{\infty} f(s)\, ds + \Sigma_i c_i .$$

But, by formulas (33) and (35), this is precisely the negative of the term free of D in the symbolic expansion of H, which, by hypothesis, is equal to zero. Therefore, H_1 is of finite range.

Moreover, since the symbolic expansion of H lacks all powers less than q of D, by Theorem 5, the expansion of H_1 lacks all powers less than $q - 1$. Applying a second time the case $q = 1$, we find that there exists a distribution operator H_2 of finite range, such that $DH_2 = H_1$, and its symbolic expansion lacks all powers less than $q - 2$ of D, and so on. By repeated application of this result, we eventually conclude that there is a distribution operator H_q of finite range, such that $D^qH_q = H$. The uniqueness of H_q follows from the fact that a distribution operator of finite range is completely determined by its qth derivative.

When the conditions of Theorem 6 are satisfied, we may without ambiguity write $D^{-q}H$ to denote the unique operator defined by the theorem. The symbolic equation $D^{-k}\delta^k = M^k$ in formula (7*b*) is an example. In such cases, it will sometimes be convenient to multiply out such a product into an expression containing various negative powers of D, even though some individual terms may not represent uniquely determined operators. For example, $\mu D^{-2} - \delta D^{-3}$ will mean $D^{-2}(\mu - M)$, since the symbolic expansion of $\mu - M$ lacks powers of D less than 2.

COMPOSITE POLYNOMIAL OPERATORS

We shall define a *composite polynomial operator* as one whose basic function is given, in successive intervals, by different polynomials, like the basic function of a polynomial interpolation formula. The definition does not require, however, that the transitions from one polynomial to another shall occur at unit intervals, or even at any regular intervals. The *degree* of a composite polynomial operator is the maximum degree occurring in any of the polynomials used to define its basic function. If P is a composite polynomial operator of degree q, it is clear that $D^{q+1}P$, when expressed in the form (25), lacks the continuous term, and we have

$$D^{q+1}P = J_{-q-1} + J_{-q}D + \ldots + J_{-1}D^q .$$

If P is of finite range, it is completely determined by its $(q + 1)$th derivative, and we may write, in accordance with the remarks following the proof of Theorem 6,

$$P = J_{-1}D^{-1} + J_{-2}D^{-2} + \ldots + J_{-q-1}D^{-q-1} . \qquad (37)$$

It follows from Theorem 6 that the symbolic expansion in powers of D of the right member of equation (37) is free from negative powers of D.

This formula provides a means of obtaining the characteristic operator corresponding to a given basic function. We take as an illustration the basic function $L(x)$ of Karup's formula, as given by the expressions (2). Differentiating these expressions successively, we obtain the data in the following table:

x	$L(x)$	$L'(x)$	$L''(x-0)$	$L''(x+0)$	$L'''(x-0)$	$L'''(x+0)$
-2.	0	0	0	-1	0	3
-1.	0	$\frac{1}{2}$	2	4	3	-9
0.	1	0	-5	-5	-9	9
1.	0	$-\frac{1}{2}$	4	2	9	-3
2.	0	0	-1	0	-3	0

From this table we compute the jumps of the second and third derivatives as follows:

x_i	b_i''	b_i'''
-2.	-1	3
-1.	2	-12
0.	0	18
1.	-2	-12
2.	1	3

By formula (26), we compute that $J_{-3} = -2\mu\delta^3$ and $J_{-4} = 3\delta^4$. By formula (37) we have

$$P = -2\mu\delta^3D^{-3} + 3\delta^4D^{-4} = 3M^4 - 2\mu M^3 = M^3(3M - 2\mu) .$$

If a distribution operator H of order r can be expressed in the form

$$H = J_lD^l + J_{l+1}D^{l+1} + \ldots + J_rD^r , \qquad (38)$$

where J_l is different from zero, we shall say that it is of *rank* l. For example, $M^r = \delta^rD^{-r}$ is of rank and order $-r$. If l is positive or zero, it is clear that the form (38) is identical with the form (36) and that H lacks the continuous term K. If l is a negative integer, this implies, in view of equation (37), that K is a composite polynomial operator of degree $-l - 1$. On the other hand, it is evident that any distribution operator having a composite polynomial operator as its continuous term K can be expressed in the form (38). The following theorem is an immediate consequence of the definition of rank.

Theorem 7. If H_1 and H_2 are distribution operators of finite range and of rank l_1 and l_2, respectively, their product is of rank $l_1 + l_2$.

PROOF OF THE RULES

Proof of Rule 1. It has been shown that a continuously defined interpolation formula is correct to rth differences if and only if its characteristic operator G maintains the degree r and is correct to rth differences. It has also been pointed out that maintenance of the degree r is equivalent to condition (a) of Rule 1, while correctness of G to rth differences is equivalent to the requirement that its symbolic expansion in powers of D be of the form $1 + kD^{r+1} + \ldots$. It remains only to show, therefore, that the latter condition is equivalent to condition (b) of Rule 1. Without going into the interpretation of a negative power of M, it will suffice to say that by the symbolic expansion of M^{-r-1} we mean merely the unique power series in D whose product with the symbolic expansion of M^{r+1} is identically 1. It follows from Theorem 5 that the symbolic expansion of G is the product of the symbolic expansions of M^{r+1} and H. Therefore, the terms in the expansion of H up to and including the one containing D^r are uniquely determined by the requirement that the symbolic expansion of G be of the form $1 + kD^{r+1} + \ldots$, and are not affected by the coefficients of D^{r+1} and higher powers in that expansion. Thus, they must agree with the corresponding terms in the expansion of M^{-r-1}. This completes the proof.

Proof of Rule 2. If (a, b) is the range of G, it follows from the remarks made earlier concerning the range of a derivative that the range of $D^{r+1}G = \delta^{r+1}H$ is also (a, b). It is then fairly clear that the range of H is[31]

$$\left(a + \frac{r+1}{2}, \qquad b - \frac{r+1}{2}\right).$$

[31] If (α, β) is the range of H, and if any of the J's in the expression for H contain terms involving $E^{-\alpha}$ and $E^{-\beta}$, this is obvious. If $E^{-\alpha}$ does not appear, then it must be true that any interval, however small, extending to the right of the argument α contains arguments for which the basic function $f(t)$ of K, the continuous part of H, is different from zero. Moreover, in a sufficiently small interval to the right of the argument

$$a - \frac{r+1}{2},$$

the basic function of $\delta^{r+1}K$ reduces to

$$(-1)^{r+1} f\left(t - \frac{r+1}{2}\right):$$

thus any such interval contains nonzero values of this basic function. Similar considerations apply at the upper end of the range.

This exemplifies a general theorem to the effect that if the ranges of two distribu-

If s denotes the span, $b - a$, of G and h is the span of H, we have, at once, $s = h + r + 1$. If s is an integer, formula (1) shows that it is the number of terms in the linear compound form of the formula, and it is clear that h is an integer. This proves Rule 2. If $h = n + f$, where n is an integer and f a proper fraction, then $s = n + r + 1 + f$, and formula (1) shows that the number of terms is $n + r + 1$ for some arguments and $n + r + 2$ for others. This proves footnote 10.

Proof of Rule 3. It follows from the definition of rank and from equation (37) that, if a continuously defined interpolation formula of finite span is of degree q, the operator K of equation (29) is of rank $-q - 1$. In view of Theorem 7, this is the rank of G, since the discrete operator U is of rank zero. Since $M^{r+1} = \delta^{r+1}D^{-r-1}$ is of rank $-r - 1$, it follows from Rule 1(a) that H is of rank $-q + r$. Similarly, if the order of contact of successive arcs is always at least p, this implies that the order of the operator K in equation (29) is at most $-p - 2$. Since the discrete operator U is of order zero, Theorem 1 shows that the characteristic operator G is of order at most $-p - 2$. Since M^{r+1} is of order $-r - 1$, it follows that H is of order at most $r - p - 1$.

On the other hand, if H is of the form (6), it is, by definition, of rank $-q + r$ and order $r - p - 1$, which implies that G, and therefore the operator K of equation (29), is of rank $-q - 1$ and order $-p - 2$. It follows that the formula is of degree q and that its successive interpolating arcs always have order of contact at least p. This proves Rule 3(a), including footnote 12.

If the formula is not a polynomial interpolation formula and $p > r$, it is still true that H is of order at most $r - p - 1$. However, only for a polynomial interpolation formula can H be expressed in the form (6) and the order determined by inspection of the subscripts of the J's, when $p > r$. For other formulas, the order may be ascertained by successive differentiation of the basic function to determine the number of continuous derivatives.

For any polynomial interpolation formula, G is a composite polynomial operator, and can be expressed in the form (37). Since it is of order $-p - 2$, it takes the form

$$G = J_{-q-1}D^{-q-1} + J_{-q}D^{-q} + \ldots + J_{-p-2}D^{-p-2}.$$

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tion operators H_1 and H_2 are (a_1, b_1) and (a_2, b_2), respectively, the range of their product is $(a_1 + a_2, b_1 + b_2)$. It is plain enough that the range of $H_1 H_2$ cannot extend beyond that indicated. To prove rigorously that it can never be smaller is somewhat difficult, and is not necessary for the purposes of this paper.

But this is just what we obtain by taking $r = -1$ in equation (6). This proves footnote 13.

It is obvious from the definitions of rank and order of a distribution operator that the rank can never exceed the order. Since the rank of H is $-q + r$ and its order is at most $r - p - 1$, we have $r - p - 1 \geq -q + r$, which gives $q \geq p + 1$. Moreover, the rank of H cannot be positive, for then the symbolic expansion of H, and therefore of G, in powers of D alone would lack the term free of D, and the latter expansion could not be of the form $1 + kD^{r+1} + \ldots$. Thus, the rank of H is at most zero. Since the rank of H is $-q + r$, this gives at once $q \geq r$. This completes the proof of Rule 3(b).

Since H is of the form (38), its continuous part K is a composite polynomial operator, and can be expressed in the form (37). The remark immediately following the latter equation establishes Rule 3(c).

Proof of Rule 4. Study of formula (1) shows that the curve of interpolated values produced by a polynomial interpolation formula can change from one polynomial to another at the argument z only if a similar change in the basic function $L(x)$ occurs at an argument which differs from z by an integer. It follows that, for an end-point formula, these "points of junction" of $L(x)$ occur only at integral arguments, while, for a mid-point formula, they occur only at arguments which are odd multiples of $\frac{1}{2}$. Now, these points of junction are precisely the arguments at which discontinuities eventually appear in some of the derivatives when $L(x)$ is successively differentiated. Further, it follows from formulas (25) and (26) that they are the negatives of the exponents of E appearing in the J's after G has been differentiated successively until the continuous operator in the right member of formula (25) has disappeared.

We note also that a Stirling operator has only integers as exponents of E when it is expressed in the form (26), while a Bessel operator has only odd multiples of $\frac{1}{2}$ as exponents when expressed in this form. Finally, we observe that multiplying a Stirling or Bessel operator by an even power of δ gives the same kind of operator as before multiplication, while multiplication by an odd power of δ changes a Stirling to a Bessel operator and vice versa. Since $D^{r+1}M^{r+1} = \delta^{r+1}$, all parts of Rule 4 follow at once from these remarks.

It will be noted that condition (b) of Rule 1 has not been used in this proof. Therefore, it applies to any operator of the form $M^{r+1}H$, whether correct to rth differences or not. This proves the statement made in Step 5(a) of the suggested procedure for obtaining a formula from its characteristic operator.

Proof of Rule 5. It was previously pointed out that a symmetrical interpolation formula is precisely one whose basic function is an even function: that is, $L(-x) = L(x)$. It is clear that the derivative of an even function is an odd function (that is, a function $f(x)$ such that $f(-x) = -f(x)$), and vice versa. We shall call a discrete operator *symmetric* if the coefficients of E^t and E^{-t} are always equal; *skew-symmetric*, if such coefficients are always negatives of each other. Further, we shall call a distribution operator H *symmetric* if, when it is expressed in the form (36), the basic function of K is an even function and the J's are symmetric or skew-symmetric according as they are coefficients of even or odd powers of D (counting D^0 as an even power). If the basic function of K is an odd function and the J's are symmetric or skew-symmetric according as they are coefficients of odd or even powers of D, we shall call H *skew-symmetric.* In view of these definitions, the general properties of distribution operators, and equation (24), it is not difficult to verify that the product of two symmetric or two skew-symmetric distribution operators is symmetric, while the product of a symmetric and a skew-symmetric distribution operator is skew-symmetric. In particular, both the operators D^{r+1} and δ^{r+1} are symmetric or skew-symmetric according as r is odd or even, and this is true also of $D^{r+1}G$ if the interpolation formula is symmetrical. Since $D^{r+1}G = \delta^{r+1}H$, it is clear that H must be a symmetric distribution operator. Finally, we remark that a discrete operator which is expressed as a Stirling operator or a Bessel operator, or a sum of both, is symmetric or skew-symmetric precisely when it contains, respectively, only even or only odd powers of δ. This completes the proof of Rule 5.

Proof of Rule 6. By definition, the trace of an interpolation formula is a discrete operator involving only integral powers of E. By means of Stirling's formula, which can be expressed symbolically as

$$E^x = 1 + x\mu\delta + \tfrac{1}{2}x^2\delta^2 + \tfrac{1}{6}x(x^2 - 1)\mu\delta^3 + \ldots, \tag{39}$$

any such operator can be expressed as a Stirling operator, since, for any integral value of x, this series terminates after a finite number of terms. If the formula is correct to rth differences, $Gt(P) = P$ for any polynomial operator P of degree r or less. However, by equation (31), this implies that $Pt(G) = P$. But this can be true for *all* polynomial operators of degree r or less only if $t(G)$, when expressed as a Stirling operator, is of the form $1 + k\delta^{r+1} + \ldots$. If the formula is also symmetrical, E^n and E^{-n} have the same coefficient, say a_n, in $t(G)$. Since the coefficients of odd powers of δ in equation (39) are odd functions of x, these terms will cancel in the sum $a_nE^n + a_nE^{-n}$. This proves Rule 6(a).

In a symmetrical formula, $L(-x) = L(x)$, and therefore the range of G

is of the form $(-b, b)$, and $s = 2b$. If this is odd, the integral arguments falling in the interval of nonzero values of $L(x)$ are those from

$$-\frac{s-1}{2} \text{ to } \frac{s-1}{2},$$

inclusive. Thus, the span of $t(G)$ is at most $s - 1$. If s is even, then at most the integral arguments from $-s/2$ to $s/2$, inclusive, lie within or at the extremities of the interval of nonzero values. However, if the formula has order of contact at least zero, $L(x)$ is free from discontinuities and $L(b) = L(-b) = 0$. Thus the integral arguments corresponding to nonzero values of $L(x)$ are at most those from

$$-\frac{s-2}{2} \text{ to } \frac{s-2}{2},$$

inclusive, and the span of $t(G)$ is at most $s - 2$. This completes the proof of Rule 6.

Proof of Rule 7. If $f(x)$ is the basic function of K and $F(x)$ the basic function of $\delta^{2i}K$, we have, by Theorem 2, $F(x) = \delta^{2i}f(x)$. Thus, by the definition of the trace,

$$t(\delta^{2i}K) = \sum_{n=-\infty}^{\infty} [\delta^{2i}f(n)]E^{-n}.$$

Expressing $\delta^{2i}f(n)$ in terms of the ordinates $f(n)$ and rearranging terms in the summation gives

$$t(\delta^{2i}K) = \sum_{n=-\infty}^{\infty} f(n)\,\delta^{2i}E^{-n} = \delta^{2i}t(K).$$

The second equation in Rule 7 is merely another way of stating the same result. Since $t(\delta^{2i}K)$ must contain the factor δ^{2i}, we obtain $t(K)$ by removing this factor. This is how δ^{-2i} in the second equation is to be interpreted. Thus, when negative powers of δ appear, as in Examples 4 and 5, their coefficients must vanish.

Rule 8 follows at once from the definition of the trace.

Proof of formulas (7). Formula (7*a*) follows from the definition of M and the elementary properties of distribution operators. Formula (7*b*) has been given previously as an example of the application of Theorem 6. Formula (7*c*) is to be accepted only with certain qualifications. If H is a distribution operator such that $D^{-k}H$ has a unique interpretation in accordance with Theorem 6, then

$$D^{-k}H = \delta^{-k}M^kH.$$

In other words, if H_k is a distribution operator of finite range such that $D^k H_k = H$, then clearly $\delta^k H_k = M^k H$. Moreover, H_k is the only distribution operator of finite range having this property. For if H'_k were another operator of finite range such that $\delta^k H'_k = M^k H$, we should have

$$D^k(M^k H'_k) = D^k(M^k H_k) = M^k H .$$

Since a distribution operator of finite range is completely determined by its kth derivative, this implies that $M^k H'_k = M^k H_k$ or $M^k(H'_k - H_k) = 0$. Multiplication by D^k then gives $\delta^k(H'_k - H_k) = 0$. This cannot be true if $H'_k - H_k$ is any nonzero operator of finite range.

SPECIAL PROPERTIES OF THE OPERATOR M^k

Schoenberg (*op. cit.*, p. 68) has given an explicit expression for the basic function of the operator M^k. It will be noted that, in the discrete case, an expression for the coefficients of the different powers of E in $[m]^k$ can be obtained from the multinomial theorem, since $[m] = E^{-(m-1)/2}(1 + E + E^2 + \ldots + E^{m-1})$, and the basic function of M^k can be deduced therefrom by taking the limit of the kth power of this expression. It can, however, be established directly by making use of a function which Schoenberg denotes by x_+^k, defined by

$$x_+^k = \begin{cases} x^k & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 . \end{cases}$$

It will be seen that, for $k \geq 0$, an indefinite integral of x_+^k is $x_+^{k+1}/(k+1)$. We shall prove by induction the formula

$$L_k(x) = \frac{1}{(k-1)!}\,\delta^k x_+^{k-1}, \tag{40}$$

where $L_k(x)$ denotes the basic function of M^k. It is evident from the definition of M that its basic function is

$$L_1(x) = \begin{cases} 1 & \text{for} \quad |x| < \frac{1}{2} , \\ 0 & \text{for} \quad |x| > \frac{1}{2} . \end{cases}$$

Clearly, equation (40) for $k = 1$ gives the same result. Now, if equation (40) is true for $k = j$, we have, by Theorem 2,

$$L_{j+1}(x) = ML_j(x) = \frac{1}{(j-1)!}\int_{-1/2}^{1/2} \delta^j (x+t)_+^{j-1}\,dt$$
$$= \frac{1}{(j-1)!}\,\delta^j \int_{-1/2}^{1/2} (x+t)_+^{j-1}\,dt$$
$$= \frac{1}{(j-1)!}\,\delta^j \left[\frac{(x+t)_+^j}{j}\right]_{-1/2}^{1/2} = \frac{1}{j!}\,\delta^{j+1} x_+^j .$$

It is evident from equation (40) that $L_k(x)$ is a composite function made up of polynomial arcs (a different polynomial in each unit interval between successive integral arguments if k is even, or between successive odd multiples of $\frac{1}{2}$ if k is odd) of degree $k - 1$ and (for $k > 2$) having $k - 2$ continuous derivatives. Its $(k - 1)$th derivative is a step-function having discontinuities at unit intervals. However, without reference to formula (40), these properties could have been deduced from the degree and order of M^k and the fact that its $(k - 1)$th derivative is $M\delta^{k-1}$.

The properties of M are analogous to those of $[m]$. The coefficients of successive powers of E in $[m]^k$ are values of a composite function made up of successive sets of m values, each set lying on a polynomial of degree $k - 1$. These successive sets "interlock" when $k > 2$, and the $(k - 1)$th finite differences, taken in sets of m, lie on a step-function. For M^k these properties are translated into the continuous analogues.

Another useful property of the operator M^k concerns its expansion in the form of an infinite Stirling or Bessel operator. By means of the expansion (39), we can obtain a symbolic Stirling expansion of any continuous operaor K of finite span, analogous to its symbolic expansion in powers of D. Similarly, using the symbolic expansion

$$E^x = \mu + x\delta + \tfrac{1}{2}(x^2 - \tfrac{1}{4})\mu\delta^2 + \tfrac{1}{6}x(x^2 - \tfrac{1}{4})\delta^3 + \ldots,$$

derived from Bessel's formula, we can obtain a Bessel expansion of K. These expansions have similar properties to those of the symbolic expansions in powers of D. When applied to any polynomial, they must give the same result as the operator K itself. Now, if K maintains the degree r, it follows from Theorem 4 and formula (31) that $Pt(K) = KP$ for every polynomial operator P of degree r or less. Therefore, the trace of K and the symbolic Stirling expansion of K give identical results when applied to any polynomial of degree r or less. Since both are Stirling operators, this can be the case only if they agree up to and including the term containing δ^r. On the other hand, $E^{-1/2}t(E^{1/2}K)$ is a Bessel operator which we may call the *pseudotrace* of K. In case K is the characteristic operator of an interpolation formula, it indicates the interpolated value produced by the formula for arguments midway between the given values. Replacing G by $E^{1/2}K$ in equation (31) and applying Theorem 4 gives $PE^{-1/2}t(E^{1/2}K) = KP$ for every polynomial operator P of degree r or less. Thus, the pseudotrace of K and its symbolic Bessel expansion must agree up to and including the term in δ^r.

Now, the operator M^r is symmetric (see proof of Rule 5), maintains the degree $r - 1$, and is of span r, so that, by Rule 7, the span of its trace does not exceed $r - 1$, and, by similar reasoning, the span of its pseudotrace (for $r > 1$) does not exceed $r - 1$. Thus, the trace and pseudotrace, when expressed as Stirling and Bessel operators, respectively, do not contain differences beyond the $(r-1)$th. They are therefore completely characterized as the terms up to and including δ^{r-1} of the symbolic Stirling and Bessel expansions.[32] Finally, it is fairly obvious that the trace of μM^r is obtained by multiplying the pseudotrace of M^r by μ. The product can then be expressed as a Stirling operator by means of the substitution $\mu^2 = 1 + \frac{1}{4}\delta^2$. The traces given in Table 3 were obtained in the manner described.

DETERMINATION OF THE BASIC FUNCTION OF A GIVEN CONTINUOUS OPERATOR

Rule 1(a) and equation (37) show that the characteristic operator of any polynomial interpolation formula can be expressed in terms of positive powers of M, positive and negative powers of D, and discrete operators. Upon multiplying the expression out, all positive powers of D can be eliminated by means of formula (7*a*). All terms not containing negative powers of D will then be of the form JM^k, where J is a discrete operator. Now, it follows from Theorem 2 and equation (40) that the basic function of such a term is

$$\frac{1}{(k-1)!} J \delta^k x_+^{k-1}. \tag{41}$$

By means of formula (7*c*), the negative powers of D can be replaced by negative powers of δ. Some of the latter may cancel out, but some will usually remain. It will be found, however, that the terms containing negative powers of δ, taken collectively, correspond to a uniquely determined basic function of finite range.

This is best explained by an example. We shall take as an illustration the operator $D^{-2}(\mu - M)$, which appears in several instances as a term in the characteristic operators shown in Table 1. By formula (7*c*), this may be written as $\delta^{-2}(\mu M^2 - M^3)$. Therefore, equation (41) gives

$$L(x) = \delta^{-2}(\mu\delta^2 x_+ - \tfrac{1}{2}\delta^3 x_+^2) = \mu x_+ - \tfrac{1}{2}\delta x_+^2 .$$

For $x > \frac{1}{2}$, the last expression is equivalent to $\mu x - \frac{1}{2}\delta x^2 = x - x = 0$, and it also vanishes for $x < -\frac{1}{2}$ by virtue of the definition of x_+. Therefore, any nonzero values of this function are confined to the interval $-\frac{1}{2} \le x \le \frac{1}{2}$. In this interval,

$$L(x) = \tfrac{1}{2}(x + \tfrac{1}{2}) - \tfrac{1}{2}(x + \tfrac{1}{2})^2 = -\tfrac{1}{2}(x^2 - \tfrac{1}{4}) .$$

[32] This was first pointed out by Schoenberg, *op. cit.*, p. 120.

RECURRENCE RELATIONS FOR THE SPECIAL OPERATORS

It will be recalled that the special operator Q_i corresponds to the expression

$$x^i u_{n+1} + y^i u_n$$

in an Everett-type formula. Bearing in mind that this expression occurs in a formula for v_{n+x}, the right member of formula (1) for this particular case would be

$$L(x-1)u_{n+1} + L(x)u_n \,.$$

Comparison of the two expressions gives:

$$L(x-1) = x^i \,, \quad L(x) = (1-x)^i \,,$$

since $y = 1 - x$. As x is restricted to the interval $0 < x < 1$, this gives for $L(x)$ the expressions

$$\begin{array}{lll} 0 & \text{for} & x < -1 \\ (1+x)^i & \text{for} & -1 < x < 0 \\ (1-x)^i & \text{for} & 0 < x < 1 \\ 0 & \text{for} & x > 1 \,. \end{array}$$

A similar analysis for the operator T_i gives the basic function

$$\begin{array}{lll} 0 & \text{for} & x < -1 \\ (1+x)^i & \text{for} & -1 < x < 0 \\ -(1-x)^i & \text{for} & 0 < x < 1 \\ 0 & \text{for} & x > 1 \,. \end{array}$$

Application of formula (24) to these two operators gives, for $i \geq 1$, the equations

$$DQ_i = iT_{i-1} \,, \quad DT_i = iQ_{i-1} - 2 \,, \tag{42}$$

while $DQ_0 = E - E^{-1} = 2\mu\delta$, and $DT_0 = \delta^2$. Therefore, $Q_0 = 2\mu\delta D^{-1} = 2\mu M$, and $T_0 = \delta^2 D^{-1} = \delta M$. Operating with M and dividing by i on both sides of both equations (42), replacing i by $i + 1$, and making use of the fact that $M = \delta^{-1}T_0$, we have

$$MT_i = \frac{1}{i+1}\,\delta Q_{i+1} \,,$$

and

$$MQ_i = \frac{1}{i+1}\,(2\,\delta^{-1}T_0 + \delta T_{i+1}) \,.$$

By successive application of these two relations, starting with $M = \delta^{-1}T_0$ and $\mu M = \frac{1}{2}Q_0$, all the expressions in Table 4 are easily obtained.

On the other hand, multiplying both sides of both equations (42) by $\delta^{j-1}M$, we obtain:

$$\delta^j Q_i = iM(\delta^{j-1}T_{i-1}), \quad \delta^j T_i = iM(\delta^{j-1}Q_{i-1}) - 2\delta^{j-1}M.$$

The expressions in Parts IB and IIB of Table 5 are easily obtained by repeated application of these two relations, starting with $Q_0 = 2\mu M$ and $T_0 = \delta M$.

In a similar manner, we find that the basic function of V_i is

$$\begin{array}{lll} 0 & \text{for} & x < -\frac{1}{2} \\ (\frac{1}{2} + x)^i & \text{for} & -\frac{1}{2} < x < 0 \\ (\frac{1}{2} - x)^i & \text{for} & 0 < x < \frac{1}{2} \\ 0 & \text{for} & x > \frac{1}{2}, \end{array} \tag{43}$$

while that of W_i is

$$\begin{array}{lll} 0 & \text{for} & x < -\frac{1}{2} \\ (\frac{1}{2} + x)^i & \text{for} & -\frac{1}{2} < x < 0 \\ -(\frac{1}{2} - x)^i & \text{for} & 0 < x < \frac{1}{2} \\ 0 & \text{for} & x > \frac{1}{2}. \end{array} \tag{44}$$

Again, application of formula (24) gives, for $i \geq 1$,

$$DV_i = iW_{i-1}, \quad DW_i = iV_{i-1} - (\tfrac{1}{2})^{i-1}.$$

Multiplying both sides of both equations by $\delta^{j-1}M$, we obtain

$$\delta^j V_i = iM(\delta^{j-1}W_{i-1}), \quad \delta^j W_i = iM(\delta^{j-1}V_{i-1}) - (\tfrac{1}{2})^{i-1}\delta^{j-1}M. \tag{45}$$

Moreover, it is evident from the expressions (43) that $V_0 = M$, while it follows from expressions (44) and formula (24) that $DW_0 = 2\mu - 2$, or $\delta W_0 = M(2\mu - 2)$. From these expressions for V_0 and δW_0, the remaining entries in Part III of Table 5 are easily obtained by repeated application of equations (45).

DISCUSSION OF PRECEDING PAPER

KINGSLAND CAMP:

It is well to have this monumental analytic monograph permanently on record in our *Transactions* even if its difficult nature prevents mastery or frequent reference by most of us. A helpful additional schedule, whether compiled by the authors or by some student, would be one showing all the formulas of Table 1 in directly usable working form together with brief comments on their individual characteristics and, if possible, on some of the types of functions they work best with. The practical user will then most probably familiarize himself with the few particular formulas that operate dependably well on the widest variety of functions he encounters, more especially on functions not of polynomial form.

Along with such a schedule, it is likely that practical workers would welcome a simple numerical index, if our research specialists would contrive one, for rating the suitability of the convenient polynomial interpolation formulas to use on nonpolynomial functions. It must be evident that smoothness as measured by the standard deviation in the third or other difference order, and fidelity when measured in similar inbred fashion—measured, that is, by other polynomials—are no criteria of a formula's suitability for interpolating precisely computed financial functions or verified accurate experimental data. It may well be, of course, that these indexes serve very well for rating formulas that are designed to adjust and partially graduate data in five-year age groups, or population studies with their inevitable age misstatements.

For the present, the best test of a reproducing formula is the practical one: try it on trigonometric or other mathematical curves for which missing values are readily supplied. Such functions are considered as *de facto* smooth, yet they frequently present difference-series, within even short intervals, that reproducing formulas considered to have unsatisfactory "smoothness" will work better on than do the formulas more generally approved nowadays.

There may be more than one reason why (see the authors' "General Observations Concerning the Rules") "in the earlier work on smooth-junction interpolation so much emphasis was placed on formulas of minimum degree." The earlier workers were more inclined to rate their formulas by success on nonpolynomial functions because they wanted general-duty processes and because presently accepted criteria had not been devised or at least had not gained wide acceptance. Also, we nowadays work with machines for which it is easy to use the formulas in quasi-Lagrangian expressions and refer to such different ones as Sprague's, Henderson's, and Shovelton's, for example, as all being "six-point" formulas. Thus we regard them as equally easy to work, although in the old days of hand-powered machines Henderson's was the easiest because then the chief part of the work was proceeding from one value to the next within an interval, which involved fewer product-sums at each step if the osculating arcs were of lower degree.

Nevertheless, the present tendency to relegate, say, Henderson's formula to "historical interest only" (Wolfenden, *Population Statistics and Their Compilation*, p. 140) is probably a mistake: experiments and observation provide other series than some mathematical functions, that exhibit rapidly steepening curves. On such data, neither Sprague's formula with its capacity for reproducing so improbable a curve in nature as a fourth-order polynomial, nor Shovelton's with its feature of satisfying the accepted smoothness test, will work anywhere nearly as well as Henderson's. In short, for a given order of continuity, exact or approximate, a formula of minimum degree is more dependably accurate on a nonpolynomial function. Accuracy can be quite as desirable a feature as "smoothness" or as fidelity to a high-order polynomial. I tried to point this out years ago in a paper (*TASA* XXXVIII, 16) and once or twice thereafter in discussions.

Systematic bivariate interpolation is not often required in our work, but when it is required it is troublesome enough to justify more attention than it has so far received in actuarial journals. Not impossibly several other readers were anticipating, when the subject of this paper was announced, that it might include a solution. It may stimulate research, and possibly even extension and generalization of the techniques of Messrs. Greville and Vaughan, to reason out here, even if in elementary manner, what may be the simplest genuine bivariate process.

Picturing bivariate data in the usual fashion as a forest of regularly spaced ordinates $u_{x,y}$ rising above the plane $u = 0$, we require that an element of surface shall join the tops of the four ordinates at the corners of each square in such a way as to blend acceptably with the similar elements of surface for the four squares at its sides and also the four that touch its corners.

Now, in univariate interpolation, a binomial of the third order is the simplest that will satisfy two pivotal points together with a prescribed first derivative at each of them—four conditions that in Karup's osculatory (perhaps, rather, tangential) formula are determined by the four pivotals to which the interval is central. Then a bivariate process of comparable rank may logically be expected to determine all necessary conditions for a square element of surface from the sixteen to which that element is central, as in the formula to be deduced below. Meanwhile, we may observe that merely applying Karup's formula in one direction along all the rows of the given array and thereafter operating in the same fashion down all the columns to complete the job (or vice versa) would also determine each element of surface by the sixteen pivotals to which it is central, but in most unsatisfactory fashion; if derived that way, almost never will the peculiar sixth-order surface (that has neither independent variable to a higher order than the third) keep reasonably near to the unknown true surface that we wish to approximate. Even as a univariate process Karup's formula gives unsatisfactory results with many functions, and of course the errors will be compounded if the results are used as a base for cross-interpolation. (Obviously, smoothing formulas that involve graduation and do not satisfy the given pivotals are still more unsatisfactory if we want accuracy.)

It will be better to construct our bivariate formula out of univariate expressions that leave the derivatives to be determined, such as:

$$u_{x+f} = \left\{ \begin{array}{l} (1-3f^2+2f^3)\,u_x + f\,(1-f)^2 Du_x \\ +\,(3f^2-2f^3)\,u_{x+1} - f^2(1-f)\,Du_{x+1} \end{array} \right\}. \tag{1}$$

Replacing Du_x and Du_{x+1} by $\mu\delta u_x$ and $\mu\delta u_{x+1}$ respectively, and rearranging, gives Karup's formula; but as mentioned, we must find derivatives more nearly appropriate to the surface. If we seek a polynomial surface of small area $U_{x,\,y}$ centered at each bivariate pivotal, say $u_{0,\,0}$ for example, we find that no order of such surface can be fitted symmetrically about a central pivotal on a rectangular pattern without distorting assumptions. For instance, a second-order surface involves six constants; it cannot really fit either a five-pivotal cross or a nine-pivotal square array. But we can prescribe a second-order surface that minimizes the squares of its deviations from such a nine-pivotal array. Suppose that by $U_{x,\,y}$ we designate such a surface centered at (0, 0) with the equation

$$\begin{aligned} U_{x,\,y} = {} & \{u_{0,\,0} - \tfrac{1}{9}\,\delta_x^2\delta_y^2 u_{0,\,0}\} + \frac{x}{3}\mu\,\delta_x(u_{0,\,-1} + u_{0,\,0} + u_{0,\,+1}) \\ & + \frac{y}{3}\,\mu\,\delta_y(u_{-1,\,0} + u_{0,\,0} + u_{+1,\,0}) + xy\mu\,\delta_x(\mu\,\delta_y u_{0,\,0}) \\ & + \frac{x^2}{6}\,\delta_x^2(u_{0,\,-1} + u_{0,\,0} + u_{0,\,+1}) + \frac{y^2}{6}\,\delta_y^2(u_{-1,\,0} + u_{0,\,0} + u_{+1,\,0})\,. \end{aligned} \tag{2}$$

Then its partial derivatives at $(x, y) = (0, 0)$ are:

$$\frac{\partial}{\partial x} U_{0,\,0} = \tfrac{1}{3}\mu\,\delta_x\,(u_{0,\,-1} + u_{0,\,0} + u_{0,\,+1}) \tag{3}$$

$$\frac{\partial}{\partial y} U_{0,\,0} = \tfrac{1}{3}\mu\,\delta_y\,(u_{-1,\,0} + u_{0,\,0} + u_{+1,\,0}) \tag{4}$$

$$\frac{\partial^2}{\partial x \partial y}\left(= \frac{\partial^2}{\partial y \partial x}\right) U_{0,\,0} = \mu\,\delta_x\,(\mu\,\delta_y u_{0,\,0})\ . \tag{5}$$

These derivatives will probably approach those of the unknown true surface more nearly than if surface $U_{x,\,y}$ passed exactly through any selected pivotals, since such restriction would probably obscure any furling tendency there might be in the true surface. Note that if the true surface should happen to be either of the second order or a plane, the resulting zero values of the corresponding difference orders would make $U_{x,\,y}$ reproduce that second-order surface or plane exactly.

Substituting into formula (1) the partial first derivatives of formulas (3) and (4) enables construction of a lattice-work of interpolated values $u_{x+f,\,y}$ and $u_{x,\,y+\phi}$ that interconnects all the pivotals except those at the very edges of the given array. The open square areas within the lattice may be completed in whichever direction is preferable, but for correct results by this method we must first supply by interpolation the several series of correct partial first derivatives at the new junction points for these secondary interpolations. Assuming that these are across the several $u_{x+f,\,y}$ series, then we must derive the several corresponding derivative series:

$$\begin{aligned}\frac{\partial}{\partial y} U_{x+f,\,y} = \frac{\partial}{\partial y} U_{x,\,y} &+ (3f^2 - 2f^3)\left(\frac{\partial}{\partial y} U_{x+1,\,y} - \frac{\partial}{\partial y} U_{x,\,y}\right) \\ &+ f(1-f)^2 \frac{\partial^2}{\partial x \partial y} U_{x,\,y} - f^2(1-f)\frac{\partial^2}{\partial x \partial y} U_{x+1,\,y}\,.\end{aligned} \tag{6}$$

The results will then be identically the same as if done across the several $u_{x,\,y+\phi}$ series with similarly prepared derivatives

$$\frac{\partial}{\partial x} U_{x,\,y+\phi}\,.$$

Obviously all this adds up to a great deal of work on anything else than controlled-sequence equipment, and for such equipment it is simpler to schedule the sets of sixteen multipliers each, for expressing each interpolated result within a lattice-square (*i.e.*, element of surface) as a product-sum involving the sixteen nearest pivotals.

Ordinarily the results within each lattice-square would form a sixth-order surface of the kind previously described, but in the improbable event of the entire body of data forming either a second-order surface or a plane, the entire mosaic of results would reproduce that surface or plane exactly. To this extent the method satisfies a fidelity test somewhat corresponding to those for Sprague's and Karup's univariate formulas, that would reproduce any series that happened to be a polynomial up to the fourth or the second order respectively. It should be mentioned, in passing, that Karup's formula applied both ways across the array would also produce such a surface and meet such a test; but ordinarily it would not conform so well to any other true surface.

Programming, of course, would depend on the particular equipment available, but with an electronic calculator having a good "memory" by present standards it is easy to visualize (1) systematic storage of the given pivotals into an adequate section of addresses reserved for that purpose, (2) storage of the schedule of multipliers described above into another section, and (3) storing into a third section the instructions for accumulating the necessary product-sums, for punching each total (*i.e.*, interpolated result) into a card or printing it in place on a sheet, and for then proceeding to work the next result, ad infinitum. Thus the results, usually too numerous for any machine's "memory," would be released as soon as produced.

So far, no provision has been made for marginal areas of the array that would correspond, of course, to end-intervals in univariate series. If it is important to fill them in, and impossible to obtain outside pivotals otherwise, such pivotals may be extrapolated for by a preliminary controlled-sequence program applied to the array of given pivotals systematically stored as suggested. Perhaps the thought will be clearer from a diagram that assigns each given pivotal $u_{x,\,y}$ to an address within the rectangle 01, 01 to $\omega - 1$, $w - 1$ and leaves room in columns $x = 00$ and $x = \omega$, and rows $y = 00$ and $y = w$, for the pivotals to be extrapolated and stored into position for use in the interpolation program previously outlined.

$$
\begin{array}{l|ll|l}
00,00 & 01,00 & \ldots\ldots\omega-1,00 & \omega,00 \\
\cline{2-3}
00,01 & 01,01 & \ldots\ldots\omega-1,01 & \omega,01 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
00,w-1 & 01,w-1 & \ldots\ldots\omega-1,w-1 & \omega,w-1 \\
\cline{2-3}
00,w & 01,w & \ldots\ldots\omega-1,w & \omega,w
\end{array}
$$

Higher-order surfaces may be reasoned out in similar fashion, perhaps by better reasoning technique, but such surfaces need not be based on square arrays: third-order $U_{x,y}$ surfaces, determined for each pivotal by the compact group (not square) of thirteen that include it, should furnish all the derivatives necessary for twenty-four point-elements of surface that blend to the second order.

HARWOOD ROSSER:

Not too many years ago, one of the authors expressed himself as feeling "like one who has just written the epitaph of osculatory interpolation."[1] But it has turned out to be a lively corpse.

Both authors are to be highly congratulated on a tremendous piece of work. The monumental quality of this paper will probably be more apparent to physicists and astronomers than to most actuaries. It is regrettable that none of the former are likely to submit discussions. Very few of the latter will. The average actuary is apt to say: "Interpolation formulas? Don't need any; I've got one."

But the point is that the authors are not peddling a new interpolation formula. They are offering us criteria for selecting or developing a formula that will be appropriate to the particular circumstances. Mr. Beers[2] stated the earlier situation very neatly when he said: "The literature of interpolation contains many formulas but few critical comparisons of their respective results." Messrs. Greville and Vaughan have now given us the means to make such comparisons. Their elegant use of symbolic operators, once it is thoroughly understood, materially shortens the process.

For most readers, the heart of the paper lies in the "Rules," in the "Computation of Interpolation Coefficients" and in the examples illustrating their application. As a test of the pedagogy of the authors, I set myself the task of actually working some of their examples before looking at their solutions—which is standard procedure for actuarial students. I will not say how many wrong answers I obtained; but I did get some right answers. Hence I may state that their expository talents have been subjected to a stern test and not found wanting. Interpolation is not usually considered an easy subject, once you get beyond straight-line methods.

While doing this work, I set out, for my own convenience, the data in Chart 1. This is included here with the thought that it might focus some attention on the number of attributes to be considered in choosing a formula for interpolation. In addition to the characteristics expressed numerically, formulas may be either ordinary or modified (reproducing or smoothing), and end-point or midpoint.

If a "modified" formula is being considered, a very practical question

[1] *TASA* XLVI, 98.

[2] *RAIA* XXXIII, 245.

is the amount of smoothing incorporated into the formula. Considerable information is obtained from the "trace" of the formula. The authors have not dwelt upon this, probably because the subject is well covered in earlier papers.[3]

It pleases me to note that s is defined as the "number of terms in the linear compound form." This is more readily understandable than the equivalent expression in one of Dr. Greville's earlier papers:[4] "The highest order of differences involved (explicitly or implicitly)." But I am not so sure that an increase in clarity results from substituting "correct to rth differences" for "degree of reproduction." The former phrase is of long standing—and also of long misunderstanding. The students I have taught usually found it easier to comprehend the concept that a formula would

CHART 1

Item	Symbol	Example						
		1	2	3	4	5 Trial	5 Final	6
Span of H	h	1	1	2	2	2	1	1
Order of contact	p	1	2	2	2	1	1	1
Degree of formula	q	4	5	3	5	3	3	2
Degree of reproduction	r	4	4	3	2	−1	0	2
Number of terms used	s	6	6	6	5	2	2	4

reproduce all polynomials up to and including the rth degree. Also, the classification, in Table 1, into reproducing formulas and smoothing formulas would seem to argue in favor of "degree of reproduction" as the definition of r.

On Example 5, a remark or two is in order. Following Rule 3(a), footnote 13, we take $r = -1$. Presumably, the interpretation of this is that the formula would not even reproduce a constant function. That is, such a formula, used to interpolate between values, all on a straight line parallel to the x-axis, would produce interpolated values not on that line. This is inconsistent with common sense. However, when we reach the final result, we find, by inspecting it, that the span of H is only one. (Since the symbol D denotes differentiation, we may ignore it in determining the span.) Then, by reapplying Rule 2, we find that r is actually zero, as the authors note.

The purpose of these "tangential" comments is to emphasize that

[3] Cf. *TASA* XLV, 224–225 and 230–232; also *TASA* XLVI, 95–96.

[4] *TASA* XLV, 211.

$r = -1$ is simply a trial value. It is my belief that this would never be the final value in any formula intended for actual use. If this be correct, why not start with $r = 0$ in such cases?

Actuarial readers who attempt to look up the "standard Steffensen formula" by that name may encounter difficulty. Steffensen himself, in his *Interpolation*,[5] calls it Everett's second formula. It also appears by that name in Freeman.[6] To convert Freeman's form to that implied by the authors, add n to all subscripts, replace p by x and q by y, and make the substitution $\Delta = \delta E^{1/2}$.

The actuarial profession, at least, owes a substantial debt of gratitude to Messrs. Greville and Vaughan for this paper. If I have seemed critical about some minor points, I plead the excuse of hoping to increase the understanding of it, that there may be wider awareness of this debt, and greater use of their gift.

(AUTHORS' REVIEW OF DISCUSSION)

T. N. E. GREVILLE AND HUBERT VAUGHAN:

We wish to thank both Mr. Camp and Mr. Rosser for their constructive and illuminating discussions. We are particularly interested in Mr. Camp's investigation of bivariate interpolation, as it clearly indicates a possible extension of the methods of the paper which had not occurred to us. A process of bivariate interpolation like those described by Mr. Camp is represented by a basic function which is a function of two variables, $f(x, y)$. This is represented geometrically by a bell-shaped surface having a maximum at the origin and eventually merging into the xy-plane in all directions from the origin. When separate interpolations are performed in the two directions successively, as in the first process mentioned by Mr. Camp, the basic function is a product of two univariate functions: $f(x, y) = f_1(x)f_2(y)$. Similarly, the characteristic function is a product of two univariate characteristic functions. Thus, in Mr. Camp's example, it is

$$M_x^3M_y^3(3M_x - 2\mu_x)(3M_y - 2\mu_y) .$$

It will be seen that this is merely the product of the univariate characteristic functions of Karup's formula for interpolation in the two directions.

The methods of this paper can also be applied to Mr. Camp's second process. Thus, the characteristic function of the interpolated values produced by his formula (1) is

$$12MD^{-2}(\mu - M)U + D^{-1}[12D^{-2}(\mu M - 1) - 2M^2]W , \qquad (1)$$

[5] P. 31, formula (28). [6] *Mathematics for Actuarial Students*, Vol. II, p. 67.

where U is the characteristic function of the given values, and W is the characteristic function of the assumed values of the derivatives at the pivotal points. Mr. Camp's formula (3) may be written in the form

$$\frac{\partial}{\partial x} U_{0,\,0} = \mu\,\delta_x\,(1 + \tfrac{1}{3}\delta_y^2)\,u_{0,\,0}\,.$$

Substituting this expression for W and 1 for U in our formula (1) and simplifying gives, for the characteristic function of the interpolation process to obtain the values of $u_{x+f,\,y}$,

$$M_x^3(3M_x - 2\mu_x)(1 + \tfrac{1}{3}\delta_y^2) - 4MD_x^{-2}(\mu_x - M_x)\delta_y^2\,. \tag{2}$$

Similarly, applying our formula (1) to the interpolation of the first partial derivatives with respect to y, using as "given values" and "given derivatives" the values produced by Mr. Camp's formulas (4) and (5), we obtain, for the characteristic formula of this interpolation process,

$$12MD_x^{-2}(\mu_x - M_x)\,\mu\,\delta_y + D_x^{-1}\,[12\,D_x^{-2}\,(\mu M_x - 1) - 2M_x^2]\,\mu\,\delta_x\mu\,\delta_y\,,$$

which reduces to

$$M_x^3(2\mu_x - M_x)\mu\delta_y\,. \tag{3}$$

Finally, using the analogue of our formula (1) for interpolation in the y-direction, taking the results of our formulas (2) and (3) as representing the given values and given derivatives, we have, for the characteristic function of the entire interpolation process,

$$\begin{aligned}&[M_x^3\,(3M_x - 2\mu_x)\;(1 + \tfrac{1}{3}\delta_y^2) - 4M\,D_x^{-2}\,(\mu_x - M_x)\,\delta_y^2]\\ &\qquad\times\,[12MD_y^{-2}\,(\mu_y - M_y)\,] + M_x^3\,(2\mu_x - M_x)\,\mu\,\delta_y D_y^{-1}\\ &\qquad\qquad\times\,[12\,D_y^{-2}\,(\mu M_y - 1) - 2M_y^2]\end{aligned}$$

or

$$\begin{aligned}&M_x^3M_y^3\,(-15M_xM_y + 14\mu_xM_y + 14M_x\mu_y - 12\mu_x\mu_y)\\ &\qquad - 48M_xM_y\,(M_x^2D_y^{-2} + M_y^2D_x^{-2})\;(\mu_x - M_x)\;(\mu_y - M_y)\,.\end{aligned}$$

Knowing the characteristic function for the process would, we believe, facilitate the computation of the 16 sets of multipliers mentioned by Mr. Camp to be applied to the 16 pivotals in the 4×4 square array surrounding the square in which interpolation is being performed. As Mr. Camp mentions, both processes have the property that if the true surface should happen to be of the second degree or a plane, it will be exactly reproduced.

However, there is an interesting difference between the two. The "double Karup" process will reproduce any bivariate polynomial in which neither variable separately occurs to a degree greater than two (for example, x^2y^2). The second process, on the other hand, will reproduce only functions of the second degree in both variables taken together, such as $x^2 + 2xy - 3y^2$.

We are inclined to agree with Mr. Camp that the last word has not been said on appropriate criteria for rating interpolation formulas. We would suggest that in judging the effectiveness of particular types of formulas it may be well to keep in mind the purpose for which they are intended. For example, if the only need for interpolation were that of providing finer subdivisions in tables of known mathematical functions, interpolation formulas designed specifically to secure smooth junction would never have been proposed. For this purpose, there is no question that plain finite-difference interpolation is generally more accurate and provides a closer estimate of the error involved. The kinds of formulas mainly dealt with in the paper were intended for use with empirical functions not believed to follow any mathematical law, or for which the law, if any, is complicated and not evident from examination of the data. A typical example, of course, is that of population data in 5-year age groups which Mr. Camp cites. To illustrate how the intended application enters into the choice of a formula, it may be pointed out that a sine curve would be considered perfectly smooth from the mathematician's point of view, but one is inclined to be suspicious of curves of somewhat similar form which are sometimes encountered when mortality rates are computed from population data.

We are not sure that the distinction between polynomial and nonpolynomial functions is as sharp as Mr. Camp seems to believe. In fact, a well-known theorem of Weierstrass states that under very general conditions a continuous function can be expressed as the limit of a sequence of polynomials. According to our investigations, the formula of Henderson mentioned in Mr. Camp's fifth paragraph is a very close approximation to the ordinary fifth-difference formula and could be used for convenience in cases where the latter was suitable. If the ordinary fifth-difference formula were unsatisfactory in a particular case, the same would normally apply to this formula of Henderson.

With reference to a question of terminology raised by Mr. Rosser, the substitution of the expression "correct to rth differences" for "degree of reproduction" was intended to avoid possible confusion between the concepts of reproduction of a stipulated degree of polynomial and reproduction of the given values. People sometimes seem to have difficulty in understanding how a formula can, for example, reproduce a third-degree polynomial when the given values are values of such a polynomial, and yet fail to reproduce the given values at all when they are not of this nature. We admit, however, Mr. Rosser's point that the expression we have chosen is liable to misunderstanding. We also agree with him that formulas with $r = -1$ are not very practical and that one may as well start with $r = 0$. Our Example 5 was not really intended to be practical but was given as an illustration of an unusual situation to which the rules might sometime have to be applied.

Milne-Thomson[1] has applied the name "Steffensen's formula" to the one so designated in our paper. We are inclined to think that Steffensen was the first to give this formula in print and that he referred to it as "Everett's second formula" because of its obvious analogy to Everett's formula, and without any intention of suggesting that Everett was its real originator.

[1] *The Calculus of Finite Differences*, p. 74.