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# A GENERAL METHOD OF CALCULATING EXPERIENCE NET EXTRA PREMIUMS BASED ON THE STANDARD NET AMOUNT AT RISK

WALTER SHUR

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J. B. MACDONALD

Mr. Shur is to be congratulated on his elegant simplification of what was formerly a very complicated calculation. However, the simplicity does not occur until after the calculation of the auxiliary mortality table and the commutation functions based upon it. A separate auxiliary mortality table will be required for each type of substandard mortality for which extras are required. In the substandard experience table itself only $q_x$'s and $l_x$'s are required, and no commutation columns.

Several insurance companies have for years based their net extras upon the difference in net premiums between the substandard and standard experience tables, or, in Mr. Shur's notation, $P^B - P^A$. This involves substandard commutation columns, just as Mr. Shur's involves auxiliary commutation columns, but no auxiliary $l_x$'s need be calculated. It would be of interest to consider the implications of the two methods.

This second method (which I shall call Method T) implies that reserves are being held based upon the substandard table B, as computing premiums on a table implies reserves should be held based upon that table. As, however, standard premiums are computed on A while reserves are held on C, it is plausible that when premiums are, in effect, computed on B the implication is that reserves should be held upon the appropriate modification of C. In actual practice the reserve held is often based on Table C plus an approximate adjustment for the extra mortality. Often this approximation for the extra mean reserve is half the extra premium, which is true only when the standard and substandard terminal reserves are the same.

Mr. Shur's method (which I shall call Method S) assumes that the reserve being held is the regular reserve based on Table C, as is usually the case, and provides for the excess of substandard experience (B) over standard experience (A) mortality with a death benefit of the difference between the face amount and the Table C reserve. (Incidentally, it is worth remarking the reserves need not be calculated at the same interest rate as is used in computing the extra.) Naturally some reserve must be held for this extra mortality and it would be preferable to hold some sort of approximate reserve. Otherwise, it might be simpler to hold reserves based upon a multiple table and compute extras by the second method.

I computed net extras, using both Methods S and T, based upon my company's standard and substandard experience tables at $3\frac{1}{2}\%$ (the only rate at which commutation columns were available). The reserve was assumed to be the CSO $3\frac{1}{2}\%$ net level premium reserve. Extras per thousand are shown based upon 200% and 500% of standard (Table 1).

TABLE 1

| Age | 200% | | 500% | |
|---|---|---|---|---|
| | *S* | *T* | *S* | *T* |
| | Ordinary Life | | | |
| 25 | 3.84 | 3.89 | 11.92 | 12.05 |
| 35 | 6.02 | 6.05 | 18.74 | 18.83 |
| 45 | 10.38 | 10.35 | 33.28 | 33.24 |
| 55 | 18.97 | 18.81 | 63.27 | 62.83 |
| | Twenty Pay Life | | | |
| 25 | 4.44 | 4.66 | 12.66 | 13.21 |
| 35 | 5.91 | 6.16 | 17.17 | 17.73 |
| 45 | 9.08 | 9.28 | 28.95 | 29.29 |
| 55 | 16.52 | 16.50 | 58.41 | 58.05 |
| | Twenty Year Endowment | | | |
| 25 | 1.08 | 1.08 | 4.37 | 4.35 |
| 35 | 2.11 | 2.08 | 8.50 | 8.42 |
| 45 | 5.37 | 5.30 | 21.66 | 21.38 |
| 55 | 13.87 | 13.62 | 54.45 | 53.62 |
| | Twenty Year Term | | | |
| 25 | 2.18 | 2.19 | 8.58 | 8.61 |
| 35 | 4.57 | 4.58 | 17.44 | 17.49 |
| 45 | 10.72 | 10.76 | 38.08 | 38.22 |
| 55 | 22.43 | 22.55 | 72.18 | 72.44 |

There is not very much difference between the two sets of extras, which suggests that, for my company at least, Method S is a needless refinement. In fact, the probable error in assessing the extra mortality whether it be 190% or 200% or 210% is greater than the difference in the extras. Needless to say, however, we were most pleased to discover the close agreement in the extras.

HARWOOD ROSSER

Mr. Shur is to be congratulated upon developing a short-cut for a fairly common actuarial calculation, that of substandard extra premiums. His method hinges upon the clever use of special commutation functions. One is reminded of Cody's resorting to the same device, also to save work, in dealing with decreasing term contracts involving two different rates of interest.[1]

At the same time, it is hoped that the sheer elegance of his method, together with his claim of "calculating without approximation," will not mislead the casual reader. He labels as "approximate" the method of Bassett's paper;[2] but many of Hoskins' remarks, in discussing that paper, apply equally well here, especially the paragraph that begins thus: "His method involves other approximations."

Stated otherwise, it seems to me that Mr. Shur has lifted a formula out of context. He has taken a special case of the well known "Equation of Equilibrium"—the one where special policy values and normal policy values are equal—and regarded it as the general case. There are good reasons why this assumption of equivalence is often made in practice—or, more correctly, why the difference is often ignored; but the nature of the approximation involved should be realized.

There is no need to repeat here what is already in print. Excellent furher treatment of this point can be found in *The Practice of Life Assurance*, by Coe and Ogborn, pages 206–208, or in "Substandard Business," by Richardson.[3] (These are required reading for British and American Fellowship students, respectively.) Again, Hoskins' discussion of the latter is quite pertinent, particularly his suggestion of adding $K/e_x$ to the standard table. Feay's discussion includes, at page 626, a neat criterion for equivalence of values.

Some of the economy of Mr. Shur's method disappears if any one of the mortality tables labeled A, B, and C is select. Valuation tables are usually ultimate or aggregate,[4] but duration is often deemed important in figuring the extra risk on substandard cases. Also, my company, like many others,

[1] *TASA* XLIX, 72.

[2] *TSA* II, 1.

[3] *RAIA* XXX, 122; especially pages 155–58.

[4] Cf. *RAIA* XXXIII, 272–73.

uses, for premium determination, a select and ultimate table that approximates our experience.

Mr. Shur's formulas still apply if select functions are introduced, but it becomes more complicated to construct the synthetic mortality table D. First, the ultimate column of $l$'s must be formed, by the method outlined by Mr. Shur. Then the select portion is obtained by the following relationship:

$$l^{\mathrm{D}}_{[x]+t} = \frac{l^{\mathrm{D}}_{[x]+t+1} + l^{\mathrm{B}}_{[x]+t}\,(q^{\mathrm{B}}_{[x]+t} - q^{\mathrm{A}}_{[x]+t})}{1 - q^{\mathrm{C}}_{[x]+t}}\,.$$

The number of such calculations is the number of ages in the table times the number of years in the select period. The columns of D's and N's are similarly expanded several times. All this is repeated for each variation in substandard mortality that the company chooses to recognize.

It is no detraction from Mr. Shur's ingenuity to note that his method will not save as much time today as it would have, say, twenty years ago. This is the increasingly common fate of most actuarial short-cuts, including commutation columns, under the impact of electronic computers. In the paper immediately following, Mr. Cueto states: "It was found more convenient by means of the electronic equipment to calculate net premiums for all ages rather than for quinquennial ages only and interpolate." Elsewhere in the same number of the *Transactions*, the Univac is prominently featured as a work-saver.

The most advanced equipment in our office is an IBM 604. Even on this, we found it practical to compute extra premiums, from first principles, for each of a number of impairments, using data from the new *Medical Impairment Study.* We assumed standard values and used select functions. To do so, we made the same assumption that Richardson did.[5] We also gave account to persistency, which Mr. Shur's method does not do. Essentially, we employed an adaptation of the "present value approach."[6] This amounted to differencing two profit margin calculations, one with standard, and the other with substandard, mortality.

However, a fair number of calculations must be required before it becomes more economical to program them for electronic machinery. For small companies without such equipment, or for actuaries who prefer a small number of broad substandard categories, Mr. Shur's method should be invaluable. If it is assumed that only the mortality is different, and that the extra mortality is constant, or else a percentage of a standard table that is not select, his method is ideal.

[5] *RAIA* XXX, 157.

[6] Cf. *TASA* III, 187.

(AUTHOR'S REVIEW OF DISCUSSION)

WALTER SHUR:

I should like to thank Mr. MacDonald and Mr. Rosser for their discussions.

Mr. MacDonald has compared values of $P^B - P^A$ with values of $\pi$ based on his company's experience tables; as indicated in his discussion, these values are quite close. It would be interesting to consider the reasons for this similarity. Consider the following two equations:

$$\pi = \frac{v \sum_{t=0}^{m-1} D^B_{x+t} (q^B_{x+t} - q^A_{x+t}) (1 - {}_{t+1}V^C)}{N^B_x - N^B_{x+n}} \tag{1a}$$

$$P^B - P^A = \frac{v \sum_{t=0}^{m-1} D^B_{x+t} (q^B_{x+t} - q^A_{x+t}) (1 - {}_{t+1}V^A)}{N^B_x - N^B_{x+n}} \tag{2a}$$

(It should be emphasized that equation (2*a*) does not imply that standard and substandard reserves are equal at all durations. The only assumptions regarding reserves which are needed to prove equation (2*a*) are that ${}_0V^B = {}_0V^A$, and that ${}_mV^B = {}_mV^A$. As a matter of fact, the derivation of equation (7) in the paper proves equation (2*a*) if the superscripts B and A are substituted for D and C, respectively, in equations (7) to (15) in the paper.)

From equations (1*a*) and (2*a*), then, it is apparent that the difference between $\pi$ and $P^B - P^A$ depends entirely on the difference between Table A and Table C reserves. In Mr. MacDonald's case, the reserves on his standard experience table must generally be quite close to the CSO reserves, which would account for the closeness of his Method T and Method S premiums.

Such similarity is not always the case, however. Method T and Method S premiums were calculated using our own company's experience tables, the CSO table, and $2\frac{1}{2}\%$ interest. The calculations were made at ages 25, 35, 45, and 55, for 200% and 500% extra mortality. The following table (Table 1) shows the percentage excess of Method T over Method S premiums for each plan with 200% extra mortality. The results for 500% extra mortality were quite similar.

Mr. Rosser states in the third paragraph of his discussion that "he has taken a special case of the well known 'Equation of Equilibrium'—the one where special policy values and normal policy values are equal—and regarded it as the general case." I do not believe that there is any implication in the definition of $\pi$ that standard and substandard reserves must be equal. $\pi$ is the extra premium required on a net experience basis to cover the cost of the extra mortality in a situation where standard reserves are based on Table C. Additional reserves for the extra mortality may be held on any basis deemed appropriate.

Perhaps Mr. Rosser believes that the use of the amount at risk in extra premium calculation implies that standard and substandard reserves must be equal at all durations. This is not the case, however, as indicated in the parenthetical comment following equation (2*a*) above.

TABLE 1

| Plan | Percentage Excess of Method T over Method S Premiums—200% Extra Mortality | | | |
|---|---|---|---|---|
| | Age 25 | Age 35 | Age 45 | Age 55 |
| Ordinary Life | 4.4% | 3.9% | 3.3% | 2.8% |
| 20 Pay Life | 12.0 | 10.2 | 6.5 | 2.6 |
| 20 Year Endowment | −3.0 | − .6 | −1.5 | −1.5 |
| 20 Year Term | .7 | .3 | 1.1 | 2.4 |
| 10 Pay Life | 12.9 | 13.0 | 11.4 | 8.0 |
| Single Premium Life | 13.9 | 14.3 | 15.0 | 16.2 |

Mr. Edwin Steinberg, a student of the society, has developed an alternate method of calculating $\pi$ which requires less calculation than the method presented in the paper. His method requires two auxiliary commutation columns, the construction of which is considerably easier than the construction of Table D. The extra premium $\pi$ may then be calculated by a formula which does not require any further summation. In the following derivation of Mr. Steinberg's formulas, the notation is the same as defined in the paper.

The reserve may be written as follows:

$$_{t+1}V^C = \frac{P^C(N_x^C - N_{x+t+1}^C) - (M_x^C - M_{x+t+1}^C)}{D_{x+t+1}^C} \qquad t \le n-1 \quad (4a)$$

$$_{t+1}V^C = \frac{P^C(N_x^C - N_{x+n}^C) - (M_x^C - M_{x+t+1}^C)}{D_{x+t+1}^C} \qquad t \ge n \quad (5a)$$

Equations (4$a$) and (5$a$) may be transformed into

$$ {}_{t+1}V^{C} = \frac{P^{C}N_x^{C} - M_x^{C}}{D_{x+t+1}^{C}} + 1 - (P^{C} + d)\,\ddot{a}_{x+t+1}^{C} \qquad t \leq n-1 \quad (6a)$$

$$ {}_{t+1}V^{C} = \frac{P^{C}(N_x^{C} - N_{x+n}^{C}) - M_x^{C}}{D_{x+t+1}^{C}} + 1 - d\,\ddot{a}_{x+t+1}^{C} \qquad t \geq n \quad (7a)$$

Substituting these values of ${}_{t+1}V^{C}$ into the expression for $\pi$ given by equation (1$a$), and defining

$$E_{x+t} = D_{x+t}^{B}\,(q_{x+t}^{B} - q_{x+t}^{A})\,\ddot{a}_{x+t+1}^{C} \qquad (8a)$$

$$F_{x+t} = D_{x+t}^{B}\,(q_{x+t}^{B} - q_{x+t}^{A})\,\frac{1}{D_{x+t+1}^{C}}\,, \qquad (9a)$$

we obtain

$$\begin{aligned}\pi = {} & \frac{v}{N_x^{B} - N_{x+n}^{B}} \sum_{t=0}^{n-1} [\,(P^{C} + d)\,E_{x+t} - (P^{C}N_x^{C} - M_x^{C})\,F_{x+t}] \\ & + \frac{v}{N_x^{B} - N_{x+n}^{B}} \sum_{t=n}^{m-1} [dE_{x+t} - \{P^{C}(N_x^{C} - N_{x+n}^{C}) - M_x^{C}\}F_{x+t}]\,.\end{aligned} \qquad (10a)$$

If two auxiliary commutation columns, $G_x$ and $H_x$, are defined by the equations

$$G_x = \sum_{s=x}^{\infty} E_s \qquad (11a)$$

$$H_x = \sum_{s=x}^{\infty} F_s\,, \qquad (12a)$$

equation (10$a$) becomes

$$\begin{aligned}\pi = {} & \frac{v}{N_x^{B} - N_{x+n}^{B}} [\,(P^{C} + d)\,(G_x - G_{x+n}) - (P^{C}N_x^{C} - M_x^{C})\,(H_x - H_{x+n}) \\ & + d\,(G_{x+n} - G_{x+m}) - \{P^{C}(N_x^{C} - N_{x+n}^{C}) - M_x^{C}\}\,(H_{x+n} - H_{x+m})\,]\,.\end{aligned} \qquad (13a)$$

This general formula simplifies for particular plans of insurance. For example,

O.L. $$\pi = \frac{v\,(P^{C} + d)\,G_x}{N_x^{B}}$$

$n$-Pay Life

$$\pi = \frac{v}{N_x^B - N_{x+n}^B}\left[ (P^C + d)(G_x - G_{x+n}) - (P^C N_x^C - M_x^C)(H_x - H_{x+n}) + dG_{x+n}\right]$$

Coterminous Plan

$$\pi = \frac{v}{N_x^B - N_{x+n}^B}\left[ (P^C + d)(G_x - G_{x+n}) - (P^C N_x^C - M_x^C)(H_x - H_{x+n})\right]$$

Single Premium Life $\pi = \dfrac{v\,dG_x}{D_x^B}$

Mr. Steinberg's method will handle all of the usual plans of insurance and is applicable in the situation mentioned by Mr. MacDonald where the reserve interest rate is different from the interest rate used in the premium calculation. The method is not applicable to some of the special cases considered in the appendix of the paper, but these are generally of minor importance.

The author would like to take this opportunity to express his thanks to those at his company, particularly Miss Nora Beattie, who reviewed the preliminary drafts of the paper and whose suggestions were very helpful.