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ACTUARIAL NOTE: RESERVES BY DIFFERENT MORTALITY TABLES

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INTRODUCTION

With the widespread adoption of the CSO Table, the equation

$$q'_x = q_x + \frac{.05}{\mathring{e}_x} \tag{1}$$

has become a part of the routine knowledge of every actuary. This is also true, perhaps to a lesser degree, of the equation

$$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}} \tag{2}$$

which is the proud father of the first equation. It is interesting, therefore, to find that the textbook treatment of (2) is far from complete and is very confusing to students. In fact, the material presented here results from my own embarrassment, when attempting to explain the textbook treatment to a student, at realizing the clouds of confusion I carried over from my student days.

Since this note is designed to help students, I have avoided some theoretical details and limited this note to fundamentals.

THE STANDARD THEOREM

Spurgeon and other writers have presented proofs of the standard theorem for equal whole-life reserves on different mortality tables (but using the same interest rate). In this section is presented a somewhat shorter proof than that generally available for the standard Theorem I.

Theorem I: If

$$_t\mathrm{V}_x = {_t\mathrm{V}'_x}$$

for all values of x and t, then

$$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}}$$

for all values of x.

Proof:
Since

$$_t\mathrm{V}_x = {_t\mathrm{V}'_x}$$

for all values of x and t,

$$\ddot{a}_x = (1+k)\ddot{a}'_x$$

for all values of x;

$$\therefore 1 + vp_x\ddot{a}_{x+1} = (1+k)\left(1 + vp'_x\frac{\ddot{a}_{x+1}}{1+k}\right)$$

$$\therefore p_x - p'_x = \frac{k}{v\ddot{a}_{x+1}}$$

$$\therefore q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}}$$

for all values of x.

CONVERSE OF THE STANDARD THEOREM

The converse of Theorem I is generally assumed without proof, and my brief review of actuarial literature failed to disclose such a proof.

This remark may lead some of our members to review the classic paper, "Mortality Tables Giving the Same Policy Values," by Dr. S. Dumas. This paper (translated by G. L. Lidstone from the original French) was once part of the required reading in our course of examinations. Dr. Dumas proves the standard theorem by the usual methods and states that the converse can be proved "by similar methods." I have not been able to prove Theorem II "by similar methods" and am inclined to think that a proof "by similar methods" would involve a bit of circular reasoning. Perhaps some light will be shed on Dr. Dumas's reasoning in the course of the discussion it is hoped this note will evoke.

Theorem II: If

$$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}}$$

for all values of x, then

$${}_tV_x = {}_tV'_x$$

for all values of x and t.

Proof:

Since

$$p_x = p'_x + \frac{k}{v\ddot{a}_{x+1}}$$

for all values of x,

$$1 + vp_x\ddot{a}_{x+1} = 1 + k + vp'_x\ddot{a}_{x+1}\ ;$$

$$\therefore\ \ddot{a}_x = 1 + k + vp'_x\ddot{a}_{x+1}$$

for all values of x.

$$\begin{aligned}\therefore\ \ddot{a}_x &= 1 + k + vp'_x(1 + k + vp'_{x+1}\ddot{a}_{x+2}) \\ &= 1 + k + vp'_x + vp'_xk + v^2\,{}_2p'_x(1 + k + vp'_{x+2}\ddot{a}_{x+3})\ .\end{aligned}$$

Continuing, and collecting terms,

$$\ddot{a}_x = \ddot{a}'_x + k\ddot{a}'_x ,$$

or

$$\ddot{a}_x = (1 + k)\ddot{a}'_x$$

for all values of x.

$$\therefore\ {}_tV_x = {}_tV'_x$$

for all values of x and t.

EXTENSION TO UNEQUAL RESERVES

The method of proof used in the preceding section is essentially that of recursion formulas. A similar method may be used to extend the theory to the case of unequal reserves. The algebraic procedure is fairly straightforward, although in some respects quite unusual. Because the procedure is unusual, some students will feel repaid for the labor.

Theorem III: If

$$q'_x = q_x + \frac{\theta_x}{v\,\ddot{a}_{x+1}}$$

for all values of x, and if θ_x is an increasing function, then

$${}_tV'_x > {}_tV_x$$

for all values of x and t.

Proof:

Since

$$p_x = p'_x + \frac{\theta_x}{v\,\ddot{a}_{x+1}}$$

for all values of x,

$$1 + vp_x\ddot{a}_{x+1} = 1 + \theta_x + vp'_x\ddot{a}_{x+1} ;$$

$$\therefore\ \ddot{a}_x = 1 + \theta_x + vp'_x\ddot{a}_{x+1}$$

for all values of x.

$$\begin{aligned}\therefore\ \ddot{a}_x &= 1 + \theta_x + vp'_x(1 + \theta_{x+1} + vp'_{x+1}\ddot{a}_{x+2}) \\ &= 1 + \theta_x + vp'_x + vp'_x\theta_{x+1} + v^2\,{}_2p'_x(1 + \theta_{x+2} + vp'_{x+2}\ddot{a}_{x+3}) .\end{aligned}$$

Continuing, and collecting terms,

$$\ddot{a}_x = \ddot{a}'_x + (\theta_x + vp'_x\theta_{x+1} + v^2\,{}_2p'_x\theta_{x+2} + \ldots) .$$

Now let

$$\ddot{a}_x = (1 + z_x)\ddot{a}'_x$$

for all values of x; then

$$z_x\ddot{a}'_x = \theta_x + vp'_x\theta_{x+1} + v^2\,{}_2p'_x\theta_{x+2} + \ldots .$$

Therefore, if θ_x is an increasing function,

$$z_x > \theta_x .$$

Furthermore,

$$z_x \ddot{a}_x' = \theta_x + vp_x'(\theta_{x+1} + vp_{x+1}'\theta_{x+2} + \ldots)$$

$$= \theta_x + vp_x' z_{x+1}\ddot{a}_{x+1}' .$$

$$\therefore\ \theta_x = z_x\ddot{a}_x' - z_{x+1}a_x'$$

$$\therefore\ z_x - \theta_x = (z_{x+1} - z_x)a_x'.$$

Since $z_x - \theta_x$ is positive, $z_{x+1} - z_x$ is also positive;

$$\therefore\ z_{x+1} > z_x$$

for all values of x.

Therefore, z_x is an increasing function and

$${}_tV_x' > {}_tV_x$$

for all values of x and t.

THE GENERAL THEOREM

The method of proof in the preceding section can also be applied when the increasing function θ_x is replaced by a decreasing function. This is left as an exercise for the student, and we are now in a position to state the general Theorem IV.

Theorem IV: If

$$q_x' = q_x + \frac{\theta_x}{v\,\ddot{a}_{x+1}}$$

for all values of x, then

(*a*) ${}_tV_x' = {}_tV_x$ if θ_x is constant ;

(*b*) ${}_tV_x' > {}_tV_x$ if θ_x is an increasing function ;

(*c*) ${}_tV_x' < {}_tV_x$ if θ_x is a decreasing function.

ILLUSTRATIONS OF THE GENERAL THEOREM

The general Theorem IV may be used in simple fashion to obtain the results derived in Spurgeon's textbook by means of the "Equation of Equilibrium," which is a *bête noire* to students.

Example 1. Suppose that

$$q_x' = q_x + c$$

for all values of x. Then,

$$q'_x = q_x + \frac{c\, v\ddot{a}_{x+1}}{v\ddot{a}_{x+1}} .$$

If c is positive, θ_x is a decreasing function (with minor exceptions) and ${}_tV'_x < {}_tV_x$.

Example 2. Suppose that

$$q'_x = q_x + kq_x$$

with k positive; find the condition that

$${}_tV'_x > {}_tV_x$$

for all x and t. Here

$$\theta_x = kvq_x\ddot{a}_{x+1} ,$$

and θ_x is increasing if

$$vq_{x+1}\ddot{a}_{x+2} > vq_x\ddot{a}_{x+1}$$

or if

$$a_{x+1} - v\ddot{a}_{x+2} < a_x - v\ddot{a}_{x+1}$$

or if

$$a_{x+1} - a_x < v(\ddot{a}_{x+2} - \ddot{a}_{x+1})$$

or if

$$(1 + i)\Delta a_x < \Delta a_{x+1}$$

or if

$$i\Delta a_x < \Delta a_{x+1} - \Delta a_x$$

or if

$$i\Delta a_x < \Delta^2 a_x ,$$

which is the required condition.

Example 3. Suppose

$$p'_x = (1 + k)p_x .$$

This becomes

$$q'_x = q_x - kp_x .$$

so that

$$\theta_x = -kvp_x\ddot{a}_{x+1} = -ka_x .$$

Therefore, if k is positive, θ_x is increasing and ${}_tV'_x > {}_tV_x$. If k is negative, then ${}_tV'_x < {}_tV_x$.

EFFECT OF CHANGE IN INTEREST RATE

It is well known that an increase in the assumed interest rate, without a change in the mortality table, has the effect of reducing whole-life reserves, while a decrease in the interest rate has the opposite effect. Various methods of proof of this fact are available in actuarial literature. A rather simple method, which does not appear to have been noted previously, and which is a by-product of the preceding section of this note, may be of interest to students.

Since

$$\ddot{a}'_x = 1 + v'p_x + (v'p_x)(v'p_{x+1}) + (v'p_x)(v'p_{x+1})(v'p_{x+2}) + \ldots,$$

$$\therefore\ \ddot{a}'_x = 1 + vp'_x + (vp'_x)(vp'_{x+1}) + (vp'_x)(vp'_{x+1})(vp'_{x+2}) + \ldots$$

where

$$p'_x = \frac{v'}{v}\, p_x = \frac{1+i}{1+i'}\, p_x$$

for all values of x. This shows that a change in the interest rate, without a change in the mortality table, has the same effect on annuity values, and hence on whole life reserves, as the transformation

$$p'_x = \frac{1+i}{1+i'}\, p_x\,.$$

Write the transformation as

$$p'_x = (1 + k)p_x\,.$$

If $i > i'$, then k is positive and ${}_tV'_x > {}_tV_x$. If $i < i'$, then k is negative and ${}_tV'_x < {}_tV_x$. (These results follow from Example 3 in the preceding section.)