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# ACTUARIAL NOTE: THE EQUATION OF EQUILIBRIUM

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THE effect on level premium reserves of a change in the interest or mortality basis can often be predicted by the results obtained in Spurgeon's *Life Contingencies*, Chap. X, pp. 185–190. His method of reaching the results, however, has never seemed entirely satisfying to me, since the only equations employed assume the *equality* of reserves on the two bases. From this equality he sets out to show by verbal argument what will happen when the two reserves are not equal. It is my hope that the analysis below will put those arguments on a firmer footing and may even clarify them a little.

Let the interest rates, tables of mortality rates, net level premiums, and $t$th terminal reserves, be $i$, $q$, P, and ${}_t\mathrm{V}$ on the one basis and $i'$, $q'$, P$'$, and ${}_t\mathrm{V}'$ on the other. The sum assured is one unit and the policy may be whole life, endowment or term. Limited payment policies we shall consider under a separate heading. Our two basic equations are

$$({}_t\mathrm{V}+\mathrm{P})(1+i)-q(1-{}_{t+1}\mathrm{V})={}_{t+1}\mathrm{V} \tag{1}$$

$$({}_t\mathrm{V}'+\mathrm{P}')(1+i')-q'(1-{}_{t+1}\mathrm{V}')={}_{t+1}\mathrm{V}'\,. \tag{2}$$

Subtracting (1) from (2) and making use of the relations

$$\begin{aligned}{}_t\mathrm{V}'+\mathrm{P}'&={}_t\mathrm{V}+\mathrm{P}+({}_t\mathrm{V}'-{}_t\mathrm{V})+(\mathrm{P}'-\mathrm{P})\\ 1+i'&=1+i+(i'-i)\\ 1-{}_{t+1}\mathrm{V}'&=1-{}_{t+1}\mathrm{V}-({}_{t+1}\mathrm{V}'-{}_{t+1}\mathrm{V})\\ 1-q'&=p'\end{aligned}$$

we obtain

$$\left.\begin{aligned}&({}_t\mathrm{V}+\mathrm{P})(i'-i)+(\mathrm{P}'-\mathrm{P})(1+i')-(q'-q)(1-{}_{t+1}\mathrm{V})\\ &\quad=p'({}_{t+1}\mathrm{V}'-{}_{t+1}\mathrm{V})-({}_t\mathrm{V}'-{}_t\mathrm{V})(1+i')=\text{say, }R_t\,,\end{aligned}\right\}\tag{3}$$

the quantity Spurgeon calls the Remainder. His verbal definition agrees with the left side of (3) but he does not employ the right side. Multiplying the latter by $v'\mathrm{D}'_{x+t}$, we have

$$\left.\begin{aligned}v'\mathrm{D}'_{x+t}R_t&=\mathrm{D}'_{x+t+1}({}_{t+1}\mathrm{V}'-{}_{t+1}\mathrm{V})-\mathrm{D}'_{x+t}({}_t\mathrm{V}'-{}_t\mathrm{V})\\ &=\Delta[\mathrm{D}'_{x+t}({}_t\mathrm{V}'-{}_t\mathrm{V})]\,,\end{aligned}\right\}\tag{4}$$

whence

$$\left.\begin{aligned}\sum_{t=0}^{n-1} v'\mathrm{D}'_{x+t}R_t &= \mathrm{D}'_{x+n}({}_n\mathrm{V}' - {}_n\mathrm{V}) - \mathrm{D}'_x({}_0\mathrm{V}' - {}_0\mathrm{V}) \\ &= \mathrm{D}'_{x+n}({}_n\mathrm{V}' - {}_n\mathrm{V}),\end{aligned}\right\} \quad (5)$$

the reserves at terminal duration 0 being 0. Now if we take $x + m$ to be the age at which the policy matures or expires, then either ${}_m\mathrm{V}' = 1 = {}_m\mathrm{V}$ or ${}_m\mathrm{V}' = 0 = {}_m\mathrm{V}$, and we have

$$\sum_0^{m-1} v'\mathrm{D}'_{x+t}R_t = 0 \,. \quad (6)$$

We shall consider, with Spurgeon, three possible ways in which $R_t$ may behave as $t$ increases. Using the notation $\uparrow$ for "increase(s) with $t$" and $\downarrow$ for "decreases as $t$ increases," we shall consider $R_t$ constant, $R_t \uparrow$, and $R_t \downarrow$. Inequalities obtained between ${}_n\mathrm{V}'$ and ${}_n\mathrm{V}$ will of course reduce to equalities at $n = 0$ and $n = m$.

I. *$R_t$ constant*

Since all the $v'\mathrm{D}'_{x+t}$ in (6) are $> 0$, then $R_t$ must be 0, and all partial sums (5) must be 0.

$$\therefore\ {}_n\mathrm{V}' = {}_n\mathrm{V} \text{ for } 0 \leq n \leq m \,.$$

We can examine two simple possibilities: $q' = q$, and $i' = i$.

(*a*) $q' = q$:

$$({}_t\mathrm{V} + \mathrm{P})(i' - i) + (\mathrm{P}' - \mathrm{P})(1 + i') = R_t = 0 \,,$$

where ${}_t\mathrm{V}$ is the only thing that can vary as $t$ increases. If ${}_t\mathrm{V}$ were constant also, then (1) would imply $q$ = constant, and in fact ${}_t\mathrm{V}$ would be 0 and $\mathrm{P} = vq$. Dismissing this kind of mortality table, we see that the only satisfactory solution is $i' = i$, whence $\mathrm{P}' = \mathrm{P}$, and the two reserve bases are identical.

(*b*) $i' = i$:

$$(\mathrm{P}' - \mathrm{P})(1 + i) - (q' - q)(1 - {}_{t+1}\mathrm{V}) = R_t = 0 \,. \quad (7)$$

Confining ourselves to whole life and endowment policies and using the notation $\ddot{a}_{+t}$ to represent either $\ddot{a}_{x+t}$ or $\ddot{a}_{x+t:\overline{m-t}|}$, we can rewrite (7) as

$$\left.\begin{aligned} q'_{x+t} &= q_{x+t} + (\mathrm{P}' - \mathrm{P})(1 + i)\,\frac{\ddot{a}_{+0}}{\ddot{a}_{+t+1}} \\ &= q_{x+t} + \left(\frac{\ddot{a}_{+0}}{\ddot{a}'_{+0}} - 1\right)\frac{(1 + i)}{\ddot{a}_{+t+1}}, \text{ using } \mathrm{P} = \frac{1}{\ddot{a}_{+0}} - d \,.\end{aligned}\right\} \quad (8)$$

In the whole life case, this is the result developed by Spurgeon, Chap. VI, (29). In the endowment case it is clear that, while $q_{x+t}$ depends only on attained age, the $\ddot{a}$'s depend on age at maturity as well. Thus, if (8) were applied to endowments maturing at the same age $z$, the addition to $q$ to get $q'$ would be of the form

$$\frac{k_z(1+i)}{\ddot{a}_{x+t+1:\overline{z-x-t-1|}}},$$

where the constant $k_z$ is independent of $t$ and equal to

$$\frac{\ddot{a}_{x:\overline{z-x|}}}{\ddot{a}'_{x:\overline{z-x|}}} - 1\,.$$

The ratio $1 + k_z$ is also independent of $x$ because

$${}_t\mathrm{V}'_{x:\overline{z-x|}} = {}_t\mathrm{V}_{x:\overline{z-x|}}$$

implies

$$\frac{\ddot{a}_{x+t:\overline{z-x-t|}}}{\ddot{a}'_{x+t:\overline{z-x-t|}}} = \frac{\ddot{a}_{x:\overline{z-x|}}}{\ddot{a}'_{x:\overline{z-x|}}},$$

a ratio clearly depending only on $z$. If $z = \omega$, then we have the whole life case already mentioned.

II. $R_t \uparrow$

In order that (6) may be true, $R_t$ must start negative and end positive. Hence (5) will be $< 0$ and ${}_n\mathrm{V}' < {}_n\mathrm{V}$. Again we consider (*a*) and (*b*).

(*a*) $q' = q$, ${}_t\mathrm{V} \uparrow$ (which excludes term policies):

Here $R_t \uparrow$ implies $i' > i$, and conversely. Hence: "the higher the interest rate, the lower the reserve."

(*b*) $i' = i$, ${}_t\mathrm{V} \uparrow$ :

Here $R_t = (\mathrm{P}' - \mathrm{P})(1 + i) - (q' - q)(1 - {}_{t+1}\mathrm{V})$, which $\uparrow$ if $(q' - q)$ is positive and does not increase with $t$; *e.g.*, when it is constant.

Hence: "the addition of a constant to $q$ leads to lower reserves."

Another possibility here would be $p' = kp$, where $0 < k < 1$. Then $q' - q = p(1 - k)$, which will not increase if $p$ does not.

Hence, over the whole mortality table after childhood, this type of addition will lead to lower reserves. It may be added that this type of mortality increase occurs whenever the *force* of mortality is increased by a positive constant, here $-\log k$.

The addition of $\frac{k(1+i)}{\ddot{a}_{x+t+1}}$ to $q_{x+t}$, which led to $R_t = 0$ for whole life policies, will give for an endowment maturing at age $z$ ($< \omega$) the value

$$R_t = (\mathrm{P}' - \mathrm{P})(1+i) - \frac{k(1+i)\,\ddot{a}_{y+1:\overline{z-y-1}|}}{\ddot{a}_{y+1}\ddot{a}_{x:\overline{z-x}|}}$$

where $y = x + t$. Since the temporary annuity diminishes more rapidly percentagewise than does the whole life annuity, or since $(1 - \mathrm{N}_z/\mathrm{N}_{y+1}) \downarrow$, it follows that $R_t \uparrow$. Hence endowment reserves are reduced by this type of addition to $q$. This remark applies to the comparison of CSO reserves with reserves by Jones's Basic Table (*TASA* XLIII, 85). Some adjustment is necessary, however, to allow for the fact that the addition in this case varies inversely as $e_y$, rather than as $\ddot{a}_{y+1}$. Accordingly, $R_t$ will $\uparrow$ if

$$\frac{\ddot{a}_{y+1:\overline{z-y-1}|}}{e_y} > \frac{\ddot{a}_{y+2:\overline{z-y-2}|}}{e_{y+1}},$$

*i.e.*, if

$$\frac{1}{\ddot{a}_{y+2:\overline{z-y-2}|}} - \frac{p_y}{e_{y+1}} > p_y - v p_{y+1}\,.$$

It appears from Jones's Model Office comparison (*ibid.*, p. 83) that, in the aggregate, 20 Year Endowment terminal reserves are appreciably lowered, 40 Year Endowment terminal reserves are barely lowered, and Whole Life (Endowment at Age $\omega$) terminal reserves are appreciably raised, in going from Jones's Reserves to CSO Reserves, at 3%.

III. $R_t \downarrow$

Here $R_t$ must start positive and end negative and (5) $> 0$ and ${}_n\mathrm{V}' > {}_n\mathrm{V}$.

(*a*) $q' = q$, ${}_t\mathrm{V} \uparrow$ :

It follows that $i' < i$ and we reach the same conclusion as in II(*a*).

(*b*) $i' = i$, ${}_t\mathrm{V} \uparrow$ :

The product $(q' - q)(1 - {}_{t+1}\mathrm{V})$ must increase, since

$$R_t = (\mathrm{P}' - \mathrm{P})(1+i) - (q' - q)(1 - {}_{t+1}\mathrm{V})$$

here. Hence $(q' - q) \uparrow$ faster than $(1 - {}_{t+1}\mathrm{V}) \downarrow$. In the common case where $q' = q(1 + c)$, this means that

$$c\,q_{x+t}\frac{\ddot{a}_{+t+1}}{\ddot{a}_{+0}} \uparrow,$$

which will be so if

$$q_{x+t+1}\ddot{a}_{+t+2} > q_{x+t}\ddot{a}_{+t+1}\,,$$

that is, if

$$\Delta\ddot{a}_{+t+1} > p_{x+t+1}\ddot{a}_{+t+2} - p_{x+t}\ddot{a}_{+t+1} \,,$$

the right hand side of which equals

$$(1+i)(\ddot{a}_{+t+1} - \ddot{a}_{+t}) = \Delta\ddot{a}_{+t} + i\Delta\ddot{a}_{+t} \,,$$

*i.e.*, if $\Delta^2\ddot{a}_{+t} > i\Delta\ddot{a}_{+t}$, the condition deduced somewhat differently by Spurgeon, X, (30). Note that $(q' - q)$ is assumed $> 0$. If $q' < q$ and $(q - q')$ does not $\uparrow$, then we can repeat the argument of II(*b*) with $q'$ and $q$ interchanged.

LIMITED PAYMENT POLICIES

Equations (1) to (6) will be true for limited payment plans, provided that we drop P and P′ to 0 at the proper duration, $r$.

IV. *$R_t$ one constant during premium period and another thereafter, the constants necessarily differing in sign, owing to (6)*

(*a*) $q' = q$, ${}_tV \uparrow$ implies $i' = i$ and identity of bases.

(*b*) $i' = i$, $P' > P$:

Here $(q' - q)(1 - {}_{t+1}V)$ must be a positive constant less than $(P' - P)(1 + i)$. In this case $R_t$ is first positive and then negative, so that $(5) > 0$ and ${}_nV' > {}_nV$. The necessary addition to $q_{x+t}$ to produce a suitable $q'_{x+t}$ would depend not only on $(x + t)$ but on the separate values of $x$ and $r$.

(*c*) $i' = i$, $P' < P$:

The argument in (*b*) is entirely reversed, as we might expect, and $V' < {}_nV$.

V. $R_t \uparrow$

(*a*) $q' = q$, ${}_tV \uparrow$, $i' > i$:

Here necessarily $P' < P$, otherwise $R_t > 0$ throughout. Thus $R_t$ must start negative and become positive not later than the end of the premium period, being just $A_{+t}(i' - i)$ after the policy is paid up. Hence $(5) < 0$ and ${}_nV' < {}_nV$.

Hence: "the higher the interest rate, the lower the reserve on limited plans."

(*b*) $i' = i$, ${}_tV \uparrow$ :

As in II(*b*), it would at first seem sufficient to have $(q' - q) > 0$ and nonincreasing. Then $R_t$ would $\uparrow$ toward $(P' - P)(1 + i)$ during the premium period; but there would be a sudden drop from $R_{r-1} > 0$ to $R_r < 0$

at the end of the period and $R_t$ would $\uparrow$ toward 0 thereafter. Note that $R_{r-1} > 0$ to satisfy (6). Since $R_t < 0$ in the paid-up period, we know that (5) $> 0$ and ${}_nV' > {}_nV$, which must obviously be necessary to cover the higher $q'$ in the paid-up period. But there are two possibilities regarding (5) during the premium period. We may have $R_t > 0$ throughout the period, whence (5) $> 0$ and ${}_nV' > {}_nV$ at all durations, or we may start with $R_t < 0$ and end with $R_{r-1} > 0$. In this case (5) $< 0$ during the early durations and $> 0$ thereafter. Thus ${}_nV'$ may start out either $> {}_nV$ or $< {}_nV$, but it will exceed ${}_nV$ before the policy has been paid up one year.

Hence: "the addition of a constant to $q$ leads to higher reserves after some duration $< r$, which may be duration 0." For example, a single premium policy would have ${}_nV' > {}_nV$ throughout, whereas a "Pay to Age 85" policy would have ${}_nV' < {}_nV$ for almost all durations.

IV and V thus far have been applicable to limited payment endowments and term policies, but the following paragraph assumes a limited payment life policy only.

As in II($b$) the addition of $\dfrac{k(1+i)}{\ddot{a}_{y+1}}$ to $q_y$ again leads to $R_t \uparrow$ for $t < r$. This is perhaps most easily seen by writing

$$1 - {}_tV = 1 - A_y + P\,\ddot{a}_{y:\overline{z-y}|} = (d+P)\,\ddot{a}_y - PN_z/D_y\,,$$

where

$$z = x + r\,,$$

whence

$$(q'-q)(1 - {}_{t+1}V) = k(1+i)(d + P - PN_z/N_{y+1})\,,$$

which $\downarrow$ toward $ki$ as $N_{y+1} \downarrow$ toward $N_z$.

Hence

$$R_t \uparrow \text{ to } R_{r-1} = (P' - P)\,(1+i) - ki$$

and then drops to a negative constant, $-ki$. To satisfy (6) we must again have $R_{r-1} > 0$ and we conclude that "an addition of the CSO type leads to ***higher*** reserves after some duration $< r$, which may be duration 0." Cf. II($b$).

VI. $R_t \downarrow$

($a$) $q' = q$, ${}_tV \uparrow$, $i' < i$:

The argument and conclusion of V($a$) is effectively repeated with $i'$ and $i$ interchanged.

($b$) $i' = i$:

As in III($b$), $R_t \downarrow$ during the premium period if $(q'-q)(1 - {}_{t+1}V) \uparrow$. We shall consider only the case $q' > q$. Then $R_t < 0$ for $t \geqslant r$, and $R_t$

must start positive, but become negative at some duration $\leqslant r$. Hence (5) $> 0$ throughout and ${}_n\mathrm{V}' > {}_n\mathrm{V}$.

As in III($b$), the case $q' = q(1 + c)$ can be analyzed using

$$1 - {}_t\mathrm{V} = d\,\ddot{a}_{+t} + \mathrm{P}\,\ddot{a}_{x+t:\overline{z-x-t|}} = d\,\ddot{a}_{+t} + \mathrm{P}\,\ddot{a}_{y:\overline{z-y|}}$$

and

$$p_{x+t}\ddot{a}_{+t+1} = (1+i)\;a_{+t} \text{ and } p_{x+t}\ddot{a}_{y+1:\overline{z-y-1|}} = (1+i)\;a_{y:\overline{z-y-1|}}$$

to obtain, as a sufficient condition for ${}_n\mathrm{V}' > {}_n\mathrm{V}$, the inequality

$$\mathrm{P}\,(\Delta^2\ddot{a}_{y:\overline{z-y|}} - i\Delta\ddot{a}_{y:\overline{z-y|}}) > d\,(i\Delta\ddot{a}_{+t} - \Delta^2\ddot{a}_{+t})\,.$$

This reduces to the condition of III($b$) when we extend the premium period to the end of the table, *i.e.*,

$$\ddot{a}_{y:\overline{z-y|}} = \ddot{a}_y = \ddot{a}_{+t}\;,$$

or when the policy is an endowment maturing at age $z$, *i.e.*,

$$\ddot{a}_{y:\overline{z-y|}} = \ddot{a}_{+t}\;.$$

Before concluding, it may be remarked that the conditions $R_t \downarrow$ and $R_t \uparrow$ are sufficient to cause ${}_n\mathrm{V}' > {}_n\mathrm{V}$ and ${}_n\mathrm{V}' < {}_n\mathrm{V}$ respectively, but they are by no means necessary. All that is necessary in either case is that the

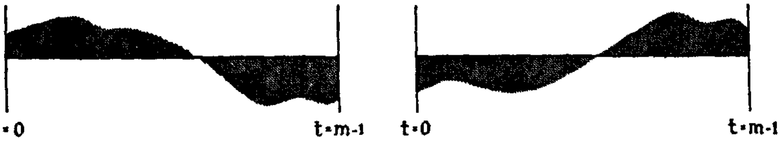


FIG. 1

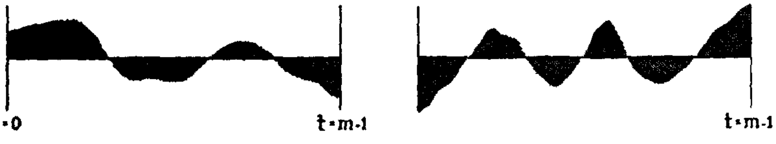


FIG. 2

partial sums (5) do not change sign as $n$ varies from 1 to $m$. This condition will always be satisfied if $R_t$ changes sign only once as in the graphs of $v^t\mathrm{D}'_{x+t}R_t$ (Fig. 1), but it could also be satisfied by $R_t$ changing sign any *odd* number of times, as in the graphs in Figure 2. To satisfy (6) the total area above the line must equal that below the line, *i.e.*, the total algebraic area from $t = 0$ to $t = n$ must be zero when $n = m - 1$. But it must not change sign prior to $m - 1$, *i.e.*, the area under the first section of the curve must not be canceled completely until the end of the last section.

Since writing this Note, I have seen a paper in *The Proceedings of the Centenary Assembly of the Institute of Actuaries, Vol. II*, entitled "On changes in policy values caused by alterations in the basis of valuation," by W. Simonsen of the University of Copenhagen.

His equation (3.16) closely corresponds to my (5), although his $K_t$ is defined somewhat differently than my $v^t R_t$. Doubtless many other readers of Spurgeon have, for their own peace of mind, put the argument on an algebraic basis. My aim has been chiefly to make the analysis easily accessible to students.