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THE MATHEMATICAL RISK OF LUMP-SUM DEATH BENEFITS IN A TRUSTEED PENSION PLAN

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It is not infrequent nowadays to find a lump-sum death benefit in a trusteed pension plan. The opinion has been expressed, however, that the presence of such a benefit exposes a small pension fund to grave risks of adverse chance deviations from the "expectation" on which the contributions have been based.

The purpose of this note is to investigate whether, *for a given level annual contribution rate*, the introduction of a lump-sum death benefit in lieu of a part of the pension otherwise payable, can result in a reduction of the mathematical risk due to chance deviations from expectation. The mathematical derivation of the results is given in some generality. However, in the illustrative example only a particular case is considered, namely that of a level term insurance combined with a deferred annuity.

Rosenthal (*RAIA* XXXVI, 6) has considered the general problem of random deviations of an insurance company's claims from the expected values foreseen in the net premium calculations. Readers are referred to his paper for a description of the underlying theory. An interesting numerical illustration of its application to the reserves of an annuity fund is to be found in a paper by R. H. Taylor, "The Probability Distribution of Life Annuity Reserves and Its Application to a Pension System" (*Proc. Confer. Actuar.*, II, 1952, 100).

Briefly, risk theory assumes that a definite probability can be ascribed to every insurable contingency that can occur to each individual of a specified group. The random variable which is of interest to the insurance company (or pension fund) is the net loss, *i.e.*, the benefit paid minus the premium (or contribution) received, both discounted to a given date.

For example, suppose the net loss that would be suffered by the company on an individual (x_j) if he were to die at time t is written as $L_t(x_j)$. In the simple case where death is the only contingency considered, the probability of death at age $x_j + t$ is

$$ {}_tp_{x_j}\mu_{x_j+t}dt $$

in the usual notation. In statistical terminology

$$ {}_tp_{x_j}\mu_{x_j+t}dt $$

is the probability density at time t and

$$\int_0^\infty {}_tp_{x_j}\mu_{x_j+t}dt = 1 .$$

The expected net loss is given by

$$L(x_j) = \int_0^\infty L_t(x_j)\, {}_tp_{x_j}\mu_{x_j+t}dt$$

and is equated to zero when the net premium is determined. The second moment of the net losses is

$$\mu_2(x_j) = \int_0^\infty \{L_t(x_j)\}^2 {}_tp_{x_j}\mu_{x_j+t}dt .$$

Since this moment is taken about the mean, its square root is the so-called standard deviation of the distribution of net losses.

As an example, it is obvious that in the case of a level unit Ordinary Life insurance subject to a continuous premium $\bar{P}_{x_j}$,

$$L_t(x_j) = v^t - \bar{P}_{x_j}\bar{a}_{\overline{t|}} \qquad 0 \leqslant t < \infty$$

and

$$\mu_2(x_j) = \int_0^\infty (v^t - \bar{P}_{x_j}\bar{a}_{\overline{t|}})^2 {}_tp_{x_j}\mu_{x_j+t}dt .$$

The latter may easily be written in terms of single premiums and interest functions.

The preceding concepts may be applied to a group of N lives (x_j), $j = 1, 2, 3, \ldots N$. In this case, from well-known theorems in mathematical statistics, we obtain for the expected total net loss, L, and for the second moment, μ_2, of such losses:

$$L = \sum_{j=1}^{N} L(x_j) \qquad \text{and} \qquad \mu_2 = \sum_{j=1}^{N} \mu_2(x_j) .$$

Furthermore, it is known that as N increases, the distribution of

$$L = \sum_{j=1}^{N} L(x_j) ,$$

the total losses, tends reasonably quickly to the normal, with zero mean and standard deviation equal to $\sqrt{\mu_2}$. This means that the probability of a given total loss exceeding L_0 is given by

$$\frac{1}{\sqrt{2\pi}}\int_{L_0/\sqrt{\mu_2}}^\infty e^{-t^2/2}dt = \Phi\left(\frac{L_0}{\sqrt{\mu_2}}\right).$$

This sketch of the theory of risk has been based on a single decrement, namely death. It can, however, easily be extended to several decrements. The probability of a given excess of benefit payments from a pension fund is thus determined, approximately, as a function of the standard deviation of all possible deviations of benefit payments from expectation.

In order, therefore, to proceed with the investigation it is only necessary to consider the variance, M^2 say, of the various possible profits and losses which may arise owing to an individual's participation in the plan. The aim is to study the behavior of M^2 regarded as a function of m, the multiple of the employee's final salary which is paid on death prior to retirement, and k_t, the proportion of the final annual salary which is paid as a pension for life if retirement occurs after t years of participation. Changes in m and k_t must occur in such a way that the annual contribution, payable continuously from entry to retirement or previous death, remains fixed at a specified level of salary.

It is assumed that the pension benefits, payable continuously, amount to $100\,k\%$ of final salary for every year of service rendered on retirement due to disabling ill-health prior to age 65 or on retirement due to the attainment of age 65; we thus have $k_t = kt$. No severances from employment are assumed to occur and the plan is supposed to be noncontributory.

Write $l_x^{aa} = 1 = l_x^{aa(1)}$, $l_x^{aa(2)} = 0$, and define $l_z^{aa(1)}$ and $l_z^{aa(2)}$ as follows:

$$0 > dl_z^{aa} = -l_z^{aa}\mu_z^a dz - l_z^{aa}\nu_z dz \qquad x \leqslant z < 65$$

$$\equiv dl_z^{aa(1)} + dl_z^{aa(2)}$$

where μ_z^a and ν_z denote, respectively, the continuous forces of mortality in active service and of disabling ill-health retirement at age z. Note that $l_z^{aa(1)}$, for instance, is the probability of a man aged x attaining age z as an active life.

It is assumed, further, that there is a discontinuity in the (otherwise continuous) curve of l_z^{aa} at $z = 65$, at which age all remaining employees retire. We write

$$l_z^{aa(1)} - l_{z+\epsilon}^{aa(1)} = 0, \qquad l_z^{aa(2)} - l_{z+\epsilon}^{aa(2)} = l_z^{aa} \qquad z = 65\,, \quad \epsilon > 0$$

and

$$l_z^{aa} \equiv 0 \qquad z > 65$$

Let the pensioners' (continuous) survivorship functions be on a "select" basis and be distinguished by the affix i. Then, for entry age x, we may express the probability that an employee must die either in service or after retirement, respectively, as

$$-\int_0^\infty dl_{x+t}^{aa(1)} - \int_0^\infty dl_{x+t}^{aa(2)} \int_0^\infty {}_r p_{[x+t]}^i \mu_{[x+t]+r}^i dr = 1$$

where the integrals are to be read in the Stieltjes sense.

The principle of equivalence of benefit and contribution values implies that the sum of the expected net losses on death in service and after retirement, respectively, should be equal to zero. Hence

$$\int_0^\infty L_t^{(d)} dl_{x+t}^{aa(1)} - \int_0^\infty dl_{x+t}^{aa(2)} \int_0^\infty {}_r p_{[x+t]}^i \mu_{[x+t]+r}^i L_{[t]+r}^{(i)} dr = 0 \quad (1)$$

where

$$L_t^{(d)} = e^{-\delta t}\left[m\, s_t - \int_0^t e^{\delta(t-u)} s_u du\right] \qquad 0 \leqslant t < 65 - x$$

and

$$L_{[t]+r}^{(i)} = e^{-\delta(t+r)}\left[kt\, s_t \bar{s}_{\overline{r}|} - \int_0^t e^{\delta(t+r-u)} s_u du\right] \quad 0 \leqslant t \leqslant 65 - x\, ;\ 0 \leqslant r.$$

The fixed contribution rate has, for convenience, been chosen at 100% of salary.

The variance of the pension fund's net losses on an individual who enters at age x with unit salary and progresses along a continuous salary scale s_t ($s_0 = 1$), is equal to the sum of the squares of the net losses on death in service and after retirement, respectively; that is

$$M^2 = -\int_0^\infty \{L_t^{(d)}\}^2 dl_{x+t}^{aa(1)}$$
$$- \int_0^\infty dl_{x+t}^{aa(2)} \int_0^\infty {}_r p_{[x+t]}^i \mu_{[x+t]+r}^i \{L_{[t]+r}^{(i)}\}^2 dr\,. \quad (2)$$

Write

$$A\,(j,\, f_t) = -\int_0^\infty e^{-j\delta t} (s_t)^j f_t\, dl_{x+t}^{aa(1)} \qquad j = 0,\, 1,\, 2$$

$$I\,(j,\, f_t) =$$
$$-\int_0^\infty t^j e^{-j\delta t} (s_t)^j f_t\, dl_{x+t}^{aa(2)} \int_0^\infty {}_r p_{[x+t]}^i \mu_{[x+t]+r}^i (\bar{a}_{\overline{r}|})^j dr \quad j = 0,\, 1,\, 2$$

and

$$S_t = \int_0^t e^{-\delta u} s_u du\,.$$

Then

$$M^2 = A\,(2,\, 1)\, m^2 + I\,(2,\, 1)\, k^2$$
$$- 2A\,(1,\, S_t)\, m - 2I\,(1,\, S_t)\, k + A\,(0,\, S_t^2) + I\,(0,\, S_t^2)$$

and (1) becomes

$$A\,(1,\,1)\,m + I\,(1,\,1)\,k = A\,(0,\,S_t) + I\,(0,\,S_t)\,. \qquad (3)$$

Substituting the value for k from this equation into the relation for M^2 we find

$$M^2 = \left\{ A\,(2,\,1) + I\,(2,\,1)\frac{A^2\,(1,\,1)}{I^2\,(1,\,1)} \right\} m^2 - 2\Big[A\,(1,\,S_t) + \{ A\,(0,\,S_t) + I\,(0,\,S_t) \} \frac{A\,(1,\,1)\,I\,(2,\,1)}{I^2\,(1,\,1)} - A\,(1,\,1)\frac{I\,(1,\,S_t)}{I\,(1,\,1)} \Big] m + A\,(0,\,S_t^2) + I\,(0,\,S_t^2) + \{ A\,(0,\,S_t) + I\,(0,\,S_t) \}^2 \frac{I\,(2,\,1)}{I^2\,(1,\,1)} - 2\{ A\,(0,\,S_t) + I\,(0,\,S_t) \} \frac{I\,(1,\,S_t)}{I\,(1,\,1)}\,. \qquad (4)$$

M^2 is thus a quadratic in m and assumes its minimum value when m equals the quantity in "square" brackets divided by the coefficient of m^2. Now if, and only if, the coefficient of m is negative, M^2 decreases as m increases from zero, and attains its minimum for a positive m-value. We note that the function in "square" brackets in the coefficient of m may be expressed as a quadratic in I (1, 1). If, therefore, it is to be positive for all values of I (1, 1),

$$\frac{I^2\,(1,\,S_t)}{I\,(2,\,1)} < 4\,\frac{A\,(1,\,S_t)}{A\,(1,\,1)}\,\{ A\,(0,\,S_t) + I\,(0,\,S_t) \} = -4\,\frac{A\,(1,\,S_t)}{A\,(1,\,1)} \int_0^\infty S_t\, dl^{aa}_{x+t}\,.$$

This is, therefore, a sufficient condition for an advantageous conversion of part of the pension rate, k, into a death benefit.

The upper limit of the range of permissible variation for m is given by (3) on putting $k = 0$; the result is

$$m = -\{ A\,(1,\,1) \}^{-1} \int_0^\infty S_t\, dl^{aa}_{x+t}\,.$$

If this value is less than the value of m corresponding to the minimum M^2, the grant of a death benefit alone is less "risky" than the provision of any combination of pension and death benefits.

Some numerical examples of the above arguments will now be provided. They are based on the special case where disability retirements prior to age 65 are assumed not to exist and where the salary scale is "level." Consider a "combination" insurance consisting of a deferred annuity of k' per annum commencing at age 65 and an immediate

$(65 - x)$-year term insurance for a sum insured of m. The net annual premium, payable continuously for $65 - x$ years or until previous death, is fixed at one unit.

We thus have

$$\nu_{x+t} \equiv 0 \qquad 0 \leqslant t < 65 - x \,; \qquad s_t \equiv 1 \qquad 0 \leqslant t \leqslant 65 - x \,;$$

$$(65 - x)\, k = k' \,; \qquad l^{aa}_{x+t} = {}_t p_x \qquad 0 \leqslant t \leqslant 65 - x \,; \qquad {}_r p^{i}_{[65]} = {}_r p_{65}$$

$$0 < r < \infty \;;$$

and, if "primed" functions refer to a rate of interest $(1 + i)^2 - 1$,

$$S_t = \bar{a}_{\overline{t}|} \qquad A\,(1,\,1) = \bar{A}^{\,1}_{x:\overline{65-x}|} \qquad A\,(2,\,1) = \bar{A}'^{\,1}_{x:\overline{65-x}|}$$

$$A\,(0,\,S_t) = \bar{a}_{x:\overline{65-x}|} - {}_{65-x}p_x \bar{a}_{\overline{65-x}|} \qquad A\,(1,\,S_t) = \frac{1}{\delta}\,(\bar{A}^{\,1}_{x:\overline{65-x}|} - \bar{A}'^{\,1}_{x:\overline{65-x}|})$$

$$A\,(0,\,S_t^2) = \frac{1}{\delta}\{\,A\,(0,\,S_t) - A\,(1,\,S_t)\,\}$$

$$I\,(1,\,1) = (65 - x)\, A_{x:\overline{65-x}|}^{\;\;\;1}\, \bar{a}_{65}$$

$$I\,(2,\,1) = (65 - x)^2 A'^{\;\;\;1}_{x:\overline{65-x}|}\,\frac{2}{\delta}\,(\bar{a}_{65} - \bar{a}'_{65})$$

$$I\,(0,\,S_t) = {}_{65-x}p_x \bar{a}_{\overline{65-x}|} \qquad I\,(1,\,S_t) = (65 - x)\, A_{x:\overline{65-x}|}^{\;\;\;1}\, \bar{a}_{\overline{65-x}|}\, \bar{a}_{65}$$

$$I\,(0,\,S_t^2) = {}_{65-x}p_x \bar{a}^{2}_{\overline{65-x}|}\,.$$

Relation (4) then becomes

$$M^2 = \left\{ \bar{A}'^{\,1}_{x:\overline{65-x}|} + 2\,\frac{\bar{a}_{65} - \bar{a}'_{65}}{{}_{65-x}p_x\,\delta\,\bar{a}^2_{65}}\,\bar{A}^{\,2}_{x:\overline{65-x}|} \right\} m^2 - \frac{2}{\delta}\left[\left\{ \bar{a}_x \bar{a}_{65} \right.\right.$$

$$\left.\left. + \,(\bar{a}_{65} - 2\,\bar{a}'_{65})\, \bar{a}_{x:\overline{65-x}|} \right\} \frac{\bar{A}^{\,1}_{x:\overline{65-x}|}}{{}_{65-x}p_x \bar{a}^2_{65}} - \bar{A}'^{\,1}_{x:\overline{65-x}|} \right] m$$

$$+ \frac{2}{\delta}\left[\frac{\bar{a}_{65} - \bar{a}'_{65}}{{}_{65-x}p_x \bar{a}^2_{65}}\, \bar{a}^2_{x:\overline{65-x}|} - (\bar{a}'_{x:\overline{65-x}|} - v^{65-x} \bar{a}_{x:\overline{65-x}|}) \right]$$

and k' is determined from

$$k' = (A_{x:\overline{65-x}|}^{\;\;\;1}\, \bar{a}_{65})^{-1}\,(\bar{a}_{x:\overline{65-x}|} - \bar{A}^{\,1}_{x:\overline{65-x}|}\, m)\,.$$

The numerical values in Tables 1 and 2 are based on the *a*-1949 Table with $2\frac{1}{2}\%$ interest (*TSA* I, 369). Taylor's paper already cited contains functions at 5.0625% interest for ages 60 and over. The necessary extension to lower ages was effected in the usual way.

A number of interesting conclusions may be drawn from the second of these tables. For the special case of the "combination" insurance described on pages 139–40:

1. The most favorable lump-sum death benefit from the "risk" point of view amounts to about 7.8 times the annual pension rate. This multiple represents

TABLE 1

EXPRESSION OF M^2 AND k' IN TERMS OF m

x	M^2			k'	
	Coeff. of m^2	Coeff. of m	Constant term	Coeff. of m	Constant term
20.......	.06378	−6.872	344.8	−.03241	8.531
25.......	.07679	−6.880	292.2	−.03138	6.998
30.......	.09205	−6.716	238.7	−.03023	5.648
35.......	.1094	−6.326	186.8	−.02888	4.461
40.......	.1279	−5.654	137.0	−.02719	3.419
45.......	.1448	−4.652	92.0	−.02488	2.507
50.......	.1507	−3.330	54.1	−.02118	1.715
55.......	.1347	−1.865	25.0	−.01571	1.036
60.......	.08724	− .6250	5.70	−.008578	.4667

TABLE 2

CHARACTERISTIC VALUES OF M^2 AND m/k'

x	m and k' Chosen So That Variance of Net Losses a Minimum		Values of M^2 for $m=0$	Values of m/k' Corresponding to M^2 of Previous Column	k', the Amount of Deferred Annuity, Equal to Zero ($m/k'=\infty$)
	M^2	m/k'			M^2
20.......	159.7	7.94	344.8	0, 21.4	2,954
25.......	138.1	8.01	292.2	0, 21.4	2,577
30.......	116.2	8.03	238.7	0, 21.2	2,198
35.......	95.4	7.98	186.8	0, 20.7	1,820
40.......	74.6	7.84	137.0	0, 19.9	1,449
45.......	54.7	7.62	92.0	0, 18.8	1,094
50.......	35.7	7.46	54.1	0, 17.7	773
55.......	18.5	7.46	25.0	0, 16.9	488
60.......	4.6	8.23	5.7	0, 17.7	230

the average over the nine entry ages for which calculations were made. The variation between the different ages is, however, surprisingly small.

2. The same "risk" is run by a fund paying no death benefit as by one providing a lump-sum death benefit of about 19.5 times the annual pension rate. Here, again, the variation of the multiple between the different entry ages is relatively small.
3. The risk of granting a death benefit alone ($k' = 0$) is considerably in excess of that assumed when a pension of equivalent value is provided ($m = 0$).

An alternative method of comparison may be illustrated using the results for age at entry 30. Table 3 shows the probabilities of various

TABLE 3

POSITIVE OR NEGATIVE DEVIATIONS FROM EXPECTATION

Probability of Exceeding Deviation Shown	Death Benefit 8.03 Times Annual Pension	No Death Benefit	No Pension Benefit
.4	2.73	3.91	11.88
.3	5.65	8.10	24.59
.2	9.07	13.00	39.46
.1	13.81	19.80	60.08
.05	17.73	25.41	77.12
.01	25.08	35.94	109.07
.001	33.31	47.74	144.88

(eventual) deviations from "expectation" on the assumption that a large number of "combination" insurances are effected simultaneously at age 30, each for the same sum insured and deferred annuity. The deviations are expressed in contribution-units, so that if, for example, the total age-at-entry contribution for the group were \$100,000 (comprising, say, 10,000 insured each paying \$10 a year), the numbers in the body of the table would all require multiplication by 100,000 to obtain results in dollars.