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ACTUARIALLY EQUIVALENT BENEFITS

by

Jeff L. Schwartzmann, ASA, MAAA
and
Ralph Garfield, ASA, FIA, MAAA

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I. INTRODUCTION

In the ongoing administration of defined benefit plans, the concept of actuarial equivalence arises when it becomes necessary to change a pension payable on one basis into a pension payable on another basis.

Some examples of typical changes are:

- o A pension commencing at one age into a pension commencing at a different age.
- o A pension payable on a life only basis into a pension payable for life with a period certain.
- o A pension payable on a life only basis into a pension payable on a joint and survivor basis.

The fundamental principle underlying the above calculations is the "equation of value". The "equation of value" means that the actuarial present values of the pensions on each basis are equal.

The four most important plan provisions used to determine the present value of a pension are:

- o The dollar amount of the pension and how frequently it is paid (e.g. monthly or annually). This provision is most commonly found in the definition of the Normal Retirement Benefit.
- o The age at which the pension commences. This provision is most commonly found in the definition of the Normal Retirement Age.
- o The period over which the pension will be paid. This provision is most commonly found in the definition of the Normal Form of Payment.
- o The definition of Actuarial Equivalence.

In this study note, we will present some common examples of converting benefits for normal, early and late retirement. We will also analyze how changes in a plan's actuarial equivalence assumptions can affect the magnitude of the converted benefits.

II. NORMAL RETIREMENT

A. Introduction

All defined benefit plans specify a Normal Retirement Age. The Normal Retirement Age is usually defined as a function of age, years of service, or years of participation. Some examples of Normal Retirement Age are:

- The attainment of age 65 (age requirement only).
- The later of the attainment of age 60 and the completion of 10 years of service (age and service requirement).
- The later of the attainment of age 55 and the completion of 5 years of participation (age and participation requirement).
- The completion of 20 years of service (service requirement only).

It should be noted that Internal Revenue Code (IRC) Section 411(a)(8) requires the Normal Retirement Age in a tax qualified retirement plan to be the earlier of:

- The Normal Retirement Age under the plan.
- The later of the attainment of age 65 and the completion of 5 years of participation.

Unless specified otherwise, the examples in this study note will incorporate the following assumptions:

- The Normal Form of Payment is a single life annuity. In many examples, we will determine the appropriate factor to convert the single life annuity benefit payable at Normal Retirement Age to the desired optional form of benefit payable at the desired retirement age.
- The primary annuitant is male.

- o In some pension plans, the interest rate specified in the definition of Actuarial Equivalence is distinguished between ages prior to the attainment of Normal Retirement Age ("pre-retirement") and ages commencing with the attainment of Normal Retirement Age ("post-retirement"). In this study note, we will assume the pre-retirement interest rate and the post-retirement interest rate are the same and the interest rate is constant.

For clarity, we define the following notation:

y = Normal Retirement Age of annuitant.

$(AP)_y$ = Monthly pension payable at age y.

$(BP)_y$ = Actuarially equivalent monthly pension payable on the given optional form of payment.

B.1 Certain and Life Annuity

Assume the certain period is n years. Then the equation of value is:

$$12\,(AP)_y * \ddot{a}_y^{(12)} = 12\,(BP)_y * \left(\ddot{a}_{\overline{n|}}^{(12)} + \frac{N_{y+n}^{(12)}}{D_y}\right)$$

Solving for $(BP)_y$, we obtain:

$$(BP)_y = (AP)_y * \ddot{a}_y^{(12)} / \left(\ddot{a}_{\overline{n|}}^{(12)} + \frac{N_{y+n}^{(12)}}{D_y}\right)$$

We will call the quantity which is multiplied by $(AP)_y$ the Conversion Factor (CF). We define the CF to be:

$$CF = \frac{\ddot{a}_y^{(12)}}{\ddot{a}_{\overline{n|}}^{(12)} + \frac{N_{y+n}^{(12)}}{D_y}} \qquad (1)$$

Example 1

A defined benefit plan provides a retiring participant a $1,000 monthly pension for life commencing at a normal retirement age of 62. The retiring participant's age is exactly 62.

You are given the following commutation functions:

x	D_x	N_x	$N_x^{(12)}$
30	16196	246134	238711
45	6121	84993	82188
60	2141	25674	24693
62	1853	21549	20700
65	1468	16406	15733
67	1256	13586	13010
72	835	8205	7806
77	537	4663	4417

You are also given the following annuity certain values:

$$\ddot{a}_{\overline{5}|}^{(12)} = 4.45$$
$$\ddot{a}_{\overline{10}|}^{(12)} = 7.93$$
$$\ddot{a}_{\overline{15}|}^{(12)} = 10.66$$

(a) Compute the monthly pension commencing at age 62 and payable on a 5 year certain and life basis.

(b) Compute the monthly pension commencing at age 62 and payable on a 10 year certain and life basis.

(c) Compute the monthly pension commencing at age 62 and payable on a 15 year certain and life basis.

Solution

(a) The equation of value is:

$$1000 * 12 * \frac{N_{62}^{(12)}}{D_{62}} = 12\,(BP)_{62} * \left(\ddot{a}_{\overline{5}|}^{(12)} + \frac{N_{67}^{(12)}}{D_{62}}\right)$$

Thus, the monthly pension is \$973.85.

(b) The equation of value is:

$$1000 * 12 * \frac{N_{62}^{(12)}}{D_{62}} = 12\,(BP)_{62} * \left(\ddot{a}_{\overline{10}|}^{(12)} + \frac{N_{72}^{(12)}}{D_{62}}\right)$$

Thus, the monthly pension is \$919.99.

(c) The equation of value is:

$$1000 * 12 * \frac{N_{62}^{(12)}}{D_{62}} = 12(BP)_{62} * \left(\ddot{a}_{\overline{15}|}^{(12)} + \frac{N_{77}^{(12)}}{D_{62}}\right)$$

Thus, the monthly pension is $856.43.

B.2 Level Payment Social Security Option

Sometimes a participant retires before he is eligible to receive Old Age Social Security Benefits. The purpose of the Level Payment Social Security Option is to allow the participant to receive a series of benefit payments from the defined benefit trust so that the combined payments from the defined benefit trust and from Social Security are constant throughout his lifetime. The next two paragraphs discuss the eligibility requirements to receive Old Age Social Security Benefits.

An individual normally waits until his attainment of the Social Security Retirement Age (SSRA) to begin collecting his Old Age Social Security Benefit. This is because the initial benefit will not be reduced. Before 1983, the SSRA was uniformly set at age 65. However, Section 216(l) of the 1983 Social Security Act enacted by Congress caused the SSRA to depend on a participant's year of birth. If an individual was born before 1/1/38, his SSRA is still age 65. If an individual was born after 12/31/59, his SSRA is age 67. Finally, if an individual was born after 12/31/37 but before 1/1/60, his SSRA is a fractional age varying between age 65 and age 67, and it depends on his year of birth.

Alternatively, a participant may begin collecting a reduced Old Age Social Security Benefit as early as the attainment of age 62. This reduced benefit is determined by multiplying the benefit payable at the attainment of SSRA by ((1)-(2)-(3)):

(1) 1.

(2) 5/9% per month for each of the first 36 months preceding the SSRA.

(3) 5/12% per month for each of the next 24 months preceding the SSRA.

In developing the theory, we define the additional notation:

s = Integral age at which Old Age Social Security Benefits commence. It should be noted that $y<s$ under this payment option.

$(SS)_s$ = Monthly Old Age Social Security Benefit commencing at age s. If applicable, it should be assumed that this benefit has already been adjusted for commencement of benefits before the attainment of the SSRA.

Thus, we want the plan to pay $(BP)_y$ from age y to age s, and $[(BP)_y - (SS)_s]$ thereafter. It should be noted that the analysis makes sense only if $(BP)_y \geq (SS)_s$. If $(BP)_y < (SS)_s$, then after the participant attains age s, he would <u>owe</u> the defined benefit trust an amount each month of $[(SS)_s - (BP)_y]$.

The equation of value is as follows:

$$12\,(AP)_y * \frac{N_y^{(12)}}{D_y} = 12\,(BP)_y * \left(\frac{N_y^{(12)}-N_s^{(12)}}{D_y}\right) + 12[(BP)_y-(SS)_s] * \frac{N_s^{(12)}}{D_y}$$

Solving for $(BP)_y$, we obtain:

$$(BP)_y = (AP)_y + (SS)_s * \frac{N_s^{(12)}}{N_y^{(12)}}$$

This time we define the CF to be the quantity which is multiplied by $(SS)_s$:

$$CF = \frac{N_s^{(12)}}{N_y^{(12)}} \tag{2}$$

<u>Example 2</u>

Use the same data from Example 1, and further assume that the participant's SSRA is age 65, that the participant will elect to commence receiving his Old Age Social Security Benefit at age 65, and that $(SS)_{65}$ = \$700/month. What is the retiring participant's initial monthly pension from the defined benefit trust at age 62 under the Social Security Level Payment Option, and what will the participant's monthly pension drop to at age 65?

Solution

The equation for $(BP)_{62}$ is:

$$(BP)_{62} = (AP)_{62} + (SS)_{65} * \frac{N_{65}^{(12)}}{N_{62}^{(12)}}$$

Thus, the initial monthly pension at age 62 under the Social Security Level Payment Option is $1,532.03, and drops to a monthly pension of $832.03 at age 65.

B.3 Joint and Survivor Annuities

B.3.a Introduction

Under ERISA, defined benefit plans (and other tax qualified retirement plans that offer a life annuity as an optional form of benefit) are required to provide a pension on a "qualified joint and survivor" basis. A "qualified joint and survivor" pension is a level annuity paid to the plan participant over the participant's lifetime, and if the participant should predecease the spouse, a pension would be paid over the surviving spouse's remaining lifetime between 50% and 100% of the original pension amount.

If the original pension is not reduced after the participant's death, the annuity is customarily referred to as a "100% joint and survivor annuity". Similarly, if the original pension is reduced by 25% after the participant's death, the annuity is customarily referred to as a "75% joint and survivor annuity".

Most commonly, the qualified joint and survivor annuity is offered on one of the following bases:

- The normal form of payment is a single life annuity, and the designated qualified joint and survivor annuity is the actuarial equivalent of the normal form of payment.

- The normal form of payment is a certain and life annuity, and the designated qualified joint and survivor annuity is the actuarial equivalent of the normal form of payment (subject to the constraints of IRC Section 415 on benefit amounts and lump sums).

- The normal form of payment is a designated qualified joint and survivor annuity (subject to the constraint of IRC Section 415 on lump sums). The plan document should state the policy for determining benefit adjustments for unmarried participants.

In the next several paragraphs, we will examine examples for converting a single life annuity to a qualified joint and survivor annuity (Sections II.B.3.b and II.B.3.c), and also to a "pop-up" annuity (Section II.B.3.d).

<u>B.3.b</u> <u>100% Joint and Survivor Annuity</u>

In developing the theory, we define the additional notation:

z = Age of co-annuitant when annuitant is age y.

The equation of value is:

$$12(AP)_y * \ddot{a}_y^{(12)} = 12(BP)_y * \left(\ddot{a}_y^{(12)} + \ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}\right)$$

Solving for $(BP)_y$, we obtain:

$$(BP)_y = (AP)_y * \frac{\ddot{a}_y^{(12)}}{\ddot{a}_y^{(12)} + \ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}}$$

We will call the quantity which is multiplied by $(AP)_y$ CF_{100}. We divide the numerator and denominator of this expression by $\ddot{a}_y^{(12)}$ and then define CF_{100} to be:

$$CF_{100} = \frac{1}{1 + \left(\frac{\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}}{\ddot{a}_y^{(12)}}\right)} \qquad (3)$$

<u>B.3.c</u> <u>p% Joint and Survivor Annuity</u>

We now want to examine a p% Joint and Survivor Annuity. The equation of value is:

$$12(AP)_y * \ddot{a}_y^{(12)} = 12(BP)_y * \left(\ddot{a}_y^{(12)} + \frac{P}{100} * \left(\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}\right)\right)$$

Solving for $(BP)_y$, we obtain:

$$(BP)_y = (AP)_y * \frac{\ddot{a}_y^{(12)}}{\ddot{a}_y^{(12)} + \frac{P}{100} * (\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)})}$$

We will call the quantity which is multiplied by $(AP)_y$ CF_p. We divide the numerator and denominator of this expression by $\ddot{a}_y^{(12)}$ and then define CF_p to be:

$$CF_p = \frac{1}{1 + \frac{P}{100} * \left(\frac{\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}}{\ddot{a}_y^{(12)}} \right)} \tag{4}$$

If in Expression (3) above we solve for $\left(\frac{\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)}}{\ddot{a}_y^{(12)}} \right)$ in terms of CF_{100}, and then substitute this into the above expression for CF_p, after simplifying we obtain:

$$CF_p = \frac{CF_{100}}{CF_{100} + \frac{P}{100} * (1 - CF_{100})}$$

Example 3

Assume that the 100% joint and survivor conversion factor (e.g. CF_{100}) is 2/3. What is the 75% joint and survivor conversion factor (e.g. CF_{75})?

Solution

With CF_{100}=2/3, then $CF_{75} = \dfrac{\frac{2}{3}}{\frac{2}{3} + \frac{75}{100} * (1-\frac{2}{3})} = \dfrac{8}{11}$

Example 4

A defined benefit plan provides a participant a normal form pension of a single life monthly annuity of $1,250. Alternatively, the plan provides an actuarially equivalent 50% joint and survivor monthly annuity with an initial payment of $1,000. In lieu of the above, the participant wants $Q payable for life with $ 2/3Q continuing to the survivor for life. Determine Q.

Solution

Clearly we are given that CF_{50}=4/5, and we need to determine $CF_{66\ 2/3}$. With CF_{50}=4/5, we have the following:

$$CF_{50} = \frac{4}{5} = \frac{CF_{100}}{CF_{100} + \frac{1}{2} * (1-CF_{100})}$$

Solving for CF_{100}, we obtain CF_{100}=2/3.

Now substituting CF_{100} into the expression for $CF_{66\ 2/3}$, we obtain:

$$CF_{66\ 2/3} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{2}{3} * (1 - \frac{2}{3})} = \frac{3}{4}$$

Thus, Q = 1,250 x .75 = 937.50.

B.3.d Pop-Up Annuities

Some plans provide "pop-up annuities" as an optional form of payment. A pop-up annuity is a special type of joint and survivor annuity. Assume that the normal form of payment is a single life annuity, and that the pop-up annuity is the actuarial equivalent of the single life annuity. Then in terms of our notation, a pop-up annuity is an annuity in which a benefit of $(BP)_y$ is provided while the co-annuitant is alive, but if the co-annuitant should predecease the annuitant the benefit increases to $(AP)_y$. The equation of value is:

$$12(AP)_y * \ddot{a}_y^{(12)} = 12(BP)_y * \ddot{a}_z^{(12)} + 12(AP)_y * \left(\ddot{a}_y^{(12)} - \ddot{a}_{yz}^{(12)}\right)$$

Solving for $(BP)_y$, we obtain:

$$(BP)_y = (AP)_y * \frac{\ddot{a}_{yz}^{(12)}}{\ddot{a}_z^{(12)}}$$

C. Effect of Changes in Actuarial Equivalence Assumptions

C.1 Introduction

The interest and mortality assumptions play a key role in determining the magnitude of the actuarial equivalence factor. Periodically, the assumptions used must be reviewed and modified so as to insure that they continue to fairly assess the cost of the optional basis of payment.

To proceed with the analysis, we will examine the conversion factors defined in Expressions (1), (2), (3) and (4) above. To ease the analysis, we will disregard the "upper 12's". We will also denote "i" for the assumed annual rate of return and "q" for the assumed mortality rates. The mortality analysis is also applicable when a participant's age is found to be incorrect, or a spousal setback is changed.

C.2 Interest

C.2.a Expression (1)

Unfortunately, the authors are not aware of any straightforward analytical technique to make a general evaluation of Expression (1) on page 3. This is due to the complex interrelationship between the certain period, the current age of the participant, and the mortality and interest assumptions. However, we hope the following analysis will be helpful to the student.

In most cases we will conclude that when n is greater then the life expectancy for a participant age y (e.g. $n>e_y$), then Expression (1) <u>increases</u> for an increase in i, and vice versa.

The technique we will use is based on the following example. Consider two annuities certain with payments of 1 made at the beginning of each year for 10 years and 20 years, respectively.

The sum of the payments for the 10 year certain annuity is 10. For this annuity, it should be clear there exists a point in time t1>0, such that $\ddot{a}_{\overline{10|}}=10v^{t1}$. Solving for t1, we obtain:

$$t1 = \frac{\log(\ddot{a}_{\overline{10|}}/10)}{\log v}$$

t1 can be viewed as the point in time in the future where the sum of the payments for $\ddot{a}_{\overline{10|}}$ (e.g. 10) can be "concentrated" so that the present value of the sum of the payments yields $\ddot{a}_{\overline{10|}}$.

Similarly, it should be clear there exists a point in time t2>0, such that $\ddot{a}_{\overline{20|}}=20v^{t2}$. Intuitively, it should be clear that t2>t1. This is because we should expect the "concentration point" to be approximately 5 years and 10 years in the future for the 10 year certain and 20 year certain annuities, respectively.

Thus, we should expect the ratio $\ddot{a}_{\overline{10|}}/\ddot{a}_{\overline{20|}}$ to <u>increase</u> as i increases, and vice versa. Table I shows sample values of $\ddot{a}_{\overline{10|}}$, $\ddot{a}_{\overline{20|}}$, and $\ddot{a}_{\overline{10|}}/\ddot{a}_{\overline{20|}}$ for various interest rates.

<u>Table I</u>

i	$\ddot{a}_{\overline{10|}}$	$\ddot{a}_{\overline{20|}}$	$\ddot{a}_{\overline{10|}}/\ddot{a}_{\overline{20|}}$
6%	7.802	12.158	.642
7%	7.515	11.336	.663
8%	7.247	10.604	.683

Now we use the results from the previous example in the following way. A rough approximation for $\ddot{a}_y$ is an annuity certain with payments of 1 at the beginning of each year for

e_y years. If we choose n large enough (e.g. n substantially larger than e_y), we should expect $\ddot{a}_{\overline{n}|}$ to be a rough approximation to $\ddot{a}_{\overline{n}|} + N_{y+n}/D_y$. Thus, for n large enough we should be able to analyze Expression (1) similarly to the way we analyzed the ratio $\ddot{a}_{\overline{10}|}/\ddot{a}_{\overline{20}|}$.

Table II shows life expectancies, e_y, for various retirement ages, and using the 1971 GAM Male Mortality Table.

Table II

y	e_y
55	22.21
60	18.26
65	14.61
70	11.41

Table III shows values of $\ddot{a}_y/(\ddot{a}_{\overline{n}|} + N_{y+n}/D_y)$ for various interest rates, retirement ages and certain periods, and using the 1971 GAM Male Mortality Table.

Table III

		Normal Retirement Age			
n	i	55	60	65	70
5	6%	.994	.988	.980	.962
5	7%	.994	.989	.980	.962
5	8%	.993	.989	.979	.962
10	6%	.975	.956	.922	.867
10	7%	.973	.955	.921	.867
10	8%	.973	.954	.921	.868
20	6%	.906	.867	.787	.683
20	7%	.909	.872	.795	.695
20	8%	.913	.878	.803	.707
30	6%	.827	.753	.666	.572
30	7%	.840	.771	.686	.596
30	8%	.852	.787	.707	.618

An examination of Table III shows that no conclusion about the certain and life conversion factor can be made for $n<e_y$. However, for $n>e_y$, we see in general that as i <u>increases</u>, the certain and life conversion factor <u>increases</u>, and vice versa.

C.2.b Expression (2)

Expression (2) is N_s/N_y. We have:

$$\frac{N_s}{N_y} = \frac{N_s}{N_{s-1}} \cdot \frac{N_{s-1}}{N_{s-2}} \cdot \cdot \cdot \frac{N_{y+1}}{N_y}$$

Now consider N_{y+1}/N_y. We have:

$$\frac{N_{y+1}}{N_y} = \frac{N_{y+1}}{N_{y+1}+D_y} = \frac{1}{1+ D_y/N_{y+1}} = \frac{1}{1+ 1/a_y}$$

Now, if i increases, then a_y decreases, then $1/a_y$ increases, then $(1 + 1/a_y)$ increases, and thus N_{y+1}/N_y decreases. It should then be clear that N_{y+t}/N_{y+t-1} decreases for all $1\le t\le (s-y)$. Thus, N_s/N_y decreases as i increases.

Thus, as i <u>increases</u> the conversion factor for the Social Security Supplement <u>decreases</u>, and vice versa.

C.2.c Expression (3) and Expression (4)

Expression (3) and Expression (4) each include the term $(\ddot{a}_z-\ddot{a}_{yz})/\ddot{a}_y$. It is sufficient to analyze how this term is impacted by a change in i.

Payments under the annuity $\ddot{a}_y$ are made only while the participant is alive. However, payments under the annuity $(\ddot{a}_z - \ddot{a}_{yz})$ are made only <u>after</u> the participant's death. Thus, we should expect $(\ddot{a}_z-\ddot{a}_{yz})$ to be impacted more by a change in the interest rate than $\ddot{a}_y$.

Thus, as i increases the expression $(\ddot{a}_z-\ddot{a}_{yz})/\ddot{a}_y$ decreases.

Thus, as i <u>increases</u> the conversion factors <u>increase</u> for both the 100% joint and survivor annuity, and the p% joint and survivor annuity.

C.3 Mortality

C.3.a Expression (1)

We can analyze Expression (1) directly if there is a change in the mortality assumption. If q increases, the numerator will decrease proportionately more than the denominator. This is because the annuity certain term in the denominator remains unchanged, and thus becomes a larger proportion of the denominator.

Thus, as q increases the conversion factor decreases for the certain and life annuity.

C.3.b Expression (2)

We will use similar reasoning as that used in analyzing Expression (2) for a change of interest in C.2.b above.

As above, we consider N_{y+1}/N_y.

Now, if q increases, then a_y decreases, then $1/a_y$ increases, then $(1 + 1/a_y)$ increases, and thus N_{y+1}/N_y decreases. It should then be clear that N_{y+t}/N_{y+t-1} decreases for all $1 \leq t \leq (s-y)$. Thus, N_s/N_y decreases as q increases.

Thus, as q increases the conversion factor for the Social Security Supplement decreases, and vice versa.

C.3.c Expression (3) and Expression (4)

We will consider two cases: (1) q for the annuitant increases and q for the co-annuitant remains unchanged, and (2) q for the co-annuitant increases and q for the annuitant remains unchanged. We will again analyze the term $(\ddot{a}_z - \ddot{a}_{yz})/\ddot{a}_y$.

If q for the annuitant increases and q for the co-annuitant remains unchanged, the numerator increases and the denominator decreases. The numerator increases because the survivor payments to the co-annuitant begin sooner. It is obvious why the denominator decreases. Thus, the ratio $(\ddot{a}_z - \ddot{a}_{yz})/\ddot{a}_y$ increases, and so the conversion factor decreases.

If q for the co-annuitant increases and q for the annuitant remains unchanged, the numerator decreases and the denominator remains unchanged. The numerator decreases because the survivor payments are not as likely to be made if the co-annuitant's life expectancy decreases. It is obvious why the denominator remains unchanged. Thus, the ratio $(\ddot{a}_z - \ddot{a}_{yz})/\ddot{a}_y$ decreases, and so the conversion factor increases.

III. EARLY RETIREMENT

A. Introduction

Many defined benefit plans allow a participant to retire early prior to the attainment of the plan's normal retirement age. Usually there is an age and service (or age and participation) requirement. For example, if the requirement for normal retirement is the later of the attainment of age 65 and the completion of 10 years of service, early retirement may be permitted between ages 55 and 65 provided at least 10 years of service have been completed by the early retirement date. The plan document specifies the exact eligibility requirements for early retirement.

The plan document will also specify how the early retirement pension is to be computed. We will be concerned with early retirement pensions that are computed as the actuarial equivalent of the portion of the projected normal retirement benefit that is earned up to the early retirement date (e.g. the actuarial equivalent of the accrued benefit).

It should be noted that many plans provide subsidized early retirement benefits. Some examples of subsidized early retirement benefits are:

- The actuarial reduction is applied from an age prior to the normal retirement age. For instance, a plan's eligibility for normal retirement is the attainment of age 65. The plan provides that the full accrued benefit is payable if early retirement occurs on or after the attainment of age 62, and if early retirement occurs prior to the attainment of age 62 the actuarial reduction is determined from age 62 to the actual early retirement age.

- The plan specifies simplified early retirement reduction factors. For instance, a plan's eligibility for normal retirement is the attainment of age 65. The plan provides that the early retirement benefit is the accrued benefit reduced by .25% per month (3% per year) for each month by which the early retirement age precedes age 65.

- The plan provides that the full accrued benefit is payable on early retirement. This type of benefit is called a "fully subsidized" early retirement benefit.

B. Calculation of Actuarially Equivalent Early Retirement Benefit for Life Only Normal Form

We use the same notation as in Section II, except we introduce the following:

x = Early retirement age of annuitant. It should be noted that $x<y$.

The equation of value is:

$$12\,(AP)_y * \frac{N_y^{(12)}}{D_x} = 12\,(BP)_x * \frac{N_x^{(12)}}{D_x}$$

Solving for $(BP)_x$, we obtain:

$$(BP)_x = (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}}$$

We define the CF to be:

$$CF = \frac{N_y^{(12)}}{N_x^{(12)}} \qquad (5)$$

If it is assumed there is no mortality decrement prior to normal retirement (e.g. no pre-retirement mortality), the CF becomes:

$$CF = v^{y-x}\,\frac{\ddot{a}_y^{(12)}}{\ddot{a}_x^{(12)}} \qquad (6)$$

It should also be noted that if the plan provides a pre-retirement death benefit equal to the present value of the accrued benefit, and the present value of the accrued benefit at time of death is determined <u>without</u> a discount for pre-retirement mortality, then the early retirement conversion factor reduces to Expression (6).

The proof relies on the assumption that the death benefit is paid at the end of the year of death. The left hand side of the equation of value would be the sum of the present value of the deferred normal retirement benefit and the present value of the pre-retirement death benefit. The left hand side of the equation of value would be:

$$\left[12\,(AP)_y\; v^{y-x}\,{}_{y-x}p_x\; \ddot{a}_y^{(12)}\right]+\left[12\,(AP)_y \sum_{t=0}^{y-x-1} \left(v^{t+1}\;{}_tp_x q_{x+t}\right)\left(v^{y-x-t-1}\ddot{a}_y^{(12)}\right)\right]$$

Since, $\sum_{t=0}^{y-x-1} {}_t\,p_x q_{x+t} = 1-{}_{y-x}p_x$ the left hand side of the equation of value simplifies to:

$$\left[12\,(AP)_y\; v^{y-x}\,{}_{y-x}p_x\ddot{a}_y^{(12)}\right]+\left[12\,(AP)_y\; v^{y-x}\;(1-{}_{y-x}p_x)\,\ddot{a}_y^{(12)}\right]=12\,(AP)_y\; v^{y-x}\ddot{a}_x^{(12)}$$

Thus, the equation of value

becomes: $12\,(AP)_y\; v^{y-x}\ddot{a}_y^{(12)} = 12\,(BP)_x\; \ddot{a}_x^{(12)}$

This equation leads directly to Expression (6).

It should be noted that some plan documents or administrative past practices will specify that early retirement factors include a discount for mortality. In this case Expression (5), instead of Expression (6) would be used to calculate the early retiree's benefit. The actuary should be careful to check the language in the plan document before calculating early retirement benefits.

The reader should also be aware that Expression (5) or Expression (6) (whichever is applicable) is used to determine the IRC Section 415(b)(1)(A) dollar limit when the pension commences prior to age 62.

For participants who are 100% vested, the reader should note that the left hand side of the equation of value is the accrued liability under the Unit Credit Cost Method. An experience gain or loss will result due to a participant taking early retirement under the following two circumstances:

- If the plan uses a cost method to determine the annual plan contribution which is different from the Unit Credit Cost Method.

- If the plan uses the Unit Credit Cost Method to determine the annual plan contribution, but the actuarial assumptions utilized for funding are different than the plan's actuarial equivalence assumptions.

Example 5

A defined benefit plan provides an annual pension of $1,000 for each year of service commencing at a normal retirement age of 65, and payable monthly for life. An early retirement benefit is provided upon attaining the age of 55. The early retirement benefit is the actuarial equivalent of the accrued benefit.

A 60 year old was hired at age 30 and is about to elect to retire early.

You are given the same commutation functions as those provided in Example 1.

(a) Compute the monthly early retirement pension commencing at age 60.

(b) Compute the experience gain due to the participant taking early retirement, given the plan's cost method is Individual Entry Age Normal (with age at date of hire used as the entry age) and the plan's actuarial equivalence assumptions are the same as the funding assumptions used to determine plan contributions.

(c) Compute the experience gain due to the participant taking early retirement, given the participant was age 45 on his entry into the plan, the plan's cost method is Individual Level Premium, and the plan's actuarial equivalence assumptions are the same as the funding assumptions used to determine plan contributions.

Solution

(a) The equation of value is:

$$(1000)(30)\frac{N_{65}^{(12)}}{D_{60}} = (12)(BP)_{60}\frac{N_{60}^{(12)}}{D_{60}}$$

Thus, the monthly early retirement pension is $1,592.86.

(b) Under the Individual Entry Age Normal Cost Method, the accrued liability at age 60 is:

$$\frac{(1000)(35)N_{65}^{(12)}/D_{30}}{(N_{30} - N_{65})/D_{30}} * \left(\frac{N_{30} - N_{60}}{D_{60}}\right) = 246{,}819$$

The present value of the accrued benefit is:

$$(1000)(30)\frac{N_{65}^{(12)}}{D_{60}} = 220{,}453$$

Thus, the experience gain due to early retirement is:

246,819 - 220,453 = 26,366

(c) Under the Individual Level Premium Cost Method, the accrued liability at age 60 is:

$$\frac{(1000)(35)N_{65}^{(12)}/D_{45}}{(N_{45} - N_{65})/D_{45}} * \left(\frac{N_{45} - N_{60}}{D_{60}}\right) = 222{,}441$$

Thus, the experience gain due to early retirement is:

222,441 - 220,453 = 1,988

C.1 Conversion of Actuarially Equivalent Early Retirement Benefit Payable in Life Only Normal Form to Certain and Life Annuity

Assume the certain period is n years. Then the equation of value is:

$$12(AP)_y * \frac{N_y^{(12)}}{D_x} = 12(BP)_x * \left[\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}\right]$$

Solving for $(BP)_x$, we obtain:

$$(BP)_x = (AP)_y * \frac{\frac{N_y^{(12)}}{D_x}}{\left(\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}\right)}$$

$$= (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{\frac{N_x^{(12)}}{D_x}}{\left(\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}\right)}$$

$$= (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{\ddot{a}_x^{(12)}}{\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}}$$

We define the CF to be:

$$CF = \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{\ddot{a}_x^{(12)}}{\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}} \qquad (7)$$

It should be noted that Expression (7) is the product of the early retirement conversion factor for a life annuity (Expression (5)), and the certain and life conversion factor for a retirement age of x (same form as Expression (1)).

C.2 Conversion of Actuarially Equivalent Early Retirement Benefit Payable in Life Only Normal Form to Level Payment Social Security Option

We develop the equation of value like we did in Section II.B.2, except payments commence from age x instead of age y. The equation of value is:

$$12\,(AP)_y * \frac{N_y^{(12)}}{D_x} = 12\,(BP)_x * \left(\frac{N_x^{(12)} - N_s^{(12)}}{D_x}\right) + 12\left[(BP)_x - (SS)_s\right] * \frac{N_s^{(12)}}{D_x}$$

Solving for $(BP)_x$, we obtain:

$$(BP)_x = (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}} + (SS)_s * \frac{N_s^{(12)}}{N_x^{(12)}}$$

$$= \frac{N_y^{(12)}}{N_x^{(12)}} * \left[(AP)_y + (SS)_s \frac{N_s^{(12)}}{N_y^{(12)}}\right] \qquad (8)$$

It should be noted that Expression (8) is the product of the early retirement conversion factor for a life annuity (Expression (5)), and $(BP)_y$ for the Level Payment Social Security Option for a retirement age of y (as shown in Section II.B.2).

C.3 Conversion of Actuarially Equivalent Early Retirement Benefit Payable in Life Only Normal to p% Joint and Survivor Annuity

We develop the equation of value like we did in Section II.B.3.c, except payments commence from age x instead of age y. The co-annuitant's age when the annuitant is age x is then z+x-y (e.g. z-(y-x)). The equation of value is:

$$12\,(AP)_y * \frac{N_y^{(12)}}{D_x} = 12\,(BP)_x * \left(\ddot{a}_x^{(12)} + \frac{P}{100} * (\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)})\right)$$

Solving for $(BP)_x$, we obtain:

$$(BP)_x = (AP)_y * \frac{N_y^{(12)}/D_x}{\left(\ddot{a}_x^{(12)} + \frac{P}{100} * (\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)})\right)}$$

$$= (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{\ddot{a}_x^{(12)}}{\left(\ddot{a}_x^{(12)} + \frac{P}{100} * (\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)})\right)}$$

$$= (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{1}{\left(1 + \frac{P}{100} * \left(\frac{\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)}}{\ddot{a}_x^{(12)}}\right)\right)}$$

We then define the CF to be:

$$CF_p = \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{1}{\left(1 + \frac{P}{100} * \left(\frac{\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)}}{\ddot{a}_x^{(12)}}\right)\right)} \qquad (9)$$

It should be noted that Expression (9) is the product of the early retirement conversion factor for a life annuity (Expression (5)), and the p% Joint and Survivor Annuity Option conversion factor for a retirement age of x (same form as Expression (4)).

D. Effect of Changes in Actuarial Equivalence Assumptions

D.1 Interest

D.1.a Expression (5)

As in Section II.C above, we will examine the case of annual payments instead of monthly payments.

The analysis of Expression (5) is identical to the analysis of Expression (2) presented in Section II.C.2.b.

We leave it to the reader to show that as i increases, the early retirement conversion factor for a life only normal form decreases, and vice versa.

We can also understand this result through general reasoning. If benefit payments begin sooner, the trust fund will have less time to earn interest. This requires a reduction in benefits. The larger the assumed interest rate, the bigger will be the shortfall in earnings. Thus, there will be a bigger reduction in benefits required.

D.1.b Expression (7), Expression (8) and Expression (9)

Expression (8) is the easiest of the three Expressions to analyze. By referring to Section II.C.2.b, we leave it to the reader to show that as i increases, the early retirement conversion factor from a life only normal form to a Level Payment Social Security Supplement Option decreases, and vice versa.

To analyze Expressions (7) and (9), we first notice that each Expression is the product of two factors:

- o N_y/N_x.
- o The conversion factor from a life only normal form to the given optional form of annuity, determined at a retirement age of x instead of y.

Expression (7) is composed of N_y/N_x and a modified Expression (1). Unfortunately, since we were not able to give a general conclusion as to how Expression (1) is affected by a change in i, it is not possible to give a general conclusion as to how Expression (7) is affected by a change in i.

Expression (9) is composed of N_y/N_x and a modified Expression (4). Since these two factors change in opposite directions for a given change in i, it is not possible to give a general conclusion as to how Expression (9) is affected by a change in i.

D.2 Mortality

D.2.a Expression (5)

The analysis of Expression (5) is identical to the analysis of Expression (2) presented in Section II.C.3.b.

Thus, as q increases, the early retirement conversion factor for a life only normal form decreases, and vice versa.

D.2.b Expression (7), Expression (8) and Expression (9)

Expression (8) can be analyzed as before. By referring to Section II.C.3.b, we leave it to the reader to show that as q increases, the early retirement conversion factor from a life only normal form to a Level Payment Social Security Supplement Option decreases, and vice versa.

To analyze Expressions (7) and (9), we again make use of the fact that each Expression is the product of two factors:

- o N_y/N_x.
- o The conversion factor from a life only normal form to the given optional form of annuity, determined at a retirement age of x instead of y.

Expression (7) is composed of N_y/N_x and a modified Expression (1). For a given change in q, these two factors change in the same direction. Therefore, as q increases, the early retirement conversion factor from a life only normal form to a Certain and Life Annuity decreases, and vice versa.

Expression (9) is composed of N_y/N_x and a modified Expression (4). There are two situations to analyze.

Case 1: q for the annuitant increases and q for the co-annuitant remains unchanged. For a given change in q, these two factors change in the same direction. Therefore, as q increases, Expression (9) decreases.

Case 2: q for the co-annuitant increases and q for the annuitant remains unchanged. We then see that N_y/N_x remains unchanged, and Expression (4) increases. Therefore, as q increases, Expression (9) increases.

IV. LATE RETIREMENT

A. Introduction

Late retirement from defined benefit plans is a relatively recent concept.

Congress passed the Age Discrimination and Employment Act (ADEA) in 1967. One of the sections in ADEA provided that an employer could not force an employee to retire prior to his attainment of age 65 solely because of age. Therefore, this protection afforded by ADEA ceased upon the employee's attainment of age 65. The 1978 amendments to ADEA increased the upper age limit for the protection to age 70, and in 1986 the upper age limit of 70 was removed.

Today, an employee has several options open to him when he attains the plan's normal retirement age:

(1) Terminate employment (e.g. retire) and commence receiving benefit payments from the defined benefit trust.

(2) Continue employment and commence receiving benefit payments from the defined benefit trust. **

(3) Continue employment and postpone the commencement of benefit payments from the defined benefit trust. **

** It should be noted that a DOL regulation provides, in general, that an employer can "suspend" benefit payments to an employee if he works an average of at least 40 hours per month.

The IRS and United States Department of Labor (DOL) have published regulations on how retirement benefits are determined for employees who continue to work past the attainment of normal retirement age (e.g. options (2) and (3), above).

B. <u>Calculation of Actuarially Equivalent Late Retirement Benefit for Life Only Normal Form</u>

We use the same notation as in Section II, except we introduce the following:

x = Late retirement age of annuitant.
It should be noted that x>y.

The equation of value is:

$$12(AP)_y * \frac{N_y^{(12)}}{D_x} = 12(BP)_x * \frac{N_x^{(12)}}{D_x}$$

Solving for $(BP)_x$, we obtain:

$$(BP)_x = (AP)_y * \frac{N_y^{(12)}}{N_x^{(12)}}$$

We define the CF to be:

$$CF = \frac{N_y^{(12)}}{N_x^{(12)}} \qquad (10)$$

The reader should also be aware that Expression (10) is used to determine the IRC Section 415(b)(1)(A) dollar limit when the pension commences after the attainment of the SSRA.

Example 6

A defined benefit plan provides an annual pension of $1,000 for each year of service commencing at a normal retirement age of 65, and payable monthly for life. The plan provides that no more than 35 years of service is taken into account to determine the normal retirement benefit. A late retirement benefit is provided which is the actuarial equivalent of the normal retirement benefit.

A 67 year old was hired at age 30 and is about to elect to retire late.

You are given the same commutation functions as those provided in Example 1.

Compute the monthly late retirement pension commencing at age 67.

Solution

The equation of value is:

$$(1000)(35)\ \frac{N_{65}^{(12)}}{D_{67}} + 12\,(BP)_{67}\ \frac{N_{67}^{(12)}}{D_{67}}$$

Thus, the monthly late retirement pension is $3,527.13.

C.1 Conversion of Actuarially Equivalent Late Retirement Benefit Payable in Life Only Normal Form to Certain and Life Annuity

Assume the certain period is n years. We leave it to the reader to show that the CF is the same expression as in III.C.1, that is:

$$CF = \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{\ddot{a}_x^{(12)}}{\left(\ddot{a}_{\overline{n}|}^{(12)} + \frac{N_{x+n}^{(12)}}{D_x}\right)} \qquad (11)$$

It should be noted that Expression (11) is the product of the late retirement conversion factor for a life annuity (Expression (10)), and the certain and life conversion factor for a retirement age of x (same form as Expression (1)).

C.2 Conversion of Actuarially Equivalent Late Retirement Benefit Payable in Life Only Normal Form to p% Joint and Survivor Annuity

We develop the equation of value like we did in Section II.B.3.c, except payments commence from age x instead of age y. The co-annuitant's age when the annuitant is age x is then z+x-y (e.g. z-(y-x)). We leave it to the reader to show that the CF is the same solution as in III.C.3, that is:

$$CF = \frac{N_y^{(12)}}{N_x^{(12)}} * \frac{1}{\left(1 + \frac{P}{100} * \left(\frac{\ddot{a}_{z+x-y}^{(12)} - \ddot{a}_{x:z+x-y}^{(12)}}{\ddot{a}_x^{(12)}}\right)\right)} \tag{12}$$

It should be noted that Expression (12) is the product of the late retirement conversion factor for a life annuity (Expression (10)), and the p% Joint and Survivor Annuity Option conversion factor for a retirement age of x (same form as Expression (4)).

D. Effect of Changes in Actuarial Equivalence Assumptions

D.1 Interest

D.1.a Expression (10)

As in Section II.C above, we will examine the case of annual payments instead of monthly payments.

The analysis of Expression (10) is identical to the analysis of Expression (2) presented in Section II.C.2.b, except that here we analyze N_y/N_{y+1}.

We leave it to the reader to show that as i increases, the late retirement conversion factor for a life only normal form increases, and vice versa.

D.1.b Expression (11) and Expression (12)

To analyze Expressions (11) and (12), we first notice that each Expression is the product of two factors:

- N_y/N_x.
- The conversion factor from a life only normal form to the given optional form of annuity, determined at a retirement age of x instead of y.

Expression (11) is composed of N_y/N_x and a modified Expression (1). Unfortunately, since we were not able to give a general conclusion as to how Expression (1) is affected by a change in i, it is not possible to give a general conclusion as to how Expression (11) is affected by a change in i.

Expression (12) is composed of N_y/N_x and a modified Expression (4). For a given change in i, these two factors change in the same direction. Therefore, as i increases, Expression (12) increases, and vice versa.

D.2 Mortality

D.2.a Expression (10)

The analysis of Expression (10) is identical to the analysis of Expression (10) presented in Section IV.D.1.a.

Thus, as q increases, the late retirement conversion factor for a life only normal form increases, and vice versa.

D.2.b Expression (11) and Expression (12)

To analyze Expressions (11) and (12), we again make use of the fact that each Expression is the product of two factors:

- N_y/N_x.
- The conversion factor from a life only normal form to the given optional form of annuity, determined at a retirement age of x instead of y.

Expression (11) is composed of N_y/N_x and a modified Expression (1). Since these two factors change in opposite directions for a given change in q, it is not possible to give a general conclusion as to how Expression (11) is affected by a change in q.

Expression (12) is composed of N_y/N_x and a modified Expression (4). There are two situations to analyze.

Case 1: q for the annuitant increases and q for the co-annuitant remains unchanged. Since these two factors change in opposite directions for a given change in q, it is not possible to give a general conclusion as to how Expression (12) is affected by a change in q.

Case 2: q for the co-annuitant increases and q for the annuitant remains unchanged. We then see that N_y/N_x remains

Case 2: q for the co-annuitant increases and q for the annuitant remains unchanged. We then see that N_y/N_x remains unchanged, and Expression (4) increases. Therefore, as q increases, Expression (12) increases.

V. CONCLUSION

The concept and applications of actuarial equivalence are fundamental to the ongoing administration of defined benefit pension plans. This study note has attempted to present an introduction to this important subject. It is a subject which must be understood and dealt with on a daily basis by virtually all pension actuaries.