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THE MATHEMATICAL RISK OF LUMP-SUM DEATH BENEFITS IN A TRUSTEED PENSION PLAN

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CECIL J. NESBITT:

This paper provided an interesting topic for discussion in our actuarial seminar at Michigan. There we were impressed by the author's mathematical skill in organizing and manipulating the rather complicated expressions. However, our discussion led to some remarks which, if accepted, would mildly modify the conclusions of the author.

For simplicity these remarks will be limited to a modification of the illustrative example presented by the author. He there considers a deferred annuity of k' per annum commencing at age 65 and an immediate $(65 - x)$-year term insurance for a sum insured of m. The net annual premium is fixed at one unit. Our modification consists of taking a variable sum insured m_t which at time $t, t \leq 65 - x$, is equal to the accumulation of the premiums, that is, $m_t = \bar{s}_{\overline{t}|}$. Under this circumstance no net loss is suffered in case of death before 65, and variance can arise only in respect to net losses after age 65. Further, it may be shown that for our modified case

$$k' = \frac{\bar{s}_{\overline{65-x}|}}{\bar{a}_{65}} \qquad (a)$$

and

$$M^2 = {}_{65-x}p_x \int_0^\infty {}_rp_{65}\mu_{65+r}\, v^{2(r+65-x)} \left[\frac{\bar{s}_{\overline{65-x}|}}{\bar{a}_{65}}\ \bar{s}_{\overline{r}|} - \bar{s}_{\overline{65-x}|}\,(1+i)^r \right]^2 dr$$

or

$$M^2 = {}_{65-x}p_x \frac{\bar{a}^2_{\overline{65-x}|}}{\bar{a}^2_{65}} \left[\frac{2}{\delta}\left(\bar{a}_{65} - \bar{a}'_{65}\right) - \bar{a}^2_{65} \right]. \qquad (b)$$

Since, in this case, no variance arises in respect to the period before 65, one might expect that M^2 by formula (b) would be less than the values shown in Table 2, although the value of k' would be altered. Our calculations indicate that M^2 by formula (b) for $x = 20$, 35 and 60 would approximate 124, 76 and 4, respectively, which are lower than the author's minimum values for these ages.

The author has considered the case of the distribution of losses over the whole future lifetime of (x). It occurred to us to examine the other extreme where losses over a single year are considered. For this purpose it is convenient to adopt discrete functions. Attention is again restricted to a modification of the author's illustrative example where for year of insurance $h+1$, $h+1 \leq 65-x$, the death benefit is m_{h+1} payable at the end of the year, and the initial and terminal reserves are ${}_h\mathrm{V}+1$, and ${}_{h+1}\mathrm{V}$, respectively. Then the net loss in case of death is $vm_{h+1}-({}_h\mathrm{V}+1)$ and in case of survivorship is $v_{h+1}\mathrm{V}-({}_h\mathrm{V}+1)$. The variance for the year, M_{h+1}^2, is given by

$$M_{h+1}^2 = q_{x+h}[vm_{h+1} - ({}_h\mathrm{V}+1)]^2 + p_{x+h}[v_{h+1}\mathrm{V} - ({}_h\mathrm{V}+1)]^2$$

or

$$M_{h+1}^2 = v^2 p_{x+h} q_{x+h}[m_{h+1} - {}_{h+1}\mathrm{V}]^2 . \qquad (c)$$

From this it is obvious that $M_{h+1}^2 = 0$, if m_{h+1} is chosen equal to ${}_{h+1}\mathrm{V}$. One may also see that M_{h+1}^2 is larger for the case where the death benefit in each year is zero than for the case where there is some death benefit provided each year.

For periods of several years Hattendorf's theorem might be used to calculate the variance for the full period in terms of the variances for the years in the period.

To summarize our thinking, we reached the rather obvious conclusion that the least risky death benefit for a pension plan is one under which the sum insured is equal to the accumulated contributions.

Two practical points should be mentioned. The paper does not take into account the catastrophe hazard which, especially for a small pension fund, might be serious. Second, in the cases considered in the paper (and also in this discussion) there is no death benefit after retirement, and consequently there is a sharp break in the death benefit at retirement, a feature that would often be eliminated. Some further decrease in variance might then be possible.

Our thanks are due the author for his stimulating, thought-provoking paper.