2016 Enterprise Risk Management Symposium

April 6–8, 2016, Arlington, Virginia

Enterprise Risk-Reward Optimization:

Two Critical Approaches

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Abstract

Enterprise risk management (ERM) is increasingly viewed as an essential discipline entirely distinct from internal audit or compliance, though many organizations still view its scope as only mitigation against severe downside events. Though strategic decisions very often balance risk and reward, few ERM frameworks determine the optimal tradeoff between these two concepts. This paper develops two technical approaches for the optimization of risk and reward at a company with a solid ERM framework and risk culture in place. These methods allow ERM to be more than protection against the downside; they enable enterprise risk-reward optimization (ERRO).

In a very general setting, this paper describes a straightforward, non-parametric approach to aggregate “stand-alone” or marginal distributions with desired correlations without imposing additional assumptions on those marginals. We then develop two optimizations for the enterprise, one based on maximizing return on economic capital and the other based on a mean-semivariance efficient frontier from the investor point of view. The definition of economic capital is applicable to any insurer or bank while the efficient frontier can be used in any for-profit company. In each case, the optimization methods use closed-form solutions and do not necessitate numerical search algorithms whose results are sometimes suboptimal and often require extensive computing resources.

1. Introduction

1.1. The Next Evolution of ERM

Despite increased focus on enterprise risk management (ERM) across insurance, banking and other sectors, many practitioners still struggle with a perception problem. They strive for ERM to be perceived as a value-adding strategic discipline on top of its traditional role of downside protection. Recently, ERM has made strides in capturing the interplay between risk and reward and helping organizations understand the tradeoffs between these concepts. The next evolution for ERM consists of defining a set of objective methods to optimize risk and reward. Once accomplished, risk management at a state-of-the-art practitioner will *drive* exceptional risk-adjusted returns within the unique confines of a

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company's business goals and risk appetite. This is enterprise risk-reward optimization (ERRO) and has been described as ERM's holy grail.[2]

The paper introduces a widely applicable framework that

1. allows for calculation of enterprise economic capital and its optimal allocation across business or product lines,
2. supplements the economic capital view with an earnings-based optimization using a mean-semivariance efficient frontier, and
3. provides closed-form solutions in both cases.

A straightforward method is employed to capture correlations between risks or product lines while preserving marginal densities.

The approaches employ common financial metrics and are generally non-parametric (distribution-free) in the sense that they avoid unsubstantiated assumptions about the underlying statistical distributions of the modeled risks or earnings variability. Both the methodology and its applications are nearly universal and the concepts will resonate with key decision-makers throughout the company. When executive leadership believes in and espouses its methods and metrics, ERM drives strategic decision-making.

For concreteness, a hypothetical public insurance company (the Company) with three lines of business (LOBs) owned by a holding company (Corporate) is considered. The paper emphasizes the optimization methods and does not provide significant detail on the necessary foundation of a strong ERM organizational structure and a healthy risk culture. A company with these attributes can implement these approaches to move from traditional ERM to ERRO.

1.2. Multiple Stakeholders

Management of the Company must consider the expectations of several internal and external stakeholders when making decisions concerning risk, reward and ERM in general. Management decisions are driven by:

- The responsibility of the board of directors to protect shareholder interests, ensure adequate return and oversee the Company's ERM framework
- Investor expectation for stock price appreciation and/or an adequate stream of dividends in light of the Company's risk exposure, that is, they expect the chance for an appropriate reward for the perceived level of risk
- Regulators' prescribed minimum levels of capital and their focus on downside protection
- Rating agency assessments of the Company's ERM framework and their expectations for minimum capital levels as a function of the Company's targeted ratings

[2] Bradford Connors and Evan Borisenko, "An Interview With Larry Moews, Chief Actuary & Chief Risk Officer of SCOR Americas," *Risk Management* 30 (August 2014). Mr. Moews uses this term in reference to enterprise risk and return optimization, which is synonymous with this paper's use of the term ERRO, and he stresses that ERM practitioners should all strive to achieve ERRO.

- Customer and policyholder expectations that the Company is able to pay claims and/or policyholder benefits and keep its financial obligations under both normal and stressed environments

Of course, management has its own views when it comes to these issues. In particular, at an organization with a strong risk culture and robust risk management framework, management's view of ERM's purview is more than just "deep tail," severely adverse events; they also analyze upside opportunity and more likely scenarios of potentially lower impact. The many viewpoints and considerations of these stakeholders are illustrated in Table 1.

Table 1. Risk and Reward Stakeholders

	Board of Directors	**Investors**	**Regulators & Rating Agencies**	**Customers & Policyholders**	**Management**
Stock performance/company "reward"					
Verify/assess ERM framework strength					
ERM includes upside considerations					
Ability to pay obligations in "deep tail events"					
Required/targeted capital levels					

Note that each *column* should be considered on a stand-alone basis. Those items shown vertically (on the left) are of varying priority or importance to each of the stakeholders listed on the top, horizontal portion. For a given stakeholder, red shading indicates high priority or interest level (possibly meaning a large amount of time spent on the concept), while orange indicates a consideration of somewhat lower priority. Grey shading indicates that the stakeholder generally does not place much import on the item or does not have much awareness of it.

1.3. The Approach

This paper describes a modeling approach that builds upon a strong ERM foundation to achieve ERRO. Emphasis is placed on the details of building the ERRO modeling capability and applying the optimizations to inform management decision-making in the context of the various stakeholder priorities described in Section 1.2.

Management has a long list of goals and constraints coming from the various ERM stakeholders. The methods in this paper allow for all of these priorities to be addressed. An ERM framework capable of ERRO is able to link strategic decisions to relevant metrics and provide a lens through which business lines, product proposals or potential acquisitions can be seen in a risk *and* reward context. The methods will drive company *performance* and also help ensure solvency.

The maximization of return on economic capital ties into financial planning and helps measure performance and shape incentive compensation while taking into account capital needs and adverse scenario protection. The mean-semivariance optimization sheds light on earnings volatility and the investor perspective.

Section 2 provides an overview of approaches for the modeling of LOB earnings as a cumulative distribution function (CDF) and defines some key concepts based on model output. In Section 3, we examine a distribution-free approach to capture correlations across LOBs so they may be properly aggregated to form the enterprise distribution of earnings. Section 4 develops a definition of economic capital, a return on economic capital measure, and maximizes this return through the optimal allocation across LOBs. In Section 5, we cover the investor point of view, earnings volatility and the development of the long-only portion of the efficient frontier. Finally, some conclusions and applications to performance measurement and incentive compensation are presented in Section 6.

2. Line of Business Distributions

2.1. Distributable Earnings

Each of the Company's LOBs develops a model of the CDF for earnings. This modeling shows the full range of potential earnings outcomes and assigns probabilities to any interval within this range. The model should be informed by the risk identification and quantification processes of the ERM framework but the distribution gives a more complete (and continuous) representation of the "continuum" of possible outcomes than any discrete, scenario-based risk analysis can. It must be stressed, however, that risk scenario (hypothetical event) modeling is still an important component of a robust ERM program.

This model should include U.S. Generally Accepted Accounting Principles (GAAP) earnings, statutory earnings and the necessary detail of the balance sheet and income statement to allow for dynamic reserve and capital modeling. Note that for organizations outside the United States, the term "statutory earnings" can be substituted with "earnings as defined by the applicable, local regulatory body."

Our choice of an earnings metric is based on its role in the concept for an economic capital metric to be discussed later in Section 4. For a specific time horizon, distributable earnings (DE) is defined as:

$$\text{distributable earnings (DE)} = \text{statutory earnings} - \Delta \text{ targeted capital},$$

where *Δ targeted capital* is the change over the period of (actual, forecast or simulated) statutory capital and surplus that is deemed necessary by the Company to meet applicable regulatory and rating agency requirements as well as any other constraints (e.g., contracts). When DE is negative, it is assumed that, rather than reducing the capital need (e.g., by risk reduction), there is an infusion of capital, supplied by Corporate, into the entity in question in order for it to continue to operate as a going concern at the targeted level of capital. For an insurance company, DE is a good analogue for the notion of *free cash flows* found in many equity valuation models.

A stochastic run based on the risk models will be used to identify percentiles of interest: first percentile, second percentile, …, 99th percentile, etc. Knowing the percentiles for earnings, at the desired precision, can be viewed as equivalent to knowing the CDF for earnings of the LOB.

Before making use of the results of such a distribution of earnings, we briefly look at some potential approaches to DE modeling.

2.2. Selected Modeling Approaches

2.2.1. Data Reliant Models

For an LOB with a long history of actual DE results, including detail of the associated income statements and balance sheets, it *may* be appropriate to largely rely on this history as the basis for the distribution of DE. In particular, if it is believed that the current risk profile (for those risks affecting DE volatility) is largely similar to the profile represented in the history then the available data may be used to help model DE. The simplest approach would be to rank the historical data and observe the empirical percentiles. Then some judgment would be needed to stress some of the upper and lower percentiles to better capture the "full" distribution. A stochastic approach can sometimes make better use of the data.

Given a rich data set for one of its lines of business the Company estimates, based on this information, that in the next year the net earned premium will fall into one of five ranges NEP_1 = [a, b], NEP_2 = (b, c], NEP_3 = (c, d], NEP_4 = (d, e] and NEP_5 = (e, f] with respective probabilities of 5 percent, 15 percent, 30 percent, 35 percent and 15 percent. The Company intentionally makes the value of *a* smaller than any witnessed value and likewise *f* is larger than the maximum of the data set.

Based on these probabilities, the Company can generate a random digit from the unit interval (e.g., in Excel using the "rand()" function) and thereby simulate into which interval NEP_x, the (simulated) net earned premium, lies.

For each of those intervals, the Company uses available data to estimate the "conditional loss ratios," that is, when simulated NEP falls in a specific interval NEP_x, the loss ratio will fall in a specific interval [u_x, v_x]. In each case, it is assumed that the distribution of the variable within an interval is uniform. In other cases, it might be worth considering non-uniform distributions over these intervals such as triangular or normal but a uniform distribution makes the "smallest" assumption about the behavior of a variable within an interval. (Note: In technical terms, in the field of information theory, the uniform distribution on the interval [a, b] is known to have the *maximum entropy* among all continuous distributions supported on the interval.)

Using the uniform assumptions, a random value from (0, 1) is generated to determine the NEP interval in which earnings lies, for example, (x, y]. Then, using another random digit and the uniform assumption over (x, y], we arrive at a simulated point estimate N = simulated NEP. A similar approach simulates the loss ratio (LR) within a *conditional loss ratio interval* associated with the NEP interval. Then the product of N and LR represents simulated incurred claims.

A model equipped with sufficient accounting logic and granularity including expenses, taxes, capital effects and GAAP to statutory differences can then be used to simulate DE. Many simulations of this model are run to develop the CDF for DE.

2.2.2. A Hybrid Approach: Empirical Data and Extreme Value Theory

The method described here is useful in capturing potential black swans while still modeling the "belly" or central portion of the distribution well.

The author has previously discussed a technique, based on extreme value theory (EVT), to model the tail of a distribution as a generalized Pareto distribution, which can be summarized as follows:[3]

> EVT is a branch of statistics dealing with extreme deviations from the median of probability distributions. Under very general conditions, one of EVT's [main] results, the Pickands-Balkema-de Hann theorem, describes observations above a high, fixed threshold as a generalized Pareto distribution (GPD). Given a set historical data, one may choose a high threshold T within that data (e.g., 95th percentile) and then examine the *excesses* above T for the subset of observations above T. That collection of distances or excesses (positive real numbers) can be well modeled as a GPD which contains two parameters which are relatively easy to estimate. The resulting GPD is capable of modeling, in a statistically sound manner, the potential magnitude and likelihood of future observations which are worse than any previously seen.

This approach can be used on the left tail of a distribution as well. Assume we are interested in a model that captures the following for DE:

a. upside results,
b. typical or common results not as favorable as those in a, and
c. adverse, unlikely results below the worst result from b.

We consider a low, fixed dollar threshold, u, of DE based on the available data history. Note that u is an adverse result for DE and could possibly be a negative amount. For a value of DE, x, below u, the "excess" is defined as $u - x$. This is the (positive) distance from x to u. The selection of u requires that we have results worse than (less than) u in the observed data sample of n points.

Given a suitable choice of u one can model the excesses of those values below T as a GPD, G(y). We define the CDF $F_n(x)$ as the empirical distribution based on the n observed data points. We can then define a "hybrid" distribution, part empirical and part GPD, as the CDF for distributable earnings:[4]

$$F(x) = F_n(x) \text{ for } x \geq u, \text{ and } F(x) = F_n(u) \bullet [1 - G(u - x)] \text{ for } x < u.$$

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[3] Damon Levine, "ERM at the Speed of Thought: Mitigation of Cognitive Bias in Risk Assessment," 2015 Enterprise Risk Management monograph, https://www.soa.org/Library/Monographs/Other-Monographs/2015/june/2015-erm-symposium.aspx.

[4] Damon Levine, "Modeling Tail Behavior with Extreme Value Theory," *Risk Management* 17 (September 2009).

2.2.3. Aggregated Risk Distributions

A common approach for European insurers under Solvency II is to develop distributions (CDFs) for the impact of each risk category such as credit risk, insurance risk and operational risk, on a particular metric and then make two linearity assumptions for aggregation purposes. These assumptions are

1. the risk distributions' interrelationships can be well captured by the Pearson correlation coefficient, and
2. the aggregate impact is a linear combination of the impacts for each risk category.

The first assumption is "linear" in the sense that the Pearson correlation measures the extent to which a linear relationship exists between a pair of random variables. This assumption can be avoided through the use of rank correlation. A way to avoid the second linearity assumption and a method for correlated aggregation are discussed in detail in Section 3. That section will use LOB distributions and a non-parametric method based on a rank correlation matrix to model the enterprise distribution.

2.2.4. Ensuring Adequate Tail Size: Black Swans and Chebyshev

Given a proposed model for the distribution of DE, it is important to assess if the tails are "fat" enough; in other words, does the model produce extreme deviations with sufficient probability and magnitude? Answering this question is usually subjective to an extent. From a risk standpoint, the focus of these models is often on the left tail comprised of negative or small, positive results for a metric such as earnings or return on equity. In this situation it may be instructive to estimate the model-implied standard deviation from a run consisting of a large number of simulations and then make use of the following *one-sided version* of Chebyshev's inequality:

> For *any* random variable X with finite expected value μ and finite non-zero variance σ^2 and a real number $k > 0$:
>
> $$P\,(X \le \mu - k\,\sigma) \le 1/(1 + k^2).$$

Note that one should consider values of $k \ge 1$ or the inequality gives only a trivial result.

Suppose a company models the distribution for a metric of interest, X. Using the sample mean and standard deviation (m and s respectively) of a large model run and the individual simulation results for X, they estimate the probability $P\,(X \le m - 3s)$. This is estimated from a (large) run as the proportion of simulated values of X less than or equal to $m - 3s$. If this value happens to be about 3 percent, for example, this alone implies the distribution has a fatter left tail than a normal distribution. But the company may want an even fatter tail.

Chebyshev's inequality states that the probability could be as much as $1/(1 + 3^2) = 10\%$, for the "fattest" tail of the (vast) class of distributions with finite mean and variance. Assuming there is some degree of confidence that the sample mean well approximates the "true" mean, it is decided that the standard deviation should be altered or at least stress tested so that $P\,(X \le m - 3s)$ is "somewhat closer" to 10 percent. This may mean they aim for the probability to be 5 to 6 percent, for example.

By choosing parameters or stressing them appropriately, the models will simulate extreme, adverse results more often and of larger magnitude. They will be capable of simulating black swans.

At this point, it is assumed that (marginal) distributions of DE have been developed for each LOB. We now discuss several definitions important for our economic capital concept.

2.3. Analysis of Simulation Output

Equipped with the LOB distributions (CDFs), the Company is able simulate an annual DE result for any particular LOB by mapping a random number r from (0, 1) to $F^{-1}(r)$. This is essentially regarding r as a percentile and then identifying it with that percentile of DE for the LOB in question. As an example, if the random digit is 0.35 then the 35th percentile of the distribution is simulated. When simulations are run in this manner, it is important to establish a minimum number of simulations per run such that the various statistics of interest such as percentiles, standard deviation and means, are consistent *across* runs. For example, the Company may find that when a run has at least 5,000 simulations, the observed statistics always come out to the same values (approximately).

The Company has vetted its respective cumulative distribution function for DE for each LOB. These are labeled $F_1(x)$, $F_2(x)$ and $F_3(x)$ for LOB1, LOB2 and LOB3.

In the early stages of the plan process, next calendar year's financials, including earnings, are projected. The preliminary estimates for DE (in USD millions) are:

- LOB1: 50
- LOB2: 70
- LOB3: 120

These are created with information from the marginal distributions and, depending on guidance from management, might be defined as the mean, median or some other specified percentile of the distribution.

Each LOB estimates its average required assets plus *planned* surplus over the plan horizon. This reflects any targeted level of excess, projected reserves, rating agency and regulatory capital constraints, working capital, trapped capital etc. The values are as follows:

- $A_1 = 500$
- $A_2 = 1{,}000$
- $A_3 = 2{,}400$

Depending on the time horizon and level of detail in the model, the average might use annual or quarterly snap shot dates for these projected average asset levels. This calculation should assume the LOB's operating income during the horizon is removed from the entity immediately unless a level of surplus is part of the plan.

We examine the first percentile of DE for a particular LOB. If this value is negative, then we define its absolute value as an infusion need. If that first percentile is a positive DE value, then the infusion is defined as zero. In such a case, the company may want to look deeper into the tail of DE for all LOBs to ensure risk of infusion is captured for all LOBs and is done so in a consistent manner.

The Company finds that the first percentile DE is negative for all LOBs and the *infusion* amounts, denoted by I_1, I_2 and I_3, are as follows:

- I_1 = 20 = absolute value of first percentile DE for LOB1, based on $F_1(x)$
- I_2 = 30 = absolute value of first percentile DE, for LOB2, based on $F_2(x)$
- I_3 = 50 = absolute value of first percentile DE, for LOB3, based on $F_3(x)$

The next section discusses aggregation of the LOB distributions.

3. The Enterprise Distribution: Capturing Correlation Across Lines of Business

3.1. Spearman's Rho

The Company creates an Excel model that incorporates (in different Excel sheets or "tabs") each of the LOB distributions. Each sheet can generate a random number from the unit interval that, as discussed earlier, defines an earnings result for the particular LOB. The model then sums these results to simulate a single Company earnings outcome. It is not appropriate, however, to generate each of these random numbers in isolation to arrive at that simulation result as this implicitly assumes zero correlation across the LOB distributions. This issue is addressed by capturing correlations between LOB earnings.

The methodology makes use of the concept of rank correlation, in particular *Spearman's rho*. When risk management practitioners refer to correlation, they are typically employing the Pearson correlation coefficient. It is important to remember that Pearson correlation measures the extent of a linear relationship between two variables. It may not capture or signal non-linear relationships well even if the relationship between the variables is exact, such as $Y = X^2$ or $Y = \log X$.

The Pearson correlation of two variables, X and Y, is estimated, based on a sample of observed values of each variable, through a straightforward formula. Spearman's rho uses this same Pearson correlation formula applied to the *ranks* of observed values instead of the values themselves. In using the ranks of the variables, Spearman's rho captures both linear and non-linear monotonic relationships between X and Y. As with Pearson correlation, Spearman's rho is in the range $[-1, 1]$.

There is an additional benefit to this rank correlation measure. The statistical interpretation of significance for a sample Pearson correlation is based on an assumption that the two variables of interest follow a bivariate normal distribution. This is a strong assumption to make in general and in using Spearman's rank correlation measure, there is no assumption made about the distributions of the variables; it is truly a non-parametric measure.

Suppose the observations for a variable X are {21, 100, 40, 83}. These would be converted to (integer) ranks {4, 1, 3, 2} because 21 is of rank 4 when the observed values are ordered from largest to smallest, and 100 is of rank 1, etc. There may be ties for ranks, for example, if the observed values are 51, 30, 10 and 30. There are various procedures to assign ranks to the "tied" observations. In this example, the two instances of 30 occupy what would typically be labeled as ranks 2 and 3; one tie-breaking method simply assigns each the average rank of (2 + 3)/2 or 2.5. This is the procedure assumed in this section.

Given n observed values of X, $\{a_1, a_2, \ldots, a_n\}$, and n values for Y, $\{b_1, b_2, \ldots, b_n\}$, one converts to ranked data to arrive at $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$, where each x_k is the rank of a_k within all the $\{a_i\}$ and similarly each y_k is the rank of b_k within all the $\{b_i\}$. Note that if $x_i = y_i$ for all values of i from 1 to n, then there is perfect rank correlation and rho is 1.

Defining $d_i = x_i - y_i$, Spearman's rank correlation, ρ, can be calculated as:[5]

$$\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

if there are no ties, and as follows when tied ranks are present:

$$\rho = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}}$$

3.2. Inducing Rank Correlation[6]

This section largely follows the methodology and notation used by Ronald Iman and W.J. Conover in their paper presented at a 1981 session of the Joint Statistical Meetings.[7] The objective is to simulate observations from a set of distributions with known marginals such that the observed correlation between the distributions from a large run is close to a predefined "target" rank correlation matrix. As pointed out by Iman and Conover, the method is non-parametric (no assumptions about the marginal distributions are made), no complicated or obscure mathematics are needed, and the marginals are left unaltered in the correlation process. It is straightforward to carry out this methodology in Excel.

It is assumed there are K marginal distributions and through some combination of risk quantification, forecasting and historical data analysis the desired rank correlations between the marginals are determined and summarized in a target correlation matrix C*. Table 2 describes the matrices used in this method. Note that N is the desired number of simulated outcomes for each of the K marginals.

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[5] Laerd Statistics, "Spearman's Rank-Order Correlation," accessed June 9, 2016, https://statistics.laerd.com/statistical-guides/spearmans-rank-order-correlation-statistical-guide.php.

[6] The reader familiar with the following technique for inducing rank correlation among a group of predefined marginals can safely skip this section without loss of continuity.

[7] Ronald L. Iman and W. J. Conover, "A Distribution-Free Approach to Inducing Rank Correlation Among Input Variables," *Communications in Statistics—Simulation and Computation* 11 , no. 3 (1982).

Table 2. Matrices Used in the Correlation Method

Matrix	Description	Size	Entries	Comment
C*	Target rank correlation matrix	KxK	Rank correlations between LOB earnings distributions	Supplied by user based on forecasting and risk/historical analysis
P	Lower triangular matrix such that PP^T= C*	KxK	Real numbers; diagonal elements are positive	P can be found using the Cholesky factorization†
R	Each column is a random permutation of the N values: $\Phi^{-1}(i/(N + 1))$, i = 1, 2, …, N	NxK	Van der Waerden scores derived from inverse of standard normal distribution	Use of the inverse normal distribution does not assume any normality in marginal or aggregate distributions
R*	Defined as RP^T	NxK	Real numbers	This matrix is used to induce desired correlations
M, M*, S, W, X, Y	*See steps 1–6 for definitions for these additional matrices.*			
† *Cholesky decomposition is available online, among other places, at http://www.bluebit.gr/matrix-calculator/.*				

To simulate N observations of K marginal distributions (e.g., for each of K lines of business, simulate N annual earnings outcomes) such that the sample rank correlations are close to those defined in the target correlation matrix C*, we carry out the following steps.

Step 1. Generate M, an NxK matrix whose entries are random draws from the uniform distribution on (0, 1).

Step 2. In each column of M, replace each of the entries with its (integer) rank *within that column*, with 1 being assigned to the largest entry in the column, 2 assigned to the second largest, and so on. The smallest is assigned the value N. The resulting matrix is M*. The Excel function "rank" can perform this.

Step 3. Replace each entry m_{ij} of M* by the Van der Waerden score, $\Phi^{-1}(m_{ij}/(N + 1))$, where Φ^{-1} is the inverse of the standard normal distribution. The resulting matrix is R.

Step 4. Compute $R^* = RP^T$, an NxK matrix and replace each entry by its rank within its own column in the manner of Step 2. The resulting matrix is Y, with elements denoted $\{y_{ij}\}$.

Step 5. Similar to Step 1, generate W, an NxK matrix whose entries are random draws from the uniform distribution over (0, 1). Define a new matrix X where each element x_{ij} of X is the (y_{ij})th largest element in the jth column of W. (Note: This is equivalent to permuting the entries of each column of W in a manner such that if the entries are replaced by within-column ranks, the matrix is identical to Y.)

Step 6. (Note: This step makes use of the matrix X, from the prior step, and each of the marginals to simulate N observations for each of the K marginals.) Define a matrix S such that each element s_{ij} is equal to $F_j^{-1}(x_{ij})$ where F_j^{-1} is the inverse of the jth marginal (cumulative distribution function). The first column of S is N simulated observations of the first marginal, the second column of S is the N simulated observations of the second marginal, and so on. Rank correlations between the simulated columns of S will be close to the target rank correlation matrix C*.

It is important to note that this procedure leaves the marginal distributions fully intact. It is the specific manner in which we permute the random digits from (0, 1) that allows us to simulate observations of each marginal such that the resulting (rank) correlations are (approximately) equal to those in C*, the target correlation matrix. An example of the Company carrying out this process is found in Appendix A.

We now turn to concepts of enterprise and LOB economic capital and describe a procedure to optimize return on (enterprise) economic capital as a function of the allocation of economic capital to the LOBs.

4. Optimization of Return on Economic Capital

4.1. Key Definitions and Stage 1 Planning

Assume the Company has implemented the methodology of Section 3 and now has a risk model that simulates LOB and enterprise DE with the desired rank correlations. In a large run of the model, we examine each simulation and observe the enterprise DE, which is the sum of the simulated LOB DE values. In some adverse simulations, enterprise DE is negative and its absolute value is regarded as the aggregate infusion need for the LOBs (to be satisfied by available liquid funds from Corporate plus any positive earnings from other LOBs if they exist in the particular simulation). That aggregate infusion amount is just enough to allow those LOBs with negative DE to continue to operate at the targeted capital levels.

So the modeling captures any earnings correlations between LOBs and there is only an aggregate infusion need in a particular simulation if *enterprise DE* is negative; otherwise it is defined as zero. The Company finds the first percentile enterprise DE is negative and defines its absolute value as "I."

Definition: I = absolute value of the first percentile of enterprise DE.

Assume for illustrative purposes that the modeling shows I = 80. This amount is less than the sum of the individual LOB infusions 20, 30 and 50, which were denoted I_1, I_2 and I_3, in Section 2.3:

$$80 = I < I_1 + I_2 + I_3 = 100.$$

Note that this relation $I \leq \sum I_k$ is typically observed in practice. We now assume management wants to hold some amount of liquid capital at Corporate to be available for infusion needs at the entities. This Corporate buffer (or simply buffer) is a form of risk capital and helps ensure that when, under stress conditions, one or more LOBs needs an infusion, such needs can be covered by funds from Corporate and any LOBs having positive DE, *without* having to resort to the public market to raise capital. It is precisely during macro stress events that the cost of raising capital externally is most prohibitive, if capital is even available.

The buffer B could reasonably be set equal to I. Management is willing to consider a value of B larger than I to account for potential model risk or simply out of conservatism. They are willing to have a value of B potentially as high as I*, which could be defined as some multiple of I, kI where k > 1, for example, 110 percent. If $\sum I_k$ is larger than I (as is often the case) then one may choose a simpler option such as setting $I^* = \sum I_k$.

One may choose a more involved procedure, perhaps overlaying a specific risk scenario on top of a set of earnings results that correspond to the first percentile of enterprise DE. Finally, an additional alternative is to define I* similarly to I but at a percentile *lower* than the first percentile, in the left tail of negative enterprise DE, so that I* > I. In any event, the Company adopts a policy that B must lie in the *critical corridor* of [I, I*], that is, it requires $I \leq B \leq I^*$.

In Section 4.3, we eventually determine the optimal value of B and also point to the right levels of enterprise and LOB economic capital. Fixing a value of B leads to the following definition.

Definition: For a fixed value of B, *enterprise economic capital* is

$$EC_{ENT} = A_1 + A_2 + A_3 + B \leq A_1 + A_2 + A_3 + I^*.$$

As part of Stage 1 planning, management sets a preliminary value of B and then we have $EC_{ENT} = A_1 + A_2 + A_3 + B$. Using the LOB infusion amounts I_1, I_2, I_3 from Section 2.3 and this fixed buffer value B, we may then *allocate this capital to the LOBs*.

Definition: *Allocated Economic Capital to a Line of Business* (LOB).

Economic capital for LOB_j is

$$EC_j = A_j + \alpha_j B \text{ where } \alpha_j = I_j / \sum I_k.$$

In our example, we have:

- $EC_1 = A_1 + \alpha_1 B$
- $EC_2 = A_2 + \alpha_2 B$
- $EC_3 = A_3 + \alpha_3 B$

where $\alpha_j = I_j/(I_1 + I_2 + I_3)$ for j = 1, 2, 3. We have $\alpha_1 = 20\%$, $\alpha_2 = 30\%$ and $\alpha_3 = 50\%$.

The $\{\alpha_i\}$ sum to 100 percent so they allow for full allocation of the enterprise economic capital, EC_{ENT}, to the LOBs.

Assume the Company chose to set B equal to a "round number" above 80: B = 90. We then have EC_{ENT} = 500 + 1,000 + 2,400 + 90 = 3,990, and the following values for allocated LOB economic capital:

- $EC_1 = 500 + 20\% \bullet 90 = 518$
- $EC_2 = 1{,}000 + 30\% \bullet 90 = 1{,}027$
- $EC_3 = 2{,}400 + 50\% \bullet 90 = 2{,}445$

Therefore, $EC_{ENT} = EC_1 + EC_2 + EC_3 = 518 + 1{,}027 + 2{,}445 = 3{,}990$.

This Stage 1 analysis has led to values of enterprise economic capital and LOB allocations that are improved upon in the next section describing Stage 2 analysis. Management is willing to modify the enterprise economic capital value to a limited degree and also change the allocations to LOBs to the extent that is feasible. The eventual EC values from Stage 2 provide the best return on economic capital given management-defined constraints on the enterprise EC and feasible changes in allocations to LOBs.

4.2. Stage 2 Planning: Linking DE to Allocated Economic Capital

As far as its allocation of capital, an insurance company is somewhat like a large sailing vessel in that it cannot significantly change its course in a short amount of time. Many businesses are long-tail and reserve and capital requirements may be slow to change even if a business is put into runoff.

Two assumptions are made at this point:

1. The de facto allocation of economic capital from Stage 1 planning is likely not optimal.
2. It is feasible to make *modest changes* in the enterprise economic capital amount and its allocation ahead of the plan horizon.

This section describes an optimization of return on economic capital with the assumption that the total (enterprise) economic capital level and its allocation to the LOBs can be modestly altered, subject to management-defined constraints, from the Stage 1 values.

Stage 2 begins when each LOB is asked to provide two additional DE forecasts that correspond to a 5 percent increase and decrease in the Stage 1 allocated EC amounts. A different percentage value might be used based on the specifics of the organization's products or management viewpoint. In some cases, other factors may make it appropriate for these "above and below" EC values to be asymmetric with respect to the Stage 1 EC value. For example, for certain long-tail products, reduction of EC may be difficult in a short time span (even if sales are discontinued) so the lower level of EC might be only 2 percent less than the Stage 1 amount.

In Stage 1 planning, LOB1 projected DE of 50 with an allocated economic capital (EC_1) value of 518. LOB1 now performs additional analysis and forecasting to consider the projected DE if, instead of $EC_1 = 518$, it is set to $.95 \bullet 518 \approx 492$ or $1.05 \bullet 518 \approx 544$, and forecasts DE values of 42 and 60 respectively. The other LOBs are asked to make similar projections for their respective hypothetical above and below EC amounts.

Recall from the last section that economic capital for LOB_j is $EC_j = A_j + \alpha_j B$ where $\alpha_j = I_j/\sum I_k$. In thinking about the lower EC case, an LOB might anticipate shutting off sales in its riskiest underwriting tier allowing it to hold less capital. This might account for most of the assumed reduction in EC. If the LOB expects a change, y, in I_i (the infusion at the first percentile DE level from Stage 1) then the LOB revises its estimate of α_i to be $\alpha_i^* = (I_i + y)/(I_1 + I_2 + I_3 + y)$. B is held constant at the Stage 1 value in these calculations.

In thinking about the higher EC level, an LOB might, for example, assume it is taking more risks or expanding sales. If sales are assumed higher than the Stage 1 forecast, one must reflect any increase in associated direct or allocated expenses from Corporate (e.g., commissions to increase performance or payroll for additional marketing staff) in the forecast DE.

It is important to note that regulatory and rating agency constraints are factored into the view of changes in average asset level, A_1, and may include reserve or capital increases resulting from expanded sales.

Returning to LOB1, there are three pairs of hypothetical allocated EC values and the associated projections of DE. These can be viewed as "points" (492, 42), (544, 60) and the Stage 1 result (518, 50). Similarly, the other LOBs each end up with three such points. Based on these three points, each LOB defines a linear function that estimates DE as a function of allocated LOB EC. One might use an ordinary linear regression, some type of weighted least squares, or simply consider the line passing through the two extremes, that is, the points with EC not equal to the Stage 1 value.

4.3. Optimization of Return on Economic Capital

The Company wishes to hold a value of EC_{ENT} and an allocation to the LOBs that maximizes the return on EC_{ENT}. The board and external stakeholders help influence management's decision for an upper bound, EC_{MAX}, on EC_{ENT}. It is understood that holding less than EC_{MAX} will typically mean less DE but management accepts that there may be a better use for any such slack ($EC_{MAX} - EC_{ENT}$). Potential uses include investing in information technology, increasing dividends to shareholders, share repurchases, and mergers and acquisition activity. The Company wants the committed EC to earn the best return and therefore will strive to maximize the quantity: expected enterprise DE/EC_{ENT}. This ratio is *return on economic capital* (ROEC).

As described in the last section, projected LOB DE has been approximated as a linear function of a potential LOB EC level. If the LOB EC levels are written as x_1, x_2 and x_3 and their corresponding (linear) estimates of DE are denoted by L_1, L_2 and L_3, then ROEC can be expressed as:

$$ROEC = [L_1(x_1) + L_2(x_2) + L_3(x_3)]/(x_1 + x_2 + x_3).$$

We now turn to the optimization of ROEC. Writing each L_k as $a_k x_k + b_k$ where the $\{a_i\}$ and $\{b_i\}$ are constants determined in Stage 2 planning from Section 4.2 yields:

$$ROEC = (a_1x_1 + b_1 + a_2x_2 + b_2 + a_3x_3 + b_3)/(x_1 + x_2 + x_3).$$

Setting $b = b_1 + b_2 + b_3$ gives:

$$ROEC = (a_1x_1 + a_2x_2 + a_3x_3 + b)/(x_1 + x_2 + x_3).$$

The Stage 2 planning of Section 4.2 asked each LOB to project DE assuming hypothetical EC amounts above and below the Stage 1 value of EC_i. Writing these lower and upper values as m_i and M_i respectively allows the optimization of ROEC to be described as the following problem in the decision variables x_1, x_2, x_3:

Maximize:

$$ROEC = (a_1x_1 + a_2x_2 + a_3x_3 + b)/(x_1 + x_2 + x_3)$$

Subject to:

$$m_1 \leq x_1 \leq M_1$$

$$m_2 \leq x_2 \leq M_2$$

$$m_3 \leq x_3 \leq M_3$$

$$x_1 + x_2 + x_3 \leq EC_{MAX}$$

Observe that this objective function is the quotient of two linear functions in the decision variables and the constraints are linear in the decision variables. This is a so-called *linear-fractional programming* (LFP) problem and can be solved by reduction to a standard linear programming (LP) problem.

Let $z = 1/(x_1 + x_2 + x_3)$ and define $y_i = x_iz$ for each i. This implies the constraint $y_1 + y_2 + y_3 = 1$ and the problem can be written as:

Maximize:

$$a_1y_1 + a_2y_2 + a_3y_3 + bz$$

Subject to:

$$y_1 + y_2 + y_3 = 1$$

$$m_1z \leq y_1 \leq M_1z$$

$$m_2z \leq y_2 \leq M_2z$$

$$m_3z \leq y_3 \leq M_3z$$

$$z \geq 0$$

$$y_1 + y_2 + y_3 \leq EC_{MAX} \bullet z$$

This problem is a standard LP problem and the solution in $\{y_i\}$ gives the solution in the original decision variables via the relation $y_i = x_iz$ for each i. Note that each constraint was rewritten in light of how z was defined. There is also an *equality constraint* based on the “z-substitution” and the inequality constraint $z \geq 0$. Appendix B provides a more detailed explanation of LFP for a numerical example.

Now we illustrate the approach with an example for the Company. In this example, we use a standard linear regression for each LOB based on its three ordered pairs of (EC, DE). Additionally, it is assumed that management communicated that EC_{MAX} is 4,100.

Using linear regression, each LOB has described DE as a linear function of LOB allocated capital x_i. Table 3 shows the (values making up the) three ordered pairs used in the regressions for each LOB and the regression results. Amounts are in USD millions.

Table 3. Regression Results: Linking Earnings to Economic Capital

	LOB1	LOB2	LOB3
Stage 1 alloced EC (EC_i)	518	1,027	2,445
m_i: lower EC constraint	492	976	2,323
M_i: upper EC constraint	544	1,078	2,567
Stage 1 DE	50	70	120
lower DE projection at m_i	45	65	102
upper DE projection at M_i	52	85	150
regression slope	0.139	0.195	0.198
regression intercept	−22.86	−126.67	−359.49

Given LOB EC levels x_1, x_2 and x_3, the corresponding DE values are estimated by each of the linear functions (regression lines). The sum of those DE values is the enterprise DE forecast and is the *numerator* of the ROEC expression; the denominator is the enterprise capital $x_1 + x_2 + x_3$. The LFP problem is:

Maximize:

$$ROEC = (.139x_1 + .195x_2 + .198x_3 - 509.01)/(x_1 + x_2 + x_3)$$

Subject to:

$$492 \leq x_1 \leq 544$$

$$976 \leq x_2 \leq 1{,}078$$

$$2{,}323 \leq x_3 \leq 2{,}567$$

$$x_1 + x_2 + x_3 \leq 4{,}100$$

Letting $z = 1/(x_1 + x_2 + x_3)$, it is possible to restate as an LP problem:

Maximize:

$$.139y_1 + .195y_2 + .198y_3 - 509.01z$$

Subject to:

$$y_1 + y_2 + y_3 = 1$$

$$492z \leq y_1 \leq 544z$$

$$976z \leq y_2 \leq 1{,}078z$$

$$2{,}323z \leq y_3 \leq 2{,}567z$$

$y_1 + y_2 + y_3 \leq 4{,}100z$

$z \geq 0$

Using an LP solver gives the solution (via matrix operations) as:

$y_1 = 0.12$, $y_2 = 0.253902$, $y_3 = 0.626098$, $z = 0.000243902$.

Using the relation $y_i = x_i z$ for each i (which implies $x_i = y_i/z$) gives:

$x_1 = 492$, $x_2 = 1{,}041$ and $x_3 = 2{,}567$ and enterprise capital is $x_1 + x_2 + x_3 = 4{,}100$.

The values $x_1 = 492$, $x_2 = 1{,}041$ and $x_3 = 2{,}567$ and the regression relationships imply LOB DE values of 45.53, 76.33 and 148.78 respectively and an enterprise DE of 270.64. Dividing this by the total EC of 4,100 gives an ROEC of 6.6 percent.

Each LOB estimated its economic capital assuming projected average assets (the A_i from Section 4.2). If these three asset levels, corresponding to the above EC levels, are subtracted from 4,100, the result is the revised (Stage 2) Corporate buffer.

4.4. Other Applications

The method of the previous section can of course be applied in other settings where one seeks to optimize a particular metric of interest. In an organization where the notion of economic capital does not apply, it may make sense to optimize return on investment (ROI). Management might decide that the total investment in the existing business lines must fall in a particular range (with any unused assets being directed toward, for example, research and development, share repurchases or acquisitions) and then ask how to allocate that amount across the businesses so the ROI is maximized. If the constraints on the allocations are linear and the return for a line of business, for example, net income, can be expressed reasonably well as a linear function of the allocation to that business, then we have an LFP problem and it can be reduced to an LP problem as in Section 4.3.

Alternatively, instead of only considering investments in the existing business lines, management could include in the analysis the potential to invest in new products, additional marketing, IT infrastructure and so on. It may be appropriate to change the return metric or the time horizon for some applications.

For a company considering options that may impact shareholder's equity and (obviously) striving for high net income, the method can once again be used if the net income can be related to the decision variables in a linear manner. In this case, the ratio of interest is return on equity (ROE) and given linear constraints for equity and potential "allocations" to products, ventures, share buybacks and so on, the company sets up an LFP problem to optimize ROE.

4.5. Exact Solutions to Imprecise Problems

Employing LFP provided an exact solution to a problem that was stated with some amount of approximation (e.g., the linear *estimates* of LOB DE as a function of allocated capital). It is worthwhile to pause here to make an important but slightly theoretical point relating to these types of situations. Consider the following linear programming problem.

> A manufacturing company produces two products, X and Y, which are sold for a profit of \$4 and \$7 per pound respectively. Determine the number of pounds, x and y, of product X and product Y that should be produced to maximize the company's profit subject to constraints:
>
> $4x + 6y \leq 900,$
>
> $x + 2y \leq 280,$
>
> $x \geq 0,\ y \geq 0.$

This is a standard LP problem, to maximize 4x + 7y, and the optimal choice of x and y can be determined exactly using matrix operations. The optimal solution is x = 60 and y = 110, yielding a profit of 1,010.

Suppose the assumed profits are actually *estimates* from the marketing and finance departments of the company and the eventual price cannot be known with certainty. If that uncertainty is mathematically captured in the statement of the problem, it will no longer be an LP problem with an exact solution. However, given that the \$4 and \$7 estimates are the best possible forecasts, the solution to the above LP problem remains highly relevant and the *exact* optimization it provides is obviously still of tremendous import.

Generally speaking, if a problem does not have a closed-form (exact) method to find the solution, we often end up in one of two situations.

1. We must attempt an arduous and/or ad hoc search through an infinite feasible space, or
2. a numerical (approximate) method exists but it may not be clear how close to optimal a proposed solution is.

In financial or risk-reward optimization problems, we often estimate various parameters or assumptions to be used in the framing of the problem (e.g., returns, correlations and volatilities) and then employ an exact method to find the solution. Because the parameters are approximated, the statement of such problems are themselves approximate; however, the availability of a closed-form approach for finding the optimal solution is the best situation for which those faced with the problem can hope.

In many real-world problems relating to allocation, the use of an inexact method in the search for optimal solutions very quickly becomes unwieldy as the feasible space can be *extremely* large even with a modest number of potential allocation "buckets" or asset categories.

The methodology of Section 4 allows for an exact solution for ROEC optimization. In Section 5, we turn our attention to resource or product allocations based on mean-semivariance return optimization and discuss when an approach giving a feasible, exact solution exists. In some cases, the optimal portfolio will contain short positions. This demonstrates an unfortunate mathematical truth: the renowned efficient frontier (whether based on variance or semivariance) often contains portfolios with short positions as indicated by negative weights to these assets.

5. Mean-Semivariance Optimization for the Enterprise

5.1. The Investor Point of View and Return Volatility

A risk-intelligent insurer naturally relates to the concept of economic capital and its allocation as it attempts to provide the best return using what is typically regarded as "scarce" risk capital. Of course, most equity investors in a company are generally not very concerned with capital allocation or return on capital for various business lines. Their focus is, rightly so, on expected enterprise earnings levels and potential volatility.

As is well known, Harry Markowitz attempted to capture this view in his modern portfolio theory, which assumed that rational investors evaluate stocks (or other investment vehicles) and form portfolios based on the expected return and standard deviation of returns. Certainly less well known is that Markowitz viewed the use of standard deviation to capture return volatility as less than ideal and its use stemmed largely from the resulting tractability of the mathematics. In fact, upon receiving the Nobel Prize in 1990, Markowitz said, "Semivariance seems more plausible than variance as a measure of risk, since it is concerned only with adverse deviations."[8] Clearly an investor is more worried about downside volatility, that is, underperformance, than surprise upside results. This suggests that *semivariance* better captures an investor's view of risk.

5.2. Benchmark Returns, Semivariance and Semicovariance

We define semivariance for an asset in relation to a fixed benchmark ("hurdle") rate of return, B, as the expected value $E[(\min(R - B, 0))^2]$ where R is the unknown future return of the asset for a time horizon of interest. So semivariance is the expected value of the *squared shortfall versus B*, where a shortfall in the case of a return at least as large as B is defined as zero.

Using the definitions of Javier Estrada,[9] of Spain's University of Navarra, we estimate the semivariance (using a sample semivariance) of asset i's returns with respect to a benchmark B as:

$$SV_{iB} \approx (1/T) \sum [\min (R_{it} - B, 0)]^2 \qquad (1)$$

where T is the number of observed time periods and R_{it} is the return of asset i during each of those time periods t = 1, 2, …, T.

Define the semicovariance between two assets i and j as:

$$SC_{ijB} = E[\min (R_i - B, 0) \bullet \min (R_j - B, 0)] \qquad (2)$$

which is estimated by the sample semicovariance as:

[8] Harry M. Markowitz, "Foundations of Portfolio Theory," Nobel lecture, Dec. 7, 1990, http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1990/markowitz-lecture.pdf.

[9] Javier Estrada, "Mean-Semivariance Optimization: A Heuristic Approach," *Journal of Applied Finance* (spring/summer 2008), http://web.iese.edu/jestrada/PDF/Research/Refereed/MSO.pdf.

$$SC_{ijB} \approx (1/T) \sum [\min (R_{it} - B, 0) \bullet \min (R_{jt} - B, 0)] \quad (3)$$

where T is the number of observed time periods and R_{it} and R_{jt} are the returns of assets i and j during each of those time periods t = 1, 2, …, T.

A very useful property of portfolio *variance* is that it can be expressed as a function of the portfolio weights to each asset, and the assets' return variances and covariances and these volatility statistics do *not* vary with a change in the weights. It is natural to attempt a mean-semivariance optimization using the sample portfolio semivariance defined as $(1/T) \sum [\min (R_{Pt} - B , 0)]^2$ where R_{Pt} is the portfolio return over period t, but this poses a significant problem described by Estrada as follows:

> The main obstacle to the solution of this problem is that the semicovariance matrix is endogenous; that is, a change in weights affects the periods in which the portfolio underperforms the benchmark, which in turn affects the elements of the semicovariance matrix.[10]

Given a portfolio with weights $\{w_i\}$ and benchmark B, Estrada's solution to this problem is to approximate the portfolio semivariance as:

$$SV_{pB} \approx \sum_i \sum_j w_i w_j SC_{ijB} \quad (4)$$

where SC_{ijB} is as defined in Equation (3).

This approach allows for a very accurate approximation of the portfolio semivariance. Additionally, these definitions and formulas lead to a semicovariance matrix that is *exogenous*: Once we fix the benchmark B, the values for asset semivariances and semicovariances do *not* change as the asset weights change. Metaphorically, they exist as constants outside of our "tinkering" with the weights.

It is natural to wonder if there is a practical distinction between variance and semivariance and the answer is a resounding yes. Estrada provides historical stock returns where an investor will make a different investment decision between Oracle and Microsoft depending on which volatility measure is used.[11]

The optimization carried out in Section 4.3 maximized ROEC subject to internal constraints at the Company. It was regarded as optimal usage of scarce capital considering the plan horizon and the DE metric. It is natural to ask how the resulting EC allocation would look from the investor point of view. In other words, given a particular return benchmark for the Company's stock and an aversion to downside volatility of returns, what allocations to business opportunities, resources or products is optimal? Section 5.3 starts to explore this question.

5.3. Selection of Return Metric, Resource and Allocation Buckets

The ROEC optimization led the Company to specific levels of enterprise EC and LOB allocations. As mentioned, this may or may not be in line with the expectations of the investor in terms of expected

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[10] Ibid.

[11] Ibid.

return and volatility. The Company may wish to consider the effect of the EC allocations on the expected return and semivariance of the return. Instead of allocation of EC, many organizations will consider how much funding or investment should be assigned to each product, operating country or new business opportunity. The choice of *resource* to be allocated and the "buckets" for such allocation depend on where a company has flexibility to change allocation and the ability to measure the hypothetical effects on expected return and volatility.

An investor looks for a return on the dollars invested via the purchase of shares. A company with strong earnings is most likely to offer this return, whether in the form of dividends or stock price appreciation. This suggests a return metric worth considering is *earnings yield* defined as earnings per share divided by price per share. This ratio is the reciprocal of the price to earnings (P/E) ratio and the numerical values of earnings yield "look" like returns (e.g., they are dimensionless and might take on values such as 5 percent, 8 percent, etc.) so the interpretation is intuitive.

A company may wish to consider this question: From an investor's point of view, how should we allocate the next dollar of "unassigned" capital (i.e., liquid, deployable funds not otherwise designated as risk capital or economic capital, and not earmarked for any particular business, venture, expense or investment) to our existing products and opportunities in order to have an expected value, Y, of earnings yield such that the semivariance of earnings yield is a minimum? This type of question can be framed in many ways with various metrics so we now take a more generic approach.

The remainder of Section 5 assumes that the return metric, resource to be allocated and the possible allocation buckets have all been chosen. *The allocation buckets will be referred to as assets* but it may be instructive to think of them as the collection of a company's product lines, business segments and any other investment options.

We assume the existence of a stochastic risk-return model capable of measuring asset semivariances and semicovariances, and portfolio expected return as a function of any set of allocation weights. It is important that the model reflect rank correlations of returns across assets in the manner of Section 3.

Based on a large run from the model, it is possible, using the simulated asset returns, to estimate the semivariance for each asset and the semicovariance between any pair of assets. These are the parameters needed to evaluate the portfolio semivariance for any set of weights (allocations) to the assets.

In Section 5.4, we will fix a benchmark and use the notation s_i^2 or s_{ii} for the (sample) semivariance of asset i and s_{ij} for the (sample) semicovariance between assets i and j. We will also suppress the "sample" clarifier as well. This is a slight abuse of notation as these symbols typically indicate sample variance and covariance but there will be no confusion as this paper will *only* use the downside volatility measures of *semivariance* and *semicovariance*.

Given a return benchmark we will seek the portfolio (described by asset weights) of minimum semivariance that provides an expected return equal to the benchmark. In the next section, this is described for the case of three assets.

5.4. Lagrange Multipliers in the Three Asset Setting

Consider a fixed benchmark return, B, as defined by an existing or potential investor in a company's stock. In the mean-semivariance framework, the main assumption is that the investor expects the company to find the allocation to its N assets, (w_1, w_2, …, w_N) with respective expected returns (r_1, r_2, …, r_N) which will:

Minimize:

(approximate) portfolio semivariance $\sum_i \sum_j w_i w_j s_{ij}$

Subject to:

$\sum w_i = 1$ (the budget constraint)

$\sum w_i r_i = B$ (the return constraint)

In this problem, the $\{w_i\}$ are the decision variables, and the objective function as well as the constraints are all functions of the $\{w_i\}$. Note that the constraints are *equality* constraints. In addition, continuous first partial derivatives with respect to each of the weights exist for the objective function and the constraints. Because the objective function is not linear in the $\{w_i\}$, the problem is not LP; however, the described conditions make this problem susceptible to the method of Lagrange multipliers[12] and the optimal solution is found directly as the solution to a system of equations based on various partial derivatives.

We will consider the case of *three* assets and two equality constraints. In this situation, the method of Lagrange multipliers will find the minimum or maximum of an objective function f(x, y, z) subject to equality constraints g(x, y, z) = 0 and h(x, y, z) = 0. The extreme values will occur when x, y and z satisfy the following equations, which include the (to be determined) constants λ and μ:

$g(x, y, z) = 0$

$h(x, y, z) = 0$

$f_x = \lambda g_x + \mu h_x$

$f_y = \lambda g_y + \mu h_y$

$f_z = \lambda g_z + \mu h_z$

where a subscript next to a function denotes partial differentiation with respect to the indicated variable.

With three assets, the mean-semivariance problem from the beginning of the section can now be described as:

[12] See http://mathworld.wolfram.com/LagrangeMultiplier.html for a general explanation of Lagrange multipliers, key formulas and additional resources.

Find allocation weights w_1, w_2 and w_3 to:

Minimize:

$$f(w_1, w_2, w_3) = w_1^2 s_1^2 + w_2^2 s_2^2 + w_3^2 s_3^2 + 2(w_1 w_2 s_{12} + w_2 w_3 s_{23} + w_1 w_3 s_{13})$$

Subject to:

$$g(w_1, w_2, w_3) = w_1 + w_2 + w_3 - 1 = 0$$

$$h(w_1, w_2, w_3) = w_1 r_1 + w_2 r_2 + w_3 r_3 - B = 0$$

Note that the various semivariances and semicovariances are all *constants*. To apply the method of Lagrange multipliers, we first compute the following partial derivatives:

$$f_{w1} = 2(w_1 s_1^2 + w_2 s_{12} + w_3 s_{13})$$

$$f_{w2} = 2(w_1 s_{12} + w_2 s_2^2 + w_3 s_{23})$$

$$f_{w3} = 2(w_1 s_{13} + w_2 s_{23} + w_3 s_3^2)$$

$$g_{w1} = 1$$

$$g_{w2} = 1$$

$$g_{w3} = 1$$

$$h_{w1} = r_1$$

$$h_{w2} = r_2$$

$$h_{w3} = r_3$$

The system of equations to solve is then:

$$w_1 + w_2 + w_3 - 1 = 0$$

$$w_1 r_1 + w_2 r_2 + w_3 r_3 - B = 0$$

$$2(w_1 s_1^2 + w_2 s_{12} + w_3 s_{13}) = \lambda + \mu r_1$$

$$2(w_1 s_{12} + w_2 s_2^2 + w_3 s_{23}) = \lambda + \mu r_2$$

$$2(w_1 s_{13} + w_2 s_{23} + w_3 s_3^2) = \lambda + \mu r_3$$

Based on the last three equations, one may express each of the three weights in terms of λ and μ. Then the first two equations can be restated in terms of only λ and μ giving a system of two equations in two unknowns that is solved for λ and μ. Using those values, we can finally solve for the three weights. The last three equations can be written in matrix form as:

$$2S \bullet (w_1, w_2, w_3)^T = (\lambda + \mu r_1, \lambda + \mu r_2, \lambda + \mu r_3)^T$$

where S is the (symmetric) semicovariance matrix. Note that the element of S in the ith row and jth column, s_{ij}, is, in fact, equal to the semivariance or semicovariance term we denoted s_{ij} in Section 5.3. As a result, the weights can be expressed in term of λ and μ as:

$$(w_1, w_2, w_3)^T = ½ \bullet S^{-1} \bullet (\lambda + \mu r_1, \lambda + \mu r_2, \lambda + \mu r_3)^T.$$

This then allows for the solution of the weights based on the budget constraint and the return constraint.

The mathematically inclined reader may be interested in the more general setting. Given M constraint equations $\{g_i(x) = 0\}$, the method of Lagrange multipliers describes the set of equations required for an extreme value of the objective function f to occur at a point p as:

For i = 1, 2, …, M,

$g_i(p) = 0$ and

$\nabla f(p) = \sum \lambda_i \nabla g_i(p)$

where ∇ is the "Del operator" defined as the vector of partial derivatives $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ of the function to which it applies, the $\{\lambda_i\}$ are scalars, and the summation is from i = 1 to M. Both the constraint functions and the objective function are defined on some subspace of $\mathbf{R}^n$ (n-dimensional Euclidean space).

5.5. Numerical Example for a Three Asset Portfolio

This example will determine an allocation to three strategic options (e.g., existing product lines or an investment opportunity) described by weights w_1, w_2 and w_3 summing to 1. The options, or *assets*, have expected returns of 9 percent, 8 percent and 6.5 percent so the expected return for the company is $.09w_1 + .08w_2 + .065w_3$.

Given a specific benchmark return B, we find the portfolio (i.e., the three weights) of minimum semivariance with an expected return equal to B.

Given a value for B, we make use of a risk model that simulates a large number of returns for each of the assets, reflecting any correlation between them. Then it is a matter of arithmetic to compute all the needed semivariances and semicovariances that comprise the matrix S, as defined in Section 5.4.

We suppress B in our notation when it is clear from the context and the semicovariance matrix S is:

S_1^2	S_{12}	S_{13}
S_{12}	S_2^2	S_{23}
S_{13}	S_{23}	S_3^2

These values are found using the simulated return data to first calculate shortfalls versus the benchmark and then using expressions (1) and (3) from Section 5.2.

If one calculates the shortfalls for a particular set of simulated asset returns, say for asset i, then we may view that set of shortfall values as a vector. In that case, the dot product of the vector with itself divided by the number of elements in the vector is s_i^2.

Similarly, for shortfall vectors for assets i and j, the dot product of these two vectors divided by the number of elements in each vector is s_{ij}.

Illustrative data and shortfalls are shown in Table 4 for the benchmark value of 8 percent for a small number of simulations. Note that a *much larger* number of simulations would be used in practice to ensure a robust estimate of the semicovariance matrix S.

Table 4. Illustrative Return Data for Three Assets

Simulated Return			B = 8.0%		
Asset 1	**Asset 2**	**Asset 3**	**Shortfall vs. B: min(0, return − B)**		
7.15%	7.88%	7.33%	−0.85%	−0.12%	−0.67%
12.74%	11.22%	6.12%	0.00%	0.00%	−1.88%
1.92%	1.88%	5.52%	−6.08%	−6.12%	−2.48%
5.90%	8.14%	6.65%	−2.10%	0.00%	−1.35%
10.21%	6.38%	6.34%	0.00%	−1.62%	−1.66%
10.78%	8.33%	5.57%	0.00%	0.00%	−2.43%
8.99%	6.11%	5.21%	0.00%	−1.89%	−2.79%
9.96%	10.48%	8.09%	0.00%	0.00%	0.00%
11.02%	10.81%	6.97%	0.00%	0.00%	−1.03%
3.86%	7.48%	7.45%	−4.14%	−0.52%	−0.55%
8.17%	12.51%	8.14%	0.00%	0.00%	0.00%
10.54%	8.14%	7.14%	0.00%	0.00%	−0.86%
6.22%	6.48%	5.39%	−1.78%	−1.52%	−2.61%
5.23%	9.99%	6.44%	−2.77%	0.00%	−1.56%
7.24%	7.28%	6.83%	−0.76%	−0.72%	−1.17%
5.54%	6.59%	6.35%	−2.46%	−1.41%	−1.65%
10.94%	7.45%	7.20%	0.00%	−0.55%	−0.80%
15.82%	8.76%	8.69%	0.00%	0.00%	0.00%
4.24%	8.80%	6.91%	−3.76%	0.00%	−1.09%
6.28%	4.72%	4.27%	−1.72%	−3.28%	−3.73%
6.41%	7.42%	4.98%	−1.59%	−0.58%	−3.02%
10.36%	4.37%	4.96%	0.00%	−3.63%	−3.04%
5.77%	9.87%	6.93%	−2.23%	0.00%	−1.07%
14.18%	8.69%	6.38%	0.00%	0.00%	−1.62%
5.56%	7.21%	7.19%	−2.44%	−0.79%	−0.81%
9.28%	7.88%	6.10%	0.00%	−0.12%	−1.90%
1.53%	6.35%	5.76%	−6.47%	−1.65%	−2.24%
12.57%	12.59%	7.58%	0.00%	0.00%	−0.42%
3.33%	4.18%	5.36%	−4.67%	−3.82%	−2.64%
10.33%	8.79%	6.44%	0.00%	0.00%	−1.56%
14.53%	12.67%	7.90%	0.00%	0.00%	−0.10%
9.22%	4.55%	5.17%	0.00%	−3.45%	−2.83%
7.59%	8.08%	7.40%	−0.41%	0.00%	−0.60%
8.63%	8.73%	6.64%	0.00%	0.00%	−1.36%
10.92%	8.06%	6.97%	0.00%	0.00%	−1.03%
8.77%	7.03%	6.51%	0.00%	−0.97%	−1.49%
10.72%	11.04%	7.83%	0.00%	0.00%	−0.17%
10.42%	11.41%	7.63%	0.00%	0.00%	−0.37%
6.92%	5.17%	6.26%	−1.08%	−2.83%	−1.74%
11.35%	8.82%	7.52%	0.00%	0.00%	−0.48%
10.92%	10.25%	6.60%	0.00%	0.00%	−1.40%
4.70%	5.85%	5.03%	−3.30%	−2.15%	−2.97%
4.67%	7.66%	6.30%	−3.33%	−0.34%	−1.70%
10.00%	6.83%	7.23%	0.00%	−1.17%	−0.77%

8.17%	6.75%	7.64%	0.00%	−1.25%	−0.36%
10.37%	5.69%	5.42%	0.00%	−2.31%	−2.58%
7.54%	4.89%	5.41%	−0.46%	−3.11%	−2.59%
8.29%	9.41%	6.37%	0.00%	0.00%	−1.63%
5.78%	5.75%	5.70%	−2.22%	−2.25%	−2.30%
7.33%	6.27%	7.27%	−0.67%	−1.73%	−0.73%

For each choice of benchmark return (B), the three right-most columns must be recalculated to obtain the matrix S. Some of the results are summarized here.

Semicovariance Matrix S for B = 7.0%

s_1^2	s_{12}	s_{13}		0.022%	0.009%	0.007%
s_{12}	s_2^2	s_{23}	=	0.009%	0.013%	0.009%
s_{13}	s_{23}	s_3^2		0.007%	0.009%	0.010%

Semicovariance Matrix S for B = 7.5%

s_1^2	s_{12}	s_{13}		0.030%	0.014%	0.013%
s_{12}	s_2^2	s_{23}	=	0.014%	0.020%	0.014%
s_{13}	s_{23}	s_3^2		0.013%	0.014%	0.018%

Semicovariance Matrix S for B = 8.0%

s_1^2	s_{12}	s_{13}		0.040%	0.020%	0.021%
s_{12}	s_2^2	s_{23}	=	0.020%	0.029%	0.023%
s_{13}	s_{23}	s_3^2		0.021%	0.023%	0.031%

Note that for fixed i and j, the values of s_{ij} or s_i^2 grow larger as the value of B is increased. This is an intuitive result considering how the semivariance and semicovariance formulas make use of the shortfall measure min (0, return – B).

With a benchmark value of 8 percent (see matrix) we now determine the weights for a portfolio of minimum semivariance and expected return of 8 percent. The inverse of S is:

4,233.300	−1,719.884	−1,610.593
−1,719.884	9,645.197	−6,080.209
−1,610.593	−6,080.209	8,937.111

The following matrix equation gives the weights expressed in terms of λ, μ and the three expected asset returns:

$$(w_1, w_2, w_3)^T = ½ \bullet S^{-1} \bullet (\lambda + \mu r_1, \lambda + \mu r_2, \lambda + \mu r_3)^T.$$

This relation along with the budget and return constraints is a system of five equations in five unknowns and can be solved as described in Section 5.4. The solution is w_1 = 28.3%, w_2 = 52.8%, w_3 = 18.9% and a portfolio semivariance of 0.0252 percent.

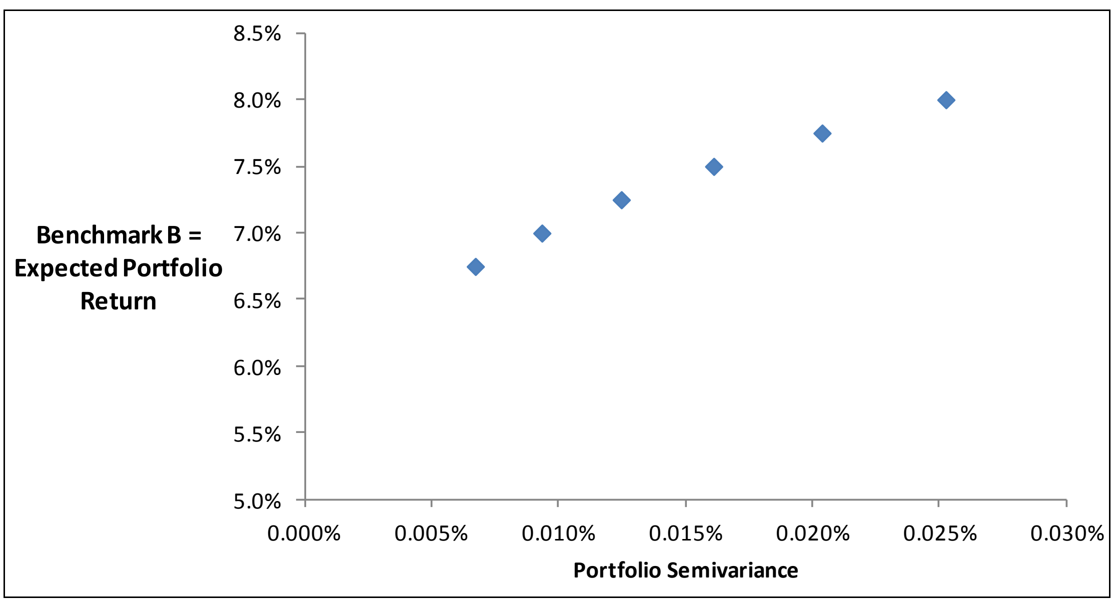
The same procedure can be used for other choices of B and some of these results are shown in Table 5. As the expected portfolio return demanded by the investor (i.e., the benchmark value B) increases, so does the portfolio semivariance. This is consistent with the usual relationship one expects in a risk-reward tradeoff: As one seeks more reward (expected return), the risk (e.g., semivariance) increases.

Observe how the weightings for assets 1 and 2 diminish as the benchmark is lowered. The weight for Asset 3 has the opposite behavior. This foreshadows an insidious problem with efficient frontiers. The benchmark and semivariance values from Table 5 are shown in Figure 1 as an efficient frontier.

Table 5. Minimum Semivariance Portfolios for Selected Benchmarks

	Minimum Semivariance Portfolio Weights			
B = benchmark	w1	w2	w3	Semivariance
6.75%	9.5%	0.8%	89.7%	0.0067%
7.00%	12.6%	12.3%	75.1%	0.0093%
7.25%	15.6%	24.0%	60.4%	0.0125%
7.50%	19.3%	34.5%	46.2%	0.0161%
7.75%	23.5%	44.2%	32.3%	0.0204%
8.00%	28.3%	52.8%	18.9%	0.0252%
8.25%	33.8%	60.3%	5.9%	0.0308%

Figure 1. Mean-Semivariance Efficient Frontier



We can use the same methodology, based on Lagrange multipliers, to add additional points between those shown or outside of them. Using benchmarks of 6.5 percent and 8.5 percent the following results are obtained.

B	w1	w2	w3	Semivariance
6.50%	5.8%	−9.6%	103.9%	0.0047%
8.50%	39.7%	67.1%	−6.9%	0.0372%

Now *negative asset weights* have crept into the answer. These are short positions for these assets and a long-only portfolio is *not* the minimum semivariance portfolio for these choices of B. In fact, for choices of B smaller than 6.5 percent or larger than 8.5 percent, there will always be short positions as indicated by negative weights for certain asset classes. In the case of the 6.5 percent benchmark, it is clear a portfolio with 100 percent in Asset 3 will have the correct expected return and the semivariance is only *very slightly* larger than that in the table. This is a "lucky" situation where a nearly optimal long-only portfolio was obvious. The case is different with the 8.5 percent benchmark and to find the long-only portfolio of minimum semivariance one may use a numerical algorithm such as quadratic programming.

Lagrange multipliers cannot be used to find the optimal *long-only* portfolio if the optimal portfolio contains short positions. If one attempts to add constraints that each weight be greater than or equal to zero then, because these are *not* equality constraints, Lagrange multipliers can no longer be used.

5.6. The Impossible Frontier

A portfolio of minimum semivariance (or variance in the classic Markowitz efficient frontier) that has short positions is rarely of use to a company. Generally such allocations are important to those capable of employing a long-short equity strategy such as a hedge fund or other sophisticated investor. Certainly in the case of the assets defined as lines of business within a company, the short positions do not make sense and such portfolios are not feasible. The celebrated efficient frontier seems to hold the promise of an exact solution for a *feasible*, minimum (semi) variance portfolio but very often only a section of the frontier consists of long-only portfolios.

In a 2009 paper,[13] Moshe Levy and Richard Roll lamented, "[I]t is well known that portfolios on the efficient frontier typically involve many short positions. … This constitutes a very dark cloud hanging over one of the most fundamental models of modern finance."

As the number of assets increases, this long-only section shrinks and may even not exist. In a 2009 research paper, Thomas Brennan and Andrew Lo[14] define the "impossible frontier" and give the following strong, if somewhat disappointing, result:

> If, for a given a set of asset-return parameters (means, variances, and covariances), the corresponding efficient frontier does not have any [long-only] portfolio, we call this an "impossible frontier" [and] as the number of assets grows large, nearly all efficient frontiers are impossible.

[13] Moshe Levy and Richard Roll, "The Market Portfolio May be Efficient After All," Jan. 14, 2009, http://www.anderson.ucla.edu/Documents/areas/fac/finance/market_portfolio_efficient.pdf.

[14] Thomas Brennan and Andrew Lo, "Impossible Frontiers," Dec. 7, 2009, http://alo.mit.edu/wp-content/uploads/2015/08/ImpossibleFrontiers2010.pdf.

> Specifically, for any arbitrary set of expected returns and for a randomly chosen covariance matrix, [it is shown] that the probability that the resulting frontier is impossible approaches one as the number of assets increases without bound.

In their paper they provide a precise, mathematical meaning for a "randomly chosen" matrix and their use of the term "nearly all" is consistent with the formal notion of "almost always" from real analysis (or probability theory).

Fortunately, it is often possible to frame portfolio optimization problems using a relatively small number of assets. In many companies, it is practical to consider allocation across business units or product lines and that number of such categories is often manageable. When the optimal long-only portfolio, for a particular benchmark return, is not on the efficient frontier, one is then left with numerical methods to find it. As previously mentioned, some variation of quadratic programming may be applied in these cases.[15] In some situations, Excel's Solver add-in can perform this task.

Section 5.5 demonstrated a case where long-only portfolios exist for most portfolio returns of interest. Based on the three asset expected returns of 9 percent, 8 percent and 6.5 percent, it was natural to consider expected portfolio returns in the interval [6.5 percent, 9 percent] as any others are clearly impossible assuming non-negative weights. In the example, long-only portfolios *did* exist on the frontier for most of that interval. Appendix C cites some key results that determine the region of the frontier where there will be such desirable portfolios. In such a region, Lagrange multipliers explicitly provide the optimal portfolio *and* it is a long-only portfolio.

6. Additional Applications and Final Thoughts

Our definition of economic capital and ROEC considered a time horizon that would typically be in the range of one to three years. Therefore the optimization seems to be in relation to a short time horizon. However, the return metric made use of an expected value (mean) of a distribution and is therefore also relevant to value creation over the *longer term* when the sample average is more likely to be close to the expected value. From this viewpoint, the ROEC method discussed can be thought of as a way to the highest *longer term* return on economic capital, assuming the relative differences in risk-return volatility profiles between lines of business remain fairly stable over the period.

In addition to its strategic importance, the ROEC methodology will serve an insurer well in its Own Risk and Solvency Assessment (ORSA) preparations. Having an approach to economic capital that applies consistently across all LOBs and captures both risk and reward is an ideal tool for the risk capital assessment and projection required by Section 3 of ORSA. Applications to Solvency II are equally apparent.

[15] For additional detail on quadratic programming, see http://www.math.uh.edu/~rohop/fall_06/Chapter3.pdf.

If optimization of ROEC is considered somewhat of an internal view, the analysis based on the mean-semivariance efficient asset allocations can be viewed as a natural supplement that captures the outside view of the investor.

Both optimizations avoid the assumption that modeled distributions are of a specific form whenever possible. Such non-parametric methods, which make very limited assumptions about the behavior of the underlying variables, have been used to a large extent. As a consequence, the results are applicable in a very wide set of circumstances and buy-in across the organization is much more attainable.

Effort in the following three areas will help ensure that an organization “lives and breathes” these enterprise risk-reward optimizations:

- Training and education
- Measurement and management of relevant metrics
- Risk-adjusted performance measurement and compensation

The training, of course, includes some objective and technical elements regarding the metrics and how to estimate or track the associated inputs. However, it must be stressed that it is no longer an absolute dollar level of a reward metric (e.g., earnings or sales) that is viewed as the primary definition of exceptional performance. This can be a significant culture change in many companies and management must “walk the talk.”

Each line of business must be given clear, consistent methods, with proper review and cross-checks, to develop earnings distributions and estimate model parameters. The ERM function must review the methods and results and is also critical in forming correlation and semicovariance assumptions. ERM is also tasked with developing the aggregate/portfolio view, creating the associated models and performing the ERRO approaches described in this paper. ERM may work with the finance and/or actuarial departments to measure and record the achieved values of the relevant metrics for performance measurement and compensation purposes.

In the case of the ROEC measure, it is natural to track actual values of LOB DE divided by the “observed” EC allocated to the LOB. The observed EC value for a given LOB can look at actual assets held during the year and the result of the infusion calculations coming from a forward-looking risk model that is updated, for example, on a quarterly basis. Incentive compensation for an LOB can in part be tied to this *achieved* ROEC.

From the mean-semivariance view, it makes sense to reward an LOB for accurate DE forecasting but not for typically beating forecasts due to systematic underestimation (e.g., “sandbagging”). It is not obvious how to achieve this concept, but one approach requires a measurement of

1. the distance from actual earnings to the mean of the (before-the-fact) DE distribution in effect before the plan year, and
2. the absolute level of earnings (in dollars).

It is important that both precision of the forecasting and a high absolute level of achieved DE are rewarded.

One scheme to capture both of these concepts is to define (a component of) an executive's bonus as a present value (PV) proportional to the (dollar) level of achieved earnings. This PV is split into cash and restricted stock awards based on the accuracy of the forecast, that is, distance from actual to mean modeled earnings. A greater share of the PV is paid in cash when the forecast is accurate. This leads to higher PVs for higher absolute earnings results but increases the *portion paid in cash* for accurate forecasting.

Both the ROEC and mean-semivariance approaches to ERRO help provide a portfolio view of a company in terms of risk-reward metrics that resonate with business leaders and the C-suite. This portfolio or "bird's eye" view of a company's risk-reward profile is universally regarded as a key ERM goal. In addition to the worthwhile "destination" of optimization of risk and reward at the enterprise level, it must be realized that the journey itself is rewarding. The benefits of the discussions, analysis and discoveries that come along the way cannot be overstated.

Appendix A. The Company's Use of the Correlation Method of Section 3.2

The Company conducts risk and forecasting analysis to determine the C* and the Cholesky decomposition C* = PP^T:

C*

	X1	*X2*	*X3*
X1	1	0.5	0.4
X2	0.5	1	0.6
X3	0.4	0.6	1

P

1	0	0
0.5	0.866	0
0.4	0.4619	0.7916

P'

1	0.5	0.4
0	0.866	0.4619
0	0	0.7916

This example assumes for illustrative purposes that we desire 50 simulations, that is, N = 50. Now the Company follows the six steps of Section 3.2. Steps 1–3 produce the matrices M, M* and R as:

M			**M***			**R**		
0.14	0.32	0.88	45	35	4	1.19	0.49	−1.42
0.90	0.65	0.43	5	22	31	−1.29	−0.17	0.27
0.89	0.05	0.00	6	48	50	−1.19	1.56	2.06
0.33	0.37	0.02	33	31	49	0.38	0.27	1.76
0.75	0.95	0.81	14	3	7	−0.60	−1.56	−1.09
0.28	0.96	0.89	36	2	3	0.54	−1.76	−1.56
0.47	0.16	0.72	23	42	13	−0.12	0.93	−0.66
0.87	0.87	0.40	8	7	34	−1.01	−1.09	0.43
0.61	0.25	0.30	20	38	39	−0.27	0.66	0.72
0.63	0.26	0.53	19	37	24	−0.33	0.6	−0.07
0.33	0.85	0.78	31	8	8	0.27	−1.01	−1.01
0.88	0.28	0.98	7	36	1	−1.09	0.54	−2.06
0.32	0.57	0.73	34	24	11	0.43	−0.07	−0.79
0.30	0.54	0.53	35	26	25	0.49	0.02	−0.02
0.04	0.00	0.12	50	49	46	2.06	1.76	1.29
0.83	0.83	0.67	10	9	16	−0.86	−0.93	−0.49
0.24	0.18	0.48	42	40	27	0.93	0.79	0.07
0.76	0.11	0.44	13	45	30	−0.66	1.19	0.22
0.65	0.14	0.45	18	44	29	−0.38	1.09	0.17
0.08	0.74	0.55	48	16	23	1.56	−0.49	−0.12
1.00	0.75	0.45	1	15	28	−2.06	−0.54	0.12
0.13	0.00	0.72	46	50	14	1.29	2.06	−0.60
0.22	0.91	0.74	43	6	10	1.01	−1.19	−0.86
0.39	0.77	0.32	27	12	36	0.07	−0.72	0.54
0.98	0.82	0.57	2	10	21	−1.76	−0.86	−0.22
0.66	0.54	0.21	16	27	44	−0.49	0.07	1.09
0.36	0.64	0.73	29	23	12	0.17	−0.12	−0.72
0.38	0.66	0.84	28	20	5	0.12	−0.27	−1.29
0.87	0.32	0.25	9	34	40	−0.93	0.43	0.79
0.26	0.81	0.64	39	11	18	0.72	−0.79	−0.38
0.66	0.76	0.42	15	14	33	−0.54	−0.60	0.38
0.10	0.71	0.61	47	18	19	1.42	−0.38	−0.33
0.06	0.09	0.96	49	47	2	1.76	1.42	−1.76
0.96	0.15	0.15	3	43	45	−1.56	1.01	1.19
0.25	0.36	0.58	40	32	20	0.79	0.33	−0.27
0.46	0.19	0.24	25	39	41	−0.02	0.72	0.86
0.27	0.34	0.65	37	33	17	0.60	0.38	−0.43
0.44	0.67	0.74	26	19	9	0.02	−0.33	−0.93
0.26	0.66	0.24	38	21	42	0.66	−0.22	0.93
0.77	0.93	0.02	12	5	48	−0.72	−1.29	1.56
0.94	0.17	0.31	4	41	38	−1.42	0.86	0.66
0.52	0.77	0.82	21	13	6	−0.22	−0.66	−1.19
0.35	0.54	0.21	30	28	43	0.22	0.12	1.01
0.24	0.73	0.36	41	17	35	0.86	-0.43	0.49

0.80	0.10	0.43	11	46	32	−0.79	1.29	0.33
0.66	0.57	0.69	17	25	15	−0.43	−0.02	−0.54
0.17	0.99	0.56	44	1	22	1.09	−2.06	−0.17
0.47	0.95	0.32	22	4	37	−0.17	−1.42	0.60
0.33	0.51	0.52	32	29	26	0.33	0.17	0.02
0.47	0.48	0.04	24	30	47	−0.07	0.22	1.42

Step 4 is performed to arrive at R* and Y:

R*			**Y**		
1.19	1.01	−0.42	6	5	34
−1.29	−0.80	−0.38	46	41	33
−1.19	0.76	1.88	45	6	2
0.38	0.43	1.67	18	15	3
−0.60	−1.65	−1.83	37	50	49
0.54	−1.25	−1.83	15	44	50
−0.12	0.74	−0.14	28	8	29
−1.01	−1.45	−0.57	43	46	37
−0.27	0.43	0.77	31	14	11
−0.33	0.36	0.09	32	18	26
0.27	−0.74	−1.15	20	39	44
−1.09	−0.08	−1.82	44	29	48
0.43	0.15	−0.48	17	23	35
0.49	0.26	0.19	16	20	23
2.06	2.55	2.66	1	1	1
−0.86	−1.23	−1.16	41	42	45
0.93	1.15	0.79	9	4	9
−0.66	0.70	0.46	38	10	16
−0.38	0.76	0.49	33	7	15
1.56	0.36	0.30	3	17	20
−2.06	−1.50	−0.98	50	48	42
1.29	2.43	1.00	5	2	6
1.01	−0.52	−0.82	8	36	40
0.07	−0.59	0.12	24	37	25
−1.76	−1.62	−1.28	49	49	46
−0.49	−0.18	0.70	35	32	12
0.17	−0.02	−0.56	22	28	36
0.12	−0.18	−1.10	23	31	43
−0.93	−0.09	0.45	42	30	17
0.72	−0.32	−0.37	12	35	32
−0.54	−0.79	−0.19	36	40	30
1.42	0.38	0.13	4	16	24
1.76	2.11	−0.04	2	3	28
−1.56	0.09	0.78	48	25	10
0.79	0.67	0.25	11	11	21
−0.02	0.61	1.00	26	13	5
0.60	0.63	0.07	14	12	27
0.02	−0.27	−0.88	25	34	41
0.66	0.14	0.90	13	24	8
−0.72	−1.48	0.35	39	47	18
−1.42	0.03	0.35	47	27	19
−0.22	−0.68	−1.33	30	38	47
0.22	0.22	0.94	21	21	7
0.86	0.05	0.53	10	26	14
−0.79	0.73	0.54	40	9	13
−0.43	−0.24	−0.61	34	33	38
1.09	−1.24	−0.65	7	43	39
−0.17	−1.31	−0.25	29	45	31
0.33	0.31	0.23	19	19	22
−0.07	0.16	1.19	27	22	4

Next the Company performs steps 5 and 6 (using the LOB distributions in Step 6) to arrive at:

W			X			S		
0.94	0.77	0.62	0.87	0.79	0.32	75.12	119.35	58.56
0.47	0.92	0.85	0.16	0.22	0.33	2.30	14.72	60.95
0.32	0.79	0.49	0.17	0.79	0.94	3.27	118.90	225.31
0.84	0.02	0.12	0.61	0.71	0.94	67.91	114.29	225.13
0.51	0.41	0.74	0.33	0.02	0.07	26.48	−26.74	−26.47
0.61	0.28	0.31	0.69	0.11	0.05	69.94	−7.34	−33.56
0.19	0.74	0.07	0.47	0.77	0.49	46.40	117.96	115.27
0.87	0.30	0.81	0.19	0.11	0.26	6.58	−8.68	39.70
0.58	0.48	0.69	0.43	0.73	0.81	40.65	115.44	208.86
0.45	0.49	0.12	0.43	0.67	0.57	39.96	112.32	147.69
0.19	0.11	0.90	0.58	0.24	0.12	64.30	18.20	−7.70
0.87	0.33	0.58	0.19	0.40	0.10	6.01	50.62	−15.04
0.88	0.34	0.85	0.62	0.52	0.31	68.22	74.87	56.98
0.48	0.97	0.26	0.66	0.58	0.62	69.33	88.89	171.45
0.19	0.75	0.64	0.96	0.98	0.97	78.04	130.60	229.14
0.93	0.98	0.15	0.25	0.22	0.12	15.49	14.58	−9.74
0.82	0.71	0.57	0.78	0.92	0.83	72.63	126.79	211.23
0.78	0.39	0.55	0.33	0.76	0.72	25.97	117.38	197.93
0.51	0.58	0.69	0.41	0.78	0.72	37.05	118.26	198.19
0.48	0.04	0.79	0.93	0.68	0.64	77.00	112.77	185.48
0.01	0.11	0.63	0.01	0.09	0.15	−21.56	−12.91	2.44
0.50	0.09	0.12	0.87	0.97	0.88	75.20	130.41	217.12
0.43	0.11	0.62	0.82	0.28	0.19	73.76	26.15	13.63
0.77	0.79	0.88	0.51	0.25	0.57	50.92	20.21	150.91
0.76	0.40	0.79	0.03	0.04	0.12	−15.89	−21.89	−10.29
0.36	0.61	0.94	0.38	0.34	0.79	32.63	38.18	206.27
0.69	0.94	0.72	0.51	0.41	0.27	51.34	52.69	41.37
0.38	0.45	0.36	0.51	0.38	0.15	50.95	45.79	−0.38
0.61	0.28	0.25	0.19	0.39	0.70	6.91	48.04	195.45
0.17	0.22	0.32	0.76	0.28	0.36	72.17	26.93	73.44
0.96	0.23	0.49	0.36	0.23	0.43	29.83	16.45	95.57
0.71	0.67	0.27	0.88	0.70	0.58	75.64	113.90	155.65
0.76	0.77	0.15	0.94	0.94	0.49	77.49	128.49	115.36
0.31	0.38	0.38	0.07	0.48	0.83	−9.59	65.65	210.98
0.40	0.22	0.10	0.77	0.75	0.63	72.25	116.54	179.58
0.51	0.09	0.20	0.48	0.74	0.90	47.45	116.17	219.40
0.66	0.70	0.12	0.71	0.74	0.55	70.64	116.36	138.56
0.25	0.24	0.57	0.50	0.30	0.18	49.47	30.01	12.33
0.08	0.74	0.97	0.76	0.49	0.85	72.09	67.08	213.42
0.33	0.78	0.83	0.32	0.09	0.69	24.83	−12.31	194.29
0.43	0.25	0.70	0.08	0.42	0.69	−8.58	54.98	194.24
0.03	0.17	0.19	0.45	0.25	0.12	42.88	19.05	−10.69
0.56	0.68	0.18	0.56	0.55	0.85	59.91	81.80	213.77
0.07	0.76	0.43	0.77	0.45	0.74	72.31	59.38	200.22
0.41	0.42	0.91	0.31	0.77	0.79	23.58	117.86	205.85
0.77	0.25	0.83	0.40	0.33	0.25	35.64	36.43	34.11
0.33	0.52	0.05	0.84	0.17	0.20	74.36	4.76	17.21
0.46	0.52	0.33	0.46	0.11	0.38	43.77	−7.72	78.86
0.62	0.55	0.72	0.61	0.61	0.62	67.79	98.85	174.13
0.16	0.73	0.94	0.48	0.52	0.91	46.97	75.82	221.59

Illustrative simulated DE results for each LOB, the enterprise, and infusion results are shown in Table 6.

Table 6. Illustrative Simulation Results

	Simulated Annual Distribution Earnings				**Scenario Infusions at:**			
Simulation #	**LOB1**	**LOB2**	**LOB3**	**Enterprise**	**LOB1**	**LOB2**	**LOB3**	**Enterprise**
1	75	119	59	253	0	0	0	0
2	2	15	61	78	0	0	0	0
3	3	119	225	347	0	0	0	0
4	68	114	225	407	0	0	0	0
5	26	−27	−26	−27	0	27	26	27
6	70	−7	−34	29	0	7	34	0
7	46	118	115	279	0	0	0	0
8	7	−9	40	38	0	9	0	0
9	41	115	209	365	0	0	0	0
10	40	112	148	300	0	0	0	0
11	64	18	−8	74	0	0	8	0
12	6	51	−15	42	0	0	15	0
13	68	75	57	200	0	0	0	0
14	69	89	171	329	0	0	0	0
15	78	131	229	438	0	0	0	0
16	15	15	−10	20	0	0	10	0
17	73	127	211	411	0	0	0	0
18	26	117	198	341	0	0	0	0
19	37	118	198	353	0	0	0	0
20	77	113	185	375	0	0	0	0
21	−22	−13	2	−33	22	13	0	33
22	75	130	217	422	0	0	0	0
23	74	26	14	114	0	0	0	0
24	51	20	151	222	0	0	0	0
25	−16	−22	−10	−48	16	22	10	48
26	33	38	206	277	0	0	0	0
27	51	53	41	145	0	0	0	0
28	51	46	0	97	0	0	0	0
29	7	48	195	250	0	0	0	0
30	72	27	73	172	0	0	0	0
31	30	16	96	142	0	0	0	0
32	76	114	156	346	0	0	0	0
33	77	128	115	320	0	0	0	0
34	−10	66	211	267	10	0	0	0
35	72	117	180	369	0	0	0	0
36	47	116	219	382	0	0	0	0
37	71	116	139	326	0	0	0	0
38	49	30	12	91	0	0	0	0
39	72	67	213	352	0	0	0	0
40	25	−12	194	207	0	12	0	0
41	−9	55	194	240	9	0	0	0
42	43	19	−11	51	0	0	11	0
43	60	82	214	356	0	0	0	0
44	72	59	200	331	0	0	0	0
45	24	118	206	348	0	0	0	0
46	36	36	34	106	0	0	0	0
47	74	5	17	96	0	0	0	0
48	44	−8	79	115	0	8	0	0
49	68	99	174	341	0	0	0	0
50	47	76	222	345	0	0	0	0
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

Appendix B. Example of Linear-Fractional Programming (LFP)

We illustrate LFP with the following optimization problem:[16]

Minimize:

$(-2x_1 + 3x_2 - 1.25)/(x_1 + x_2 + 4)$

Subject to:

$-x_1 + x_2 - 2 \leq 0$

$2x_1 + 3x_2 - 14 \leq 0$

$x_1 - x_2 - 5 \leq 0$

$x_1 \geq 0$

$x_2 \geq 0$

As a motivation for the upcoming substitution, note that the objective function can be written as:

$-2x_1/(x_1 + x_2 + 4) + 3x_2/(x_1 + x_2 + 4) - 1.25/(x_1 + x_2 + 4).$

This suggests the substitution $z = 1/(x_1 + x_2 + 4)$. We then define $y_i = x_i z$ for i = 1, 2. To capture these relations in a new problem, we add the equality constraint $y_1 + y_2 + 4z = 1$ (which follows from the definition of z, y_1 and y_2) and the constraint $z \geq 0$.

The original constraints are restated by first multiplying each side by z. For example, the first constraint $-x_1 + x_2 - 2 \leq 0$ is equivalent to $-x_1 z + x_2 z - 2z \leq 0$. This is written as $-y_1 + y_2 - 2z \leq 0$.

The problem then becomes:

Minimize:

$-2y_1 + 3y_2 - 1.25z$

Subject to:

$y_1 + y_2 + 4z = 1$

$-y_1 + y_2 - 2z \leq 0$

$2y_1 + 3y_2 - 14z \leq 0$

$y_1 - y_2 - 5z \leq 0$

[16] Mojtaba Borza, Azmin Sham Rambely and Mansour Saraj, "Solving Linear Fractional Programming Problems with Interval Coefficients in the Objective Function: A New Approach," *Applied Mathematical Sciences* 6, no. 69 (2012): 3443–52. This example is provided in addition to the topic of optimization of "fuzzy" objective functions with so-called "interval coefficients."

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$

Solving this problem (e.g., using software or an online linear programming calculator) shows the minimum occurs when $y_1 = 0.5556$, $y_2 = 0$ and $z = 0.1111$.

The relation $y_i = x_i z$ for i = 1, 2 implies the original problem is minimized when $x_1 = y_1/z = 5$ and $x_2 = 0$ and the objective function takes on the value $(-2 \bullet 5 - 1.25)/(5 + 4) = -1.25$.

Appendix C. Characterizing Long-Only Portfolios

For convenience, Table 7 duplicates material from Section 5.5.

Table 7. Minimum Semivariance Portfolios for Selected Benchmarks

	Minimum Semivariance Portfolio Weights			
B = benchmark	w1	w2	w3	Semivariance
6.75%	9.5%	0.8%	89.7%	0.0067%
7.00%	12.6%	12.3%	75.1%	0.0093%
7.25%	15.6%	24.0%	60.4%	0.0125%
7.50%	19.3%	34.5%	46.2%	0.0161%
7.75%	23.5%	44.2%	32.3%	0.0204%
8.00%	28.3%	52.8%	18.9%	0.0252%
8.25%	33.8%	60.3%	5.9%	0.0308%

In a 1992 paper,[17] Michael Best and Robert Grauer show the weight to asset i in the optimal portfolio on the mean-variance efficient frontier varies linearly with the expected portfolio return. This behavior is also seen in Table 7, which is based on semivariance. Even with some amount of rounding and approximation occurring in the methodology and its calculations, we have the following Pearson correlations and linear regression results for the linear relation *weight = slope • B + y-intercept*:

Table 8. A Linear Relation: Asset Weight and Expected Return (B)

	w1	w2	w3
Correlation with B	0.993	0.997	−1.000
slope	16.050	39.916	−55.966
x-intercept	0.062	0.067	0.083
y-intercept	−1.000	−2.667	4.677

For each weight, we may use the sign of the slope and the value of the x-intercept in Table 8 to specify when each weight is non-negative in the efficient frontier portfolio:

$w_1 \geq 0$ for B in $[0.062 \pm \varepsilon, \infty]$

$w_2 \geq 0$ for B in $[0.067 \pm \varepsilon, \infty]$

$w_3 \geq 0$ for B in $[-\infty, 0.083 \pm \varepsilon]$

Where the ± ε indicates the (lower or upper) endpoint of the interval is approximate based on the particular regression. These relations imply that all portfolio weights are non-negative when the expected return, B, is approximately in [0.067, 0.083]. This estimate proves to be very accurate: It is

[17] Michael Best and Robert Grauer, "Positively Weighted Minimum-Variance Portfolios and the Structure of Asset Expected Returns," *Journal of Financial and Quantitative Analysis* 27, no. 4 (December 1992).

observed that portfolios with short positions occur when $B \leq 0.0673$ or $B > 0.0837$.

Best and Grauer present a more rigorous treatment and show the general case for any number of assets. They explicitly derive the lower and upper bounds of expected portfolio return associated with the long-only portion of the frontier.