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NEW POSSIBILITIES IN GRADUATION

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INTRODUCTION

A RECENT development in calculating equipment is a controlled-sequence machine with unusually large capacity for either temporary or permanent storage of both working factors and instructions. This suggests the possibility of easily producing graduations that are much more elaborate than ever previously attempted and perhaps virtually beyond criticism by accepted standards. The principal operation here described will be a Whittaker-Henderson "B" graduation of the fourth order, that exerts a geometric trend where the data are light, and is explained in Part II.

The present paper will confine itself to the problem of graduating a single column of rates, such as an ultimate column. Select mortality and bivariate graduation will not be considered here. This paper is primarily an attempt to outline clearly, to the man who plans the machine operations, the series of calculating steps required. These would consume an inordinate amount of time by desk calculation, although they would present small difficulty in such programming, which is essentially straightforward with little hesitation or occasion to choose between alternatives. Theory will be introduced to give the reasons for the procedures recommended, but will not be necessary for the supervising operator to master. Actuaries will have no difficulty in writing out, if needed, the expressions for differences of any orders that appear in this paper.

Since this paper is also for the benefit of nonactuarial readers, I should add that the purpose of all the following graduation technique is to approximate, from the irregular series of ratios necessarily characteristic of a limited body of data, to the smooth series of ratios that we expect would characterize an infinite body of data (if there were or could be one) with which the given sample is homogeneous. Obviously, an acceptable process will disturb least any heavily weighted areas of experience, but will strongly support the main trend wherever the data are light and the given series is therefore erratic. The shape of this trend, as far as has ever been observed in mortality data, will be considered later.

Outline

It may be well to outline here the general plan of the program; as it includes subroutines that are sometimes useful separately, perhaps several separate machine programs are best practically:

I. Given the preliminary preparation and storing of the basic data and, if data are in amounts, secondary useful functions such as R_x^2 and l_x described below, we wish first to produce a strongly smoothed series of rates to use in furnishing both the weights and a geometric factor for the more scientific graduation job to follow. The results of this smoothing process may as well be stored for reference parallel with the crude rates, and the weights, computed from them, should be similarly stored. The best geometric factor, estimated by a subroutine of several readily constructed columns, should be stored where it will be available in program II following.

   Since these smoothed results can have other incidental usefulness (in constructing a dividend scale, for example, actuaries are likely to prefer an experience-mortality factor that irons out short-range fluctuations), it is desirable to produce also a schedule exhibiting this column of smoothed rates and the tests that it should satisfy. This schedule may either be printed directly or punched into cards for subsequent printing by another machine. In any case, all the operations so far outlined will be no great tax on the machine's capacity.

II. Retaining in its "memory" any of the above material that will be of use in subsequent operations, and having released any that will not be, the machine is now ready to receive into storage any other constants together with all the instructions that will be needed for a scientific graduating job as sketched in a preceding paragraph. This part of the program may be among the most complicated jobs ever assigned to any machine, although almost certainly feasible.

III. Rigorous comparisons and tests, both arithmetic and statistical, of the entire job II, just above.

IV. Further possibilities: improvement of the smoothing function and approach towards a "law" of mortality.

Data

Depending on the original material, we will have the exposed to risk E_x and the claims θ_x either in lives or policies or in amounts. If in lives or policies, it is necessary to examine the data and eliminate duplicates as discussed in Chapter III of *Actuarial Studies No. 2*, "Construction of Mortality Tables," by R. D. Murphy and P. C. H. Papps. If this is done, the material is in indivisible units and the accepted formulas for weights of observations and for standard deviations are valid.

When, however, the exposed and claims are furnished in dollars, $\$E_x$ and $\$\theta_x$, and mortality rates are wanted by amounts, those formulas require adjustment to take account of the variation in policy sizes.

Of the several discussions of this problem, that of Mr. D. D. Cody (*TASA* XLII, 69) is perhaps at least as simple and clear as any: if at all possible, obtain for a few well-distributed ages the lives exposed l_x, and also, for those ages, the total exposed amount $\$E_x^s$ for each of several conveniently chosen policy classes with central amounts $\$s$. Then clearly

$$\sum_s (\$E_x^s) = \$E_x .$$

Although we need not in practice subdivide l_x, the equality

$$\sum_s (l_x^s) = l_x$$

and the approximation $l_x^s \cdot \$s = \$E_x^s$ are also valid and may make clearer the relation of these functions.

With such data for any age we can derive a factor R_x^2 whose multiplication into the average size of policy produces an "equivalent average amount" that allows for variation by size when computing weights or standard deviations; alternatively, division of l_x (the number of lives exposed at age x) by R_x^2 will give an adjusted number of exposed (l'_x) with similar corrective effect. Find then

$$R_x^2 \doteq \frac{l_x \cdot \sum_s (\$s \cdot \$E_x^s)}{(\$E_x)^2} . \tag{1}$$

In a large investigation it would probably be practicable to use sampling techniques to ascertain the amount of each exposed class $\$E_x^s$ corresponding to its central amount $\$s$.

Having found R_x^2 for a few sample ages as above, fit to it a straight line or a second-order series by age. If it is not possible to obtain data for calculating this, perhaps a permissible substitute would be to accept the results for the fairly typical distribution given in Mr. Cody's actuarial note cited above, that

$$R_x = 1.59 \quad \text{and} \quad 1.84$$

$$R_x^2 = 2.53 \quad \text{and} \quad 3.39$$

throughout for amount limits of $25,000 and $50,000 respectively. It is much better, of course, to obtain R_x^2 for the material under study, by formula (1) above.

The few pivotal values recommended for this function R_x^2 do not adapt it to controlled-sequence calculation; the function is noted here only as sometimes a practicable help for adequate preparation and testing of a graduation of data by amounts. In such a case, primary data cards should carry not only the exposed and claims, but also l_x and the above R_x^2 for each age in the range to be graduated.

I. PRELIMINARY STRONG SMOOTHING PROCESS

A. Theory

The Initial Smoothing Process here proposed is generally called a Whittaker-Henderson Formula A graduation of the second order, and by many would be considered altogether sufficient for final graduation. With the working coefficients to be recommended below, however, it quickly produces a series that emphasizes smoothness rather than fidelity. Mathematically this process produces a smoothed series q_x^a such that at every age x from $x = a + 2$ to $x = \omega - 2$,

$$q_x'' = q_x^a + g_2 \cdot \delta^4 q_x^a \,, \tag{2}$$

wherein $\delta^4 q_x^a$ is the fourth central difference of the smoothed series at age x.

The Basic Working Principle is to build up an intermediate series v_x, related by compensating extrapolations to the crude q_x'' and to the smoothed q_x^a series. In effect, each q_x'' value lies on a second-order extrapolation from three successive v_x values, and each v_x on a similar extrapolation in the opposite direction from three successive q_x^a values, making the displacement (c_1 in subscripts below) of opposite sign for the two extrapolations:

$$q_x'' \doteq v_{x-c_1} \doteq q_x^a \,. \tag{3}$$

This second-order process uses Newton's forward-difference formula in each extrapolation. By the first,

$$q_x'' = v_{x-c_1} = v_x + (-c_1) \cdot \Delta v_x + c_2 \cdot \Delta^2 v_x \,, \tag{4}$$

wherein $c_2 = \frac{1}{2}(-c_1)(-c_1 - 1)$. By the second extrapolation, turning the formula around,

$$v_x = q_{x+c_1}^a = q_x^a + c_1 \cdot \Delta q_{x-1}^a + c_2 \cdot \Delta^2 q_{x-2}^a \,. \tag{5}$$

If the series q_x'' were a second-order binomial throughout, v_x and q_x^a would also be, and q_x'' would equal q_x^a exactly. No series as smooth as that would ever be submitted for graduation, of course, but the smoother the series submitted, the more nearly exact would be the approximations of formula (3) and the closer would each q_x^a value be to its q_x'', both in (2) and in (3).

Initial Values: Evidently the above extrapolation-relations are not by themselves sufficient; to calculate v_ω we must have not only q_ω'' but also $v_{\omega+1}$ and $v_{\omega+2}$; given those, indeed, we can by (4) work all the way back to v_a. Thereupon, using this v_x column and, if we find them, acceptable values of q^a_{a-2} and q^a_{a-1}, it is possible by (5) to work the q^a_x column also.

Finding suitable values for $v_{\omega+1}$ and $v_{\omega+2}$ depends, first, on a self-correcting feature of the working formula: if a series is of reasonable length and we assign even roughly suitable starting values, repeated application of the working formula all along the column will tend to "wash out" any initial error in a succession of decreasing plus and minus ripples and produce practically correct final terms.

On this principle we construct an auxiliary series $q'''_x \doteq q''_{x-c_1}$, starting (for mortality data) at the younger-age end $x = a$ where the arithmetical scale of error is likely to be small, as are the rates themselves. First, we extrapolate on a smooth curve fitted to a few initial values of the crude q''_x series, either graphically or (better) by formulas for a second-order curve given in the practical directions to follow, by which we can find

$$q'''_{a-2} \doteq q''_{a-2-c_1} \qquad \text{and} \qquad q'''_{a-1} \doteq q''_{a-1-c_1}\,.$$

Using these, we work down the q'''_x column to $x = \omega$ by the relation

$$q''_x \doteq q'''_{x+c_1} = q'''_x + c_1\Delta q'''_{x-1} + c_2\Delta^2 q'''_{x-2}\,. \tag{6}$$

This, repeated all the way down the column, will as hinted usually produce practically correct q'''_x values at the ω-end.

Another convenient feature of this simple second-order Whittaker-Henderson "A" formula: second differences vanish at and beyond either end and the related first differences will not only be constant but, since the functions q'''_x, v_x, and q^a_x are all on the same scale of magnitude with q''_x, first differences at the ends will also be equal between them:

$$\begin{aligned}\Delta q'''_{\omega-1} &= \Delta q'''_{\omega} = \Delta q'''_{\omega+1} \\ &= \Delta v_{\omega-1} = \Delta v_{\omega} = \Delta v_{\omega+1}\end{aligned} \quad \text{and} \quad \begin{aligned}\Delta v_a &= \Delta v_{a-1} = \Delta v_{a-2} \\ &= \Delta q^a_a = \Delta q^a_{a-1} = \Delta q^a_{a-2}\,.\end{aligned} \tag{7}$$

Thus we have by (3) and (7) the required starting values for v_x and q^a_x:

$$\begin{aligned} v_{\omega+1} &= q'''_{\omega} + (2c_1 + 1)\cdot\Delta q'''_{\omega-1} \\ v_{\omega+2} &= q'''_{\omega} + (2c_1 + 2)\cdot\Delta q'''_{\omega-1} \end{aligned} \tag{8}$$

and

$$q^a_{a-2} = v_a - (c_1 + 2)\cdot\Delta v_a \qquad \text{and} \qquad q^a_{a-1} = v_a - (c_1 + 1)\cdot\Delta v_a\,. \tag{9}$$

Testing q^a_x: Usually this is all the essential procedure for computing the q^a_x column; however, it can happen, with extremely short or refractory series, that even perfect arithmetical work will not quite produce the correct q^a_x column: the initial error in the q'''_x column can, though rather seldom, survive even into the q^a_x column. The accepted tests, of equal corresponding q_x'' and q_x^a values extrapolated beyond the ω-end, or of vanishing second differences of the q_x^a series there, will not be convenient for our present programming method.

It will be much better to form outside error-values $\epsilon'_{\omega+1}$ and $\epsilon'_{\omega+2}$ directly from the four q_x^a values ending at $x = \omega + 2$: if both these error-values are less than a half-unit in the last digit to be retained and their difference is less than a quarter of a unit, accept the q_x^a column as correct; otherwise work the ϵ'_x column upwards by repetition of the formula

$$0 = \epsilon'_x - c_1 \cdot \Delta\epsilon'_x + c_2 \cdot \Delta^2\epsilon'_x , \tag{10}$$

thrown into working-form, of course.

Expressions for these error-values follow from the considerations (i) that as the terminal second, and therefore also third, differences of the q_x^a series should vanish, they must be, when they do not vanish, equal to corresponding differences of the error-series: that is,

$$\Delta^2 q^a_{\omega-1} = \Delta^2\epsilon'_{\omega-1} \qquad \text{and} \qquad \Delta^3 q^a_{\omega-1} = \Delta^3\epsilon'_{\omega-1} ; \tag{11}$$

and (ii) that the "washing-out" tendency of the work-process (10) so rapidly reduces the error extrapolation towards evanescence that it is safely equated to zero as in (10). Differencing (10) we have

$$0 = \Delta\epsilon'_x - c_1 \cdot \Delta^2\epsilon'_x + c_2 \cdot \Delta^3\epsilon'_x . \tag{12}$$

Between (11) and (12), taking $x = \omega - 1$ in the latter, we obtain $\Delta\epsilon'_{\omega-1}$; thence by (10) both $\epsilon'_{\omega-1}$ and ϵ'_ω. Upon reaching $\epsilon'_{\alpha-2}$ by repeatedly applying (10), the entire q_x^a column should be replaced by

$$\text{Corrected } q_x^a = \text{Original } q_x^a - \epsilon'_x . \tag{13}$$

For completeness of treatment, the working directions to follow will take account of the very, very infrequent cases requiring ϵ''_x or even ϵ'''_x series.

B. Practical Program: The Smoothing Process

Preliminary Operations

It is obviously advantageous to standardize storage arrangements at the outset; therefore allow, for all entries that vary with age, several parallel columns accommodating, if possible, 125 ages ($x = -4$ to $x = +120$) each, as explained in the practical section B of Part II of this paper. Into such columns of addresses store the exposed E_x, the claims θ_x, and also, if those data are by amounts, the functions R_x^2 and l_x, found previously by separate calculation. Other such columns of addresses, obviously, should be blank and available for working purposes later.

Assign still other addresses in the "memory" to the instructions required by the following technique and the constant coefficients they involve, and to the limiting ages a and ω of the series to be smoothed, and assign one also to the g_z value (or its reciprocal) that will be used in the actual graduation job to be detailed in Part II. As this is written, the best value of g_z has not been determined for the fourth-order process recommended below, but a very few trial-and-error experiments should establish it. For a first trial, $g_4 = 2 \times 10^8$ is suggested. (If a third-order process were tried, $g_3 = 2 \times 10^7$ would be suggested; $g_2 = 1.5 \times 10^6$ has already proven satisfactory for the second-order process used in graduating the *Ga*-1951 Table.)

The next step is to calculate and store into one of the parallel address-columns mentioned above, the crude mortality rates $q_x'' = \theta_x \div E_x$, or $\$\theta_x \div \$E_x$. Here a minor decision is necessary: how many decimals to retain? The best answer is, only as many as we require in the graduated results; the q_x'' values so rounded off should (for reasons to appear later) be stored with two ciphers affixed for the $g_2 = 60$ smoothing process below, but with three such annexed ciphers for the $g_2 = 588$ process. Then, throughout the following calculation work, keep all figures until the correct smoothed q_x^a series is produced; those results should then be rounded off to the required number of decimals.

If the data are in amounts, the "adjusted average" number of lives, $l_x' = l_x \div R_x^2$, should also be worked out and stored.

Recommended Smoothing Processes

i. With graduating coefficient $g_2 = 60$

Other coefficients: $c_1 = 3$; $c_2 = 6$

Working constants: $s = 0.1$

$p_1 = 1.5$

$p_2 = 0.6$

ii. With graduating coefficient $g_2 = 588$

Other coefficients: $c_1 = 6$; $c_2 = 21$

Working constants: $s = \frac{1}{28} = 0.0357143$*

$p_1 = \frac{12}{7} = 1.7142857$*

$p_2 = \frac{3}{4} = 0.75$

The first of these should be nearly always practically strong enough; if the data form a long series of very erratic values, the second powerful one may be preferred. Formulas of lesser and intermediate

* All the digits shown are, of course, not usually necessary, but they cause the machine no particular difficulty.

strengths are listed on pages 45–47 of Camp's manual, *The Whittaker-Henderson Graduation Processes.*

(The above working constant s has, of course, nothing to do with the symbol $\$s$ for central amounts of policy classes, used near the beginning of this paper in connection with data by amounts. Formulas (26) and (28) in the Appendix express g_2 and these working constants in terms of the c's; the relation of c_2 to c_1 appears immediately after formula (4) above.)

Arithmetical Operations

(*a*) *General Procedure* (consult (*b*) below for formulas for outside initial and terminal values)

From initial values q'''_{a-2} and q'''_{a-1} work the q'''_x series downward by formula (6) rearranged for working:

$$s \cdot q''_x + p_1 \cdot q'''_{x-1} - p_2 \cdot q'''_{x-2} = q'''_x \,. \tag{6'}$$

Store each result as soon as computed into an address paralleling that of the corresponding q''_x. Upon reaching q'''_ω, calculate approximate values of $v_{\omega+1}$ and $v_{\omega+2}$, and store them in the same column just beyond q'''_ω.

Work the v_x series upward from $v_{\omega+1}$ and $v_{\omega+2}$ by formula (4) rearranged for working:

$$s \cdot q''_x + p_1 \cdot v_{x+1} - p_2 \cdot v_{x+2} = v_x \,. \tag{4'}$$

Store each v_x as soon as computed into the address of the corresponding q'''_x, now no longer needed. Upon reaching v_a, calculate and store just above it (thus replacing the corresponding q''' values) q^a_{a-1} and q^a_{a-2}.

Work the q^a_x series downward by relation (5) rearranged for working:

$$s \cdot v_x + p_1 \cdot q^a_{x-1} - p_2 \cdot q^a_{x-2} = q^a_x \,. \tag{5'}$$

Store each q^a_x as soon as computed into the address of the corresponding v_x, now no longer needed. Upon reaching $q^a_{\omega+2}$, compute $\epsilon'_{\omega+1}$ and $\epsilon'_{\omega+2}$, storing them in another column parallel with $q^a_{\omega+1}$ and $q^a_{\omega+2}$ respectively.

If both of these outside errors are less than half a unit in the *last decimal to be retained* and their difference is less than a quarter of a unit, then the q^a_x series already found is almost certainly correct; we should round it off and proceed to produce the geometric factor, the weights, and the final comparison-schedule for this smoothed series. If not, work the oscillating ϵ'_x series upward by relation (10) rearranged,

$$p_1 \cdot \epsilon'_{x+1} - p_2 \cdot \epsilon'_{x+2} = \epsilon'_x \,. \qquad (10')$$

Store each such result ϵ'_x into an address parallel to that of the q^a_x value it will adjust. Upon reaching ϵ'_{a-2}, replace the entire series of q^a_x entries by their corrected values:

$$\text{First-corrected } q^a_x = \text{Original } q^a_x - \epsilon'_x \,. \qquad (13')$$

Compute, for this new series, ϵ''_{a-2} and ϵ''_{a-1}; if (as is practically always the case) both are within half a unit in the last decimal to be retained and their difference is less than a quarter of a unit, this corrected q^a_x series is acceptable; if not, compute a new error series downwards,

$$p_1 \cdot \epsilon''_{x-1} - p_2 \cdot \epsilon''_{x-2} = \epsilon''_x \,, \qquad (10'')$$

and store each ϵ''_x result into the address, paralleling that of first-corrected q^a_x, previously occupied by ϵ'_x.
Upon reaching $\epsilon''_{\omega+2}$, replace entire q^a_x series by

$$\text{Second-corrected } q^a_x = \text{First-corrected } q^a_x - \epsilon''_x \,. \qquad (13'')$$

Compute then $\epsilon'''_{\omega+2}$ and $\epsilon'''_{\omega+1}$ by the same formulas as those for $\epsilon'_{\omega+2}$ and $\epsilon'_{\omega+1}$, apply the half-unit test and, if not satisfied, work the ϵ'''_x series upwards by formula (10′) above. Theoretically it could happen to be necessary to seesaw up and down this way indefinitely, but in practice even a first error series is seldom needed and very seldom really needs to be completed (it is mechanically simpler, of course, to complete an oscillating series than to ascertain and stop just where it falls permanently below half a unit).

(*b*) *Outside Values for Initiating the Working Series above* q'''_x series:

i. For the $g_2 = 60$ work-process with $c_1 = 3$, it is probably better than graphic approximation to fit a second-order curve q'_x to four q''_x values at the α-end, for which

The coefficients	Of q''_α	Of $q''_{\alpha+1}$	Of $q''_{\alpha+2}$	Of $q''_{\alpha+3}$
In q'_α are:	+ .95	+ .15	− .15	+ .05
And in $\Delta q'_\alpha$ are:	− .8	+ .4	+ .6	− .2
Then in $q'''_{\alpha-2} \doteq q'_\alpha - (2 + c_1)\Delta q'_\alpha$ they are:	+4.95	−1.85	−3.15	+1.05
And in $q'''_{\alpha-1} \doteq q'_\alpha - (1 + c_1)\Delta q'_\alpha$ they are:	+4.15	−1.45	−2.55	+0.85

ii. For the very strong $g_2 = 588$ work-process with $c_1 = 6$, it would be better to use the second-order curve best fitting five q''_x values, for which

The coefficients

	Of q''_a	Of q''_{a+1}	Of q''_{a+2}	Of q''_{a+3}	Of q''_{a+4}
In q'_a are:	+ 6.2/7	+1.8/7	− 0.6/7	− 1.0/7	+ 0.6/7
And in $\Delta q'_a$ are:	− 4.4/7	+0.8/7	+ 3.0/7	+2.2/7	− 1.6/7
Then in $q'''_{a-2} \doteq q'_a - (2+c_1)\Delta q'_a$ they are:	+41.4/7	−4.6/7	−24.6/7	−18.6/7	+13.4/7
And in $q'''_{a-1} \doteq q'_a - (1+c_1)\Delta q'_a$ they are:	+37.0/7	−3.8/7	−21.6/7	−16.4/7	+11.8/7

v_x series; by formula (3), $v_x \doteq q'''_{x+2c_1}$:

i. For $g_2 = 60$	ii. For $g_2 = 588$
$v_{\omega+1} = 8q'''_\omega - 7q'''_{\omega-1}$	$v_{\omega+1} = 14q'''_\omega - 13q'''_{\omega-1}$
$v_{\omega+2} = 9q'''_\omega - 8q'''_{\omega-1}$	$v_{\omega+2} = 15q'''_\omega - 14q'''_{\omega-1}$

q^a_x series; by formula (3), $q^a_x \doteq v_{x-c_1}$:

$q^a_{a-2} = 6v_a - 5v_{a+1}$	$q^a_{a-2} = 9v_a - 8v_{a+1}$
$q^a_{a-1} = 5v_a - 4v_{a+1}$	$q^a_{a-1} = 8v_a - 7v_{a+1}$

ϵ'_x series (and ϵ'''_x and all error-series of odd order): [From formulas (10), (11), and (12)]

	The coefficients	Of $q^a_{\omega-1}$	Of q^a_ω	Of $q^a_{\omega+1}$	Of $q^a_{\omega+2}$
i. For $g_2 = 60$	In $\epsilon'_{\omega+1}$ are:	+ 40	−110	+100	− 30
	And in $\epsilon'_{\omega+2}$ are:	+ 50	−135	+120	− 35
ii. For $g_2 = 588$	In $\epsilon'_{\omega+1}$ are:	+196	−560	+532	−168
	And in $\epsilon'_{\omega+2}$ are:	+224	−636	+600	−188

ϵ''_x series (and ϵ''''_x and all error-series of even order):

	The coefficients	Of q^a_{a-2}	Of q^a_{a-1}	Of q^a_a	Of q^a_{a+1}
i. For $g_2 = 60$	In ϵ''_{a-2} are:	− 35	+120	−135	+ 50
	And in ϵ''_{a-1} are:	− 30	+100	−110	+ 40
ii. For $g_2 = 588$	In ϵ''_{a-2} are:	−188	+600	−636	+224
	And in ϵ''_{a-1} are:	−168	+532	−560	+196

It is apparent, from the size of the numerical coefficients above, why it is advisable to initiate the $g_2 = 60$ process with two ciphers affixed to the decimals to be retained, and the $g_2 = 588$ process with three. Otherwise the decimals dropped in working would often make errors appear much larger than they are.

From the operations and formulas of subsections (*a*) and (*b*) just above, we should have the correct smoothed q_x^a series. Its primary purpose is to produce the geometric factor and the weights discussed in the next subsections.

(*c*) *The Geometric Factor*

There is probably no theoretically best way to go about finding this factor, but a good approximation should be had by ascertaining either that ten-year or that fifteen-year succession of ages for which the ratio $(q_{x+1}^a \div q_x^a)$ varies within the narrowest limits, and then taking the average value for that short series as r^2, the geometric factor we seek. The following subroutine for this should be easily within the machine's capacity:

i. For all values of x, calculate the ratio $(q_{x+1}^a \div q_x^a)$ and store parallel with q_x^a.

ii. Beginning at *each* age x, scan the 10 (or the 15) ages following it for the least value of that ratio. Store that least value for each age x in a following column parallel with q_x^a.

iii. Beginning at *each* age x, scan the 10 (or the 15) ages following, for the greatest value, and store similarly in another column.

iv. For each age x, determine the spread between these results, and store similarly.

v. Scan entire last column for that age for which that spread is least, and store that age for use in the next instruction. The columns created in steps ii, iii, and iv may be cleared now, for all practical purposes.

vi. Beginning with the age found in (v), add together the 10 (or the 15) values for $(q_{x+1}^a \div q_x^a)$, and divide the sum by 10 (or by 15). This result is the geometric factor sought, r^2, and its value to two or three decimals (further refinement is useless) should be stored for computing the working constants of the graduation process to follow. Store also its reciprocal, r^{-2}, but to several more decimals. It may be of use to note that r^2 corresponds and is close in value to the best Makeham c-constant and, like it, seems to be always less for a male experience than for a similar female experience.

(*d*) *The Weights, to Be Used in the Actual Graduation by Formula B in Part II*

By the graduation process to follow, it will be convenient to incorporate the coefficient g_z into the weights-formula thus:

$$W'_x = (E_x \text{ in lives}) \div [g_z \cdot q^a_x (1 - q^a_x)] \,, \tag{14}$$

where, if we have E_x in dollars but not in lives, we use the adjusted value of exposed l'_x, *i.e.*, the "adjusted average" number of lives, as:

$$E_x \text{ in lives} = l_x \div R^2_x \,. \tag{15}$$

The column W'_x should be worked and stored in parallel with the other columns of data and of results so far completed. (Try first, $g_4 = 2 \times 10^8$.)

C. Comparison-Schedule for the Smoothing Process

As already indicated, the purpose of this process is to find an acceptable geometric factor and a smooth weights-series. Nevertheless its possible incidental usefulness for other purposes makes it worth while to produce a printed schedule showing in parallel columns the several kinds of comparison-data that actuaries appreciate:

The crude rates q''_x
The smoothed series q^a_x
The series of ratios $q^a_{x+1} \div q^a_x$
The deviations $(q^a_x - q''_x)$ with their algebraic signs
The first sums of the latter, with signs, from age a to each age x to and including $x = \omega$
The second sums, similarly scheduled
The deviations above scheduled, divided by the g_2 coefficient used (here either 60 or 588)
The fourth central differences of q^a_x, with signs

$$\delta^4 q^a_x = q^a_{x+2} - 4\,q^a_{x+1} + 6\,q^a_x - 4\,q^a_{x-1} + q^a_{x-2}$$

For any age x, the entries in the last two columns should be equal but of opposite signs, as expressed in formula (2) of this paper. Due to the extreme smoothness of the processes recommended, there will usually not be many changes of sign in the column of deviations, but the column of first sums should show about half that many, and the second sums about one-fourth as many. In both of the sums-columns, the entries opposite $x = \omega$ should be negligible or small. All this demonstrates correctness of the arithmetical work with the process, not suitability as a graduation. Nevertheless columns of deviations and their sums by 5-year age groups may be desired by some actuaries and should in that case be produced.

Some actuaries may also prefer to have additional comparison-columns as follows:

Actual claims θ_x, from original data
Expected claims $E_x \cdot q_x^a$
Discrepancies between last two columns, with signs
First sums of the latter
Also discrepancies and sums by 5-year age groups

If the additional columns just suggested are within the machine's capacity to calculate, they should be produced also. The instructions to handle all the manipulations of these sections B and C of Part I are very numerous by this time, however, and it may be that these additional columns must be produced by an auxiliary program.

II. THE GRADUATION PROCESS: WHITTAKER-HENDERSON FORMULA B

A. Theory

An ideal graduation would produce that series of rates q_x that by statistical standards best reconciles its deviations from the crude rates q_x'' with its own variation of the general law of mortality—that is, if there exists such a thing as an acceptable mathematical expression for that law involving only a few constants that differ merely between different particular experiences. There was a time when actuaries supposed that Makeham's law met these requirements, through the adult ages at least; but continually improved mortality since that day, even more than improved statistical standards, has demonstrated that Makeham's law in simpler form cannot be depended on to fit such data below some age in the 60's or 70's. Nevertheless most investigations that include considerable data in and beyond the 70's exhibit a pronounced geometric tendency there, conforming to that extent with Makeham.

As comparison and reflection will show, every graduation method (except the graphic) relates mortality in some way to a mathematical expression, whether by the direct curve-fitting of Makeham's law or by producing a graduated series constrained towards vanishing differences of an assigned order, and thus towards a polynomial of the next lower order: this is the effect of the Whittaker-Henderson process as usually applied (and recommended for reasons of simplicity in the smoothing process of Part I above).

But other trends can be prescribed with the Whittaker-Henderson process; in particular, as Mr. Henderson himself pointed out long ago, it is possible to frame a smoothing function that constrains the results, or an assigned order of their differences, towards a geometric progression. This will be recommended for graduation by desk workers in the Appendix to this paper and in more elaborate form is also proposed for the process of this Part II.

Such a function will reflect the generally observed geometric trend at high ages where Makeham's law is appropriate and where data are most likely to be scanty if the experience to be graduated is small. If this geometric trend is prescribed for a difference-order that is negligible at the younger ages, the smoothing function so produced will almost certainly be close to the best form we can suggest. Therefore, in the most general expression for the B-formula:

$$\sum_{x=a}^{\omega} W_x (q_x - q_x'')^2 + g_z \cdot \sum_{a+z/2}^{\omega - z/2} (\delta^z q_x)^2 + g_{z-1} \cdot \sum_{a+(z-1)/2}^{\omega-(z-1)/2} (\delta^{z-1} q_x)^2 + \ldots = \text{minimum}\,, \tag{16}$$

let us require that third differences be constrained towards a geometrical progression with constant ratio r^2:

$$\sum_{x=a}^{\omega} W_x (q_x - q_x'')^2 + g_4 \cdot \sum_{a+3/2}^{\omega-3/2} (r\,\delta^3 q_{x-1/2} - r^{-1} \delta^3 q_{x+1/2})^2 = \text{minimum}\,. \tag{17}$$

Following Mr. Henderson's analysis, we differentiate this with respect to each q_x value sought, and obtain $(\omega - a + 1)$ independent equations, the majority of the form

$$W_x (q_x - q_x'') + g_4 [\delta^8 q_x - (r - r^{-1})^2 \cdot \delta^6 q_x] = 0\,, \tag{18}$$

the exceptions being four equations at each end of the series, whose different form is inferable from what follows. Meanwhile, as previously suggested, working is simplified by dividing all equations throughout by g_4; the majority then read

$$W_x' (q_x - q_x'') + [\delta^8 q_x - (r - r^{-1})^2 \cdot \delta^6 q_x] = 0\,, \tag{19}$$

with a similar simplification at the ends of the series; W_x' is the function derived by formula (14).

The working factors call for some comment: (1) the best g_4 value, as previously noted, is not known at this writing but will be easily established with a very few successive trials of the following technique, and when found it should serve equally well with this B-process for experiences of enormously greater or less volume; (2) the numerical coefficients within each ${}_xR_n$ factor (to be defined below) add up to the same total as for that factor in any other fourth-order Formula B; (3) W_x and r^2 depend on the data; if the data include both male and female experience, the r^2 for the male section has been always less, and in all cases the r^2 seems to be close to the best value for a Makeham c-constant.

The basic working principle is to build up an intermediate series that we may as well call v_x as we did the similar series in the A-process of Section I. But W_x is not unity throughout as in that process, so that the "displacement" (c_1 in Part I) varies from age to age down the series as do the working factors that are therefore much more numerous.

Taking ${}_xR_n$,* ${}_xP_n$ and ${}_xp_n$ below as zero for $n > 4$, for $x < a$ and for $x > \omega - n$, equation (19) above may be written

$$W'_x q''_x = {}_{x-4}R_4 \cdot q_{x-4} - {}_{x-3}R_3 \cdot q_{x-3} + {}_{x-2}R_2 \cdot q_{x-2} - {}_{x-1}R_1 \cdot q_{x-1} + {}_xR_0 \cdot q_x - {}_xR_1 \cdot q_{x+1} + {}_xR_2 \cdot q_{x+2} - {}_xR_3 \cdot q_{x+3} + {}_xR_4 \cdot q_{x+4}\,, \tag{20}$$

wherein ${}_xR_n$ is as below (note that r^2 and r^{-2} are the only powers of r involved):

x	${}_xR_4$	${}_xR_3$	${}_xR_2$	${}_xR_1$	${}_xR_0$
$a-4<x<a$	0	0	0	0	0
a	1	$3+r^2$	$3+3r^2$	$1+3r^2$	$r^2+W'_a$
$a+1$	1	$6+r^2+r^{-2}$	$13+6r^2+3r^{-2}$	$13+12r^2+3r^{-2}$	$6+10r^2+r^{-2}+W'_{a+1}$
$a+2$	1	$6+r^2+r^{-2}$	$16+6r^2+6r^{-2}$	$25+15r^2+12r^{-2}$	$24+19r^2+10r^{-2}+W'_{a+2}$
$a+3$	1	$6+r^2+r^{-2}$	$16+6r^2+6r^{-2}$	$26+15r^2+15r^{-2}$	$30+20r^2+19r^{-2}+W'_{a+3}$
$a+3<x<\omega-3$	1	$6+r^2+r^{-2}$	$16+6r^2+6r^{-2}$	$26+15r^2+15r^{-2}$	$30+20r^2+20r^{-2}+W'_x$
$\omega-3$	0	$3\quad+r^{-2}$	$13+3r^2+6r^{-2}$	$25+12r^2+15r^{-2}$	$30+19r^2+20r^{-2}+W'_{\omega-3}$
$\omega-2$	0	0	$3\quad+3r^{-2}$	$13+3r^2+12r^{-2}$	$24+10r^2+19r^{-2}+W'_{\omega-2}$
$\omega-1$	0	0	0	$1\quad+3r^{-2}$	$6+r^2+10r^{-2}+W'_{\omega-1}$
ω	0	0	0	0	$r^{-2}+W'_\omega$

In the solution, the $(\omega - a + 1)$ equations of form (20) are replaced by an equally long series of equations of form (23) below with ${}_xP_n$ coefficients, and another series of that length, of form (24) with ${}_xp_n$ coefficients. All of these coefficients for any particular antescript x are derived at one time from ${}_xR_n$ and several preceding coefficient values (some of which may vanish) by equations (21) and (22):

$${}_xP_n = {}_xR_n - {}_{x-1}P_{n+1} \cdot {}_{x-1}p_1 - {}_{x-2}P_{n+2} \cdot {}_{x-2}p_2 - {}_{x-3}P_{n+3} \cdot {}_{x-3}p_3 - {}_{x-4}P_{n+4} \cdot {}_{x-4}p_4 \tag{21}$$

$${}_xp_n = {}_xP_n \div {}_xP_0\,; \text{ there is no } {}_xp_0\,. \tag{22}$$

* No connection with Mr. Cody's $R_x^!$ factor previously discussed.

Note that the previously mentioned zero values for all these functions beyond the margins of their schedules (identical in extent with that for ${}_xR_n$ above), simplifies their derivation: thus, for all n, ${}_aP_n = {}_aR_n$ by equation (21); for $x = a + 1$, $a + 2$, etc., ${}_xP_n$ is expressed by not more than 2, 3, etc., terms respectively. Also ${}_xP_4 = {}_xR_4$ for all x; usually, ${}_xP_n$ equals the corresponding ${}_xR_n$ less the sum of $(4 - n)$ products of previously-computed ${}_xP$ and ${}_xp$ coefficients; furthermore the last line, for $x = \omega$, will contain an entry in only one of the nine columns: ${}_\omega P_0$.

The two series of factor equations, (23) worked downwards age by age from $x = \alpha$, and (24) worked upwards similarly from $x = \omega$, are of the forms

$$[W'_x q''_x + {}_{x-1}P_1 \cdot v_{x-1} - {}_{x-2}P_2 \cdot v_{x-2} + {}_{x-3}P_3 \cdot v_{x-3} - {}_{x-4}P_4 \cdot v_{x-4}] \div {}_xP_0 = v_x \tag{23}$$

$$v_x + {}_xp_1 \cdot q_{x+1} - {}_xp_2 \cdot q_{x+2} + {}_xp_3 \cdot q_{x+3} - {}_xp_4 \cdot q_{x+4} = q_x \tag{24}$$

Solution of these equation-series also is simpler by reason of the vanishing of outer coefficient-values; thus the leading equation of type (23) is merely $(W'_a q''_a \div {}_aP_0) = v_a$ and the next three equations of that series contain respectively only two, three and four terms in the numerator. Equation-series (24) begins with $v_\omega = q_\omega$ and the next few member-equations of its series include an increasing number of terms until the constant total of five terms is reached.

The q_x series thus arrived at in equation-series (24) is the objective of this entire program, and is practically certain to satisfy every test to be scheduled in Part III hereinafter, once a few trials of this program have established the best g_4 value to incorporate into the weights-function W'_x. That value should work with practically any experience.

B. Practical Operations: The Graduation Process

Storage

In planning this work, it is worth while to consider what may be the longest series we are ever likely to want to graduate: the maximum length of a human life. Infant mortality may, even in the near future, become practically continuous with that for later childhood; its present rate is enormously improved over even the recent past. At the opposite extreme there appears to be at least one reliable record of a human life almost attaining 114 years of age. The work-processes for equations (21), (23) and (24) will apply without change through the whole length of any series under study, if zero values are stored in the addresses of ${}_xR_n$, ${}_xP_n$ and ${}_xp_n$ functions that would be four ages beyond the x and n limits cited in the previous section. It follows that the ideal length of working column for graduating the entire known extent of life would run from $x = -4$ through zero to at least $x = 114 + 4$: say, 125 ages for good measure. As ten working columns will be needed altogether, probably most machines now in production will not have enough additional storage space for all the necessary instructions, and some compromise—preferably at the older age end—with the ideal column-length will be necessary.

Another compromise with perfection will probably also be unavoidable at present: punched cards will have to serve as part of the machine's "memory" in carrying forward such original data and such results from the smoothing process of Part I as are needed for the graduation program of this Part II and later in the comparisons and tests of Part III to follow. Perhaps no machine now on the market will carry considerable data in storage, accept into storage a large body of new instructions, and clear an arena of blank addresses in the rest of its "memory" to provide work-columns for this involved graduation process. After complete clearing, then, load in the instructions and constants involved with them, and also, from cards, the crude mortality rates q_x'' with their limiting ages α and ω (required for control purposes) as well as, from the results of Part I, the weights function W_x' and the geometric factor r^2 with its reciprocal r^{-2}.

Calculation

$W_x' q_x''$: The first calculating step in this graduation program illustrates, in almost too elementary a fashion, how the instructions can sometimes be simplified at the immaterial cost of somewhat lengthening the calculation time. From the columns q_x'' and W_x' compute the product-column $W_x' q_x''$, storing each such result in place of its q_x'', no longer needed in this Part II. As the limiting ages α and ω for a particular experience under study should usually fall within the column-length assigned for data in the machine, there will usually be blank entries for ages below α or above ω. Obviously it will make no difference in the product-column, and will save two or three minor instruction steps at the cost of only slightly longer calculating time, if the machine computes all the zero values of $W_x' q_x''$ as well as those actually required. There will be much more useful illustrations of this principle later.

${}_xR_n$: One complete column (or n-value) at a time is probably the best practical program; this takes advantage of the (ordinarily) long range ($x = \alpha + 4$ to $\omega - 4$ inclusive) for which its value (or at least its expression, with ${}_xR_0$) does not change but for which the values must nevertheless be separately stored despite the repetitions. Storage space will be economized, however, if each ${}_xR_0$, when calculated, is stored in place of the W_x' it involves, which will not be used again in this Part II. Note that every ${}_xR_n$ column is confined to the range $x = \alpha$ to $x = \omega - n$. There should be only zero values for ${}_xR_n$ beyond this range.

${}_xP_n$, ${}_xp_n$: By formulas (21) and (22), these are inextricably related with each other and with preceding values of the same as far as four lines back. All values of both for any line x must be worked and stored complete before proceeding to line $x + 1$. Two economies are practically imperative: storage is economized if each ${}_xP_n$ value on completion is stored in the address occupied by the corresponding ${}_xR_n$ value, which thereafter is no longer needed; the instruction-routine is shortened if there are four lines of zero entries stored beyond each end of the possible range to be graduated (as already suggested in discussing storage). This permits a single instruction-routine to work every line for ${}_xP_n$, ${}_xp_n$ beginning at $x = 0$; such a routine must include a test that will reject division if ${}_xP_0 = 0$ and in that case automatically initiate the next line. At the end of the maximum column length, the v_x routine must begin as below.

At the completion of this ${}_xP_n$, ${}_xp_n$ routine, the maximum storage space is being utilized: a column of $W'_xq''_x$ that may extend for the possible length of life; and five columns of ${}_xP_n$ and four of ${}_xp_n$, all nine of which should accommodate four additional lines of zero entries at each end beyond that length.

v_x: Worked towards high ages by formula (23). It simplifies this also to work by a single uniform routine from age $x = 0$ to the maximum age accommodated; again, ${}_xP_0$ must be tested and not used as divisor when it is zero. Each v_x value when computed may as well be stored into the address of the corresponding ${}_xP_0$ value; upon completion of this v_x column there is no further use for the ${}_xP_n$ schedule.

q_x: Worked towards low ages by formula (24). Store each result into the address of the v_x made useless by its completion. Again, the instruction-routine is simplified by working the entire column, this time from the maximum age accommodated back to age zero. Zero ${}_xp_n$ factors, however, do not affect correctness of this particular formula.

The final instructions of this Part II program should round off this q_x series, the objective of the entire job, to the number of decimals retained in the original q''_x series, and punch into a card for each age x the values of the corresponding product $W'_xq''_x$ and of this graduated q_x result: these are the only new items of further use, computed by all this Part II program.

It may be necessary to emphasize the necessity for all the blank addresses prescribed to accommodate the ${}_xR_n$ and thereafter the ${}_xP_n$ and ${}_xp_n$ functions: some programmers might imagine that the last-mentioned could, when computed, just as well be stored to replace the considerable body of instructions needed for producing the ${}_xR_n$ schedule—it would not crowd so tightly the storage available for other instructions. Unfortunately, if formulas (21) and (24) are programmed simply, so as to operate without change down the entire column length, values of ${}_xp_n$ outside of the range graduated must be zero at the outset; otherwise unreplaced instructions might be taken for ${}_xp_n$. The only spots in the ${}_xP_n$, ${}_xp_n$ columns that can safely accommodate instructions are ${}_{-2,-3,-4}P_1$; ${}_{-3,-4}P_2$ and ${}_{-4}P_3$ and the six corresponding ${}_xp_n$ addresses.

III. TESTS OF THE GRADUATION

An arithmetical summary of tests and comparisons will probably always be expected by students and reviewers. These tests should be produced on the same electronic calculator and transferred through punched cards to a printed schedule.

In addition to the instructions necessary for the operations below, the machine's "memory" should have in storage appropriately located columns of the functions E_x (in lives), also $\$E_x$ if given; θ_x or $\$\theta_x$; q_x''; W_x'; $W_x'q_x''$; and q_x. The geometric constant for the experience, r^2, or its related constant $(r^2 - 2 + r^{-2})$, should also be stored.

The tests should prove correctness of the arithmetical work and also show that the results satisfy accepted statistical standards.

Columns Needed in the Tests

(1) Central differences of q_x of several even orders,

$$\delta^2 q_x = q_{x+1} - 2\,q_x + q_{x-1}\,.$$

Also $\delta^4 q_x$, $\delta^6 q_x$ and $\delta^8 q_x$, all found by the relation

$$\delta^{2n+2} q_x = \delta^{2n} q_{x+1} - 2\,\delta^{2n} q_x + \delta^{2n} q_{x-1}\,.$$

Also a column of third differences, $\Delta\delta^2 q_x = \delta^2 q_{x+1} - \delta^2 q_x$.

(2) From the $\delta^6 q_x$ and $\delta^8 q_x$ columns, the function

$$\delta^8 q_x - (r^2 - 2 + r^{-2})\,\delta^6 q_x\,.$$

(3) The deviations at each age, $(q_x - q_x'')$; also columns of its first three sums to each age x.

(4) The weighted deviations, $W_x'(q_x - q_x'')$; also with columns of its first three sums.

(5) The function $W_x'(q_x - q_x'') + [\delta^8 q_x - (r^2 - 2 + r^{-2})\delta^6 q_x]$.

(6) The function $(E_x q_x - \theta_x)$, whether in dollars or in lives. Columns of first three sums to each age x. Also a column of $(E_x q_x - \theta_x)$ in 5-year age groups.

(7) For each age, the variance, or $p_x q_x \div (E_x$ in lives).

(8) The square of the deviation $(q_x - q_x'')$ at each age, divided by the variance.

The arithmetical work has been correct if (i) three sums of the weighted deviations to age ω (last item of each of the three sum-columns listed under item (4) above) are zero or negligible and (ii) the function listed in item (5) is zero within the effect of dropped fractions at every age. These arithmetical tests are valid whether the graduation is any good or not—that is, whether or not the g_4 used is anywhere near to the right value.

The columns described in items (4) and (5) should therefore be printed, for inspection purposes.

The results are practically certain to satisfy statistical tests also if the right value was used for the graduating coefficient g_4, since that is almost a natural constant for experiences of enormously different volumes. In this class of tests we may list the following:

Smoothness: The column of second differences, $\delta^2 q_x$, should appear smooth to the most cursory inspection, with few sign changes—never any, probably, beyond age 70. Third differences, $\Delta\delta^2 q_x$, beyond some point in the older ages should behave steadily more and more like a geometric progression (then so will lower difference-orders and ultimately the q_x function itself) as the volume of supporting data decreases; at younger ages they should be small and fluctuate somewhat both in sign and value.

Both the second and third-difference columns should appear in the printed summary of tests.

Deviations:

(1) Of graduated rates from crude: The column of $(q_x - q_x'')$ and of its first, second and third sums to each x should show respectively something like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{16}$ times as many sign changes as there are terms $(\omega - a + 1)$ in the series graduated. Probably the machine can be made to count and record these sign changes; the count would be good to note on the printed schedule of tests and comparisons, under the columns themselves. Although the final sum-values at age ω should not be large, they cannot vanish automatically as do those of the weighted deviations.

(2) Of expected from actual: The function $(E_x q_x - \theta_x)$, whether in dollars or in lives, should be treated and set forth in much the same way.

(3) Of expected from actual in age groups—usually 5-year age groups—is a favorite additional exhibit that in no way overtaxes the machine to calculate and provide for printing.

Pattern of Deviations: This graduation process should produce a pattern of deviations that approximates to the ideal distribution of multiples of the standard deviations. The latter function involves taking square roots, which is feasible enough; the present occasion, however, seems a good opportunity to propose the following equivalent test which is mechanically simpler and theoretically just as valid:

Square the deviation, divide result by the corresponding variance as in item (8) above, and observe whether the ratios conform approximately to the following ideal distribution:

$(\text{Deviation})^2 \div$ Variance	Ideal Proportion
Within 0.2	.3452
" 0.5	.5205
" 1.0	.6827
" 2.0	.8426
" 4.0	.9545
" 6.0	.9858
" 9.0	.9973

In this comparison we should use, of course, p_x and q_x from the actual graduation, not p_x^a and q_x^a from the smoothing process worked in Part I. "E_x in lives" means the "adjusted average" number of lives as described in the introductory section of this paper under "Data."

This last step would also seem feasible to program mechanically and include in a final printed schedule.

IV. FURTHER STUDIES

These would go beyond the province of graduation and inquire more searchingly into the basic function on which to construct the best formula. It will be worth while to discuss briefly the problems involved in this, even if such studies are almost certainly not worth while in the near future.

In the first place, what shape of trend should an ideal smoothing function express? For one thing, an indefinitely increasing element in the rate of mortality (q_x), as recommended above, is somewhat preferable to Makeham's geometric factor in the force of mortality (μ_x), which, at least theoretically, admits a remote possibility of physical immortality, something quite preposterous. If the varying length of newspaper obituary columns reflects a seasonal fluctuation in people's vitality, then annual rates are an essentially better basis for both study and theory anyhow. For another thing, any such function as the probability of death or survival must undoubtedly vary not merely by age but between individuals of the same age; however meticulous the selection exercised, every factor in the true q_x function (if we ever find it) will cluster more or less closely about a mode just as do directly measurable characteristics like height and weight, and the function must be one that expresses this variation.

We can hardly be on the threshold of discovering any such law of mortality now, but our means of study and comparison will improve when we can regard the results by Part II above as only tentative and use equipment of still greater power to find a more complete solution of equation (16): that is, regard all the several g_x values as quantities to be more precisely determined, and the q_x values found as also subject to adjustment (though very slightly).

Suppose that we can, with a proven g_4 constant and the technique described in Parts I and II above, depend on producing a smooth graduated q_x series that is also satisfactory by accepted statistical standards, from any experience large enough to be significant. The g_4, r^2 and W_x factors used, with the q_x results found, are those that produce the minimum by equation (17) above; however, a severer minimum is possible for equation (16), if we find suitable adjustments to all the quantities so far used or obtained, and permit more g-coefficients (for manageability, of lower rather than higher orders) to have other than zero values.

Writing the left side of equation (16) in the form

$$F(g_4; \quad g_3 = (r - r^{-1})^2 g_4; \quad g_2 = 0; \quad g_1 = 0; \quad q_a, q_{a+1}, \ldots q_\omega),$$

we may slightly improve it by substituting into it a new weights-series W_x from the q_x values found, and then by accepted least-squares technique compute the small positive and negative increments ($\delta' g_4$; $\delta' g_3$; $\delta' g_2$; $\delta' g_1$; $\delta' q_a, \ldots \delta' q_\omega$) to each of the quantities just indicated under F, finding a new minimum by differentiating with respect to each of these increments and equating those derivatives to zero. Some of the partial derivatives needed can be assembled from Part III above; others will require complete calculation. Given adequate equipment for such a research, and—some day perhaps—more stabilized mortality conditions, it may develop that the resulting g-coefficients will exhibit, for different experiences, a frequently repeated pattern of relationship and thus eventually suggest a probable "law" of mortality if there is one. Certainly this procedure is more promising than hit-or-miss trials of curve-fitting such as Gompertz', Makeham's or Wittstein's formulas, whatever narrowly practical use the first two have had.

APPENDIX

A really scientific graduation will take account of the relative weights of data at different ages, as in Part II of this paper. However, some workers may have to do the entire graduation job by operations such as those in Part I, on ordinary desk calculating machines.

For reasons developed in Part II, but of even more force with a working formula of as low an order as the second, a graduation of mortality or sickness data should recognize the geometric tendency at higher ages—a feature neglected in the smoothing process of Part I, because that was worked with other purposes in view. Without controlled-sequence equipment, probably the simplest and quickest approximation to the geometric factor r^2 will be as follows: (1) select a succession of several higher ages for which the data are not too scanty; (2) plot their crude q_x'' rates on semi-logarithmic paper; (3) rule a straight line as closely among those points as seems possible; (4) from its slope, estimate the average for those ages of $(q_{x+1}'' \div q_x'')$. This average is nearly the required constant ratio r^2. From it make up $(r - r^{-1})^2 = (r^2 - 2 + r^{-2})$.

When Formula A is used as in this Appendix it is necessary to consider the best value for the graduating coefficient g_2 since this depends on the volume of the experience studied, quite otherwise than with Formula B. The particular processes detailed in Part I, appropriate for strongly smoothing a series, are ordinarily too strong for satisfactory graduation and the choice is further complicated by the geometric factor, which however only slightly disturbs the relation between g_2 and the displacement-value c_1 on which the following algebra is based for simplicity. (c_1, always positive, increases from 1 to 3 as g_2 increases from 3 to 60 respectively.)

Altering equation (2) at the beginning of Part I to express the relation between a crude q_x'' series and a graduated q_x series whose first differences are constrained towards a geometric progression with a constant ratio r^2, we have

$$q_x'' = q_x + g_2 \left[\delta^4 q_x - (r - r^{-1})^2 \delta^2 q_x \right] . \qquad (25)$$

The coefficients of $\delta^4 q_x$ and of $\delta^2 q_x$, in terms of c_1 and c_2, are

$$g_2 = c_2 (1 + c_1 + c_2) \quad \text{and} \quad (r - r^{-1})^2 g_2 = c_1 (1 + c_1) - 2 c_2 . \, (26)$$

Substituting from the first into the second gives us a quadratic in c_2:

$$(r - r^{-1})^2 c_2^2 + \left[(r - r^{-1})^2 (1 + c_1) + 2 \right] c_2 - c_1 (1 + c_1) = 0 . \, (27)$$

The positive root furnishes the c_2 value corresponding to the c_1 assigned in the trial; the working factors are:

$$s = \frac{1}{1 + c_1 + c_2}; \qquad p_1 = \frac{c_1 + 2 c_2}{1 + c_1 + c_2}; \qquad p_2 = \frac{c_2}{1 + c_1 + c_2} . \quad (28)$$

After thus finding c_2 and these working factors that correspond to the assigned trial c_1, the exact g_2 value is given by the first of equations (26) and the arithmetical operations under subsection (*a*) of Part I, Section B, are followed without change. However, the outside starting values required are not the simple first-difference extrapolations by formula (3) that are given in subsection (*b*). Theoretically they should be estimated by outer first differences all in geometric progression with a constant ratio r^2; practically it will be much simpler to approximate by a second order extrapolation from only two consecutive first differences that are in geometric progression. Thus, finding q'_a and $\Delta q'_a$ by the four or five point formulas for fitting a second order curve, in paragraphs i or ii of subsection (*b*) under "Arithmetical Operations," write $\Delta q'_{a-1} = r^{-2}\Delta q'_a$ and therefore $\Delta^2 q'_{a-1} = (1 - r^{-2})\,\Delta q'_a$, with an accumulated error that will probably be negligible in

$$q'''_{a-2} \doteq q'_a - (c_1+2)\,\Delta q'_{a-1} + \tfrac{1}{2}\,(c_1+2)\,(c_1+1)\,\Delta^2 q'_{a-1}$$
$$= q'_a + \tfrac{1}{2}\,(c_1+2)\,[\,(c_1+1) - r^{-2}\,(c_1+3)\,]\,\Delta q'_a$$
$$q'''_{a-1} \doteq q'_a - (c_1+1)\,\Delta q'_{a-1} + \tfrac{1}{2}\,(c_1+1)\;c_1\Delta^2 q'_{a-1}$$
$$= q'_a + \tfrac{1}{2}\,(c_1+1)\,[\,c_1 - r^{-2}\,(c_1+2)\,]\,\Delta q'_a\ .$$

In corresponding fashion, write $\Delta q'''_\omega = r^2\Delta q'''_{\omega-1}$ and $\Delta^2 q'''_\omega = r^2(r^2-1)\Delta q'''_{\omega-1}$ and thence

$$v_{\omega+1} = q'''_\omega + (2\,c_1+1)\,\Delta q'''_\omega + \tfrac{1}{2}\,(2\,c_1+1)\,(2\,c_1)\,\Delta^2 q'''_\omega$$
$$= q'''_\omega + r^2\,(2\,c_1+1)\,[\,1 + c_1\,(r^2-1)\,]\,\Delta q'''_{\omega-1}$$

and $\qquad (8'')$

$$v_{\omega+2} = q'''_\omega + (2\,c_1+2)\,\Delta q'''_\omega + \tfrac{1}{2}\,(2\,c_1+2)\,(2\,c_1+1)\,\Delta^2 q'''_\omega$$
$$= q'''_\omega + r^2\,(c_1+1)\,[\,2 + (2\,c_1+1)\,(r^2-1)\,]\,\Delta q'''_{\omega-1}\ .$$

Also, by the same extrapolation as between q'''_x and q'_x:

$$q_{a-2} = v_a + \tfrac{1}{2}\,(c_1+2)\,[\,(c_1+1) - r^{-2}\,(c_1+3)\,]\,\Delta v_a$$
$$q_{a-1} = v_a + \tfrac{1}{2}\,(c_1+1)\,[\,c_1 - r^{-2}\,(c_1+2)\,]\,\Delta v_a\ . \qquad (9'')$$

As with the formulas recommended for smoothing, it is well to carry an extra decimal or two throughout the work in order to avoid exaggerating the error at and beyond the end of the series. Our clue to the error when the geometric relation should hold at the end of the series is:

$$0 \neq r^2\Delta q_{\omega-1} - \Delta q_\omega = r^2\Delta\epsilon'_{\omega-1} - \Delta\epsilon'_\omega = \Delta\, Y_{\omega-1}$$
$$0 \neq r^2\Delta q_\omega - \Delta q_{\omega+1} = r^2\Delta\epsilon'_\omega - \Delta\epsilon'_{\omega+1} = \Delta\, Y_\omega\ , \qquad (29)$$

wherein $Y_x = r^2\epsilon_x' - \epsilon_{x+1}'$ and from which above equations $\Delta^2 Y_{\omega-1}$ is readily found.

Now the true q_x series, if subtracted from that already obtained, leaves an error series ϵ_x' that may be regarded as second-order extrapolations to value zero:

$$0 = \epsilon_x' - c_1 \cdot \Delta\epsilon_x' + c_2\Delta^2\epsilon_x' \qquad (10''')$$

or, since Y_x is a simple product-sum of ϵ_x',

$$0 = Y_{\omega-1} - c_1\Delta Y_{\omega-1} + c_2\Delta^2 Y_{\omega-1} \,. \qquad (30)$$

Having thus $Y_{\omega-1}$ it is simple to obtain

$$\begin{aligned} Y_\omega &= Y_{\omega-1} + \Delta Y_{\omega-1} = r^2\epsilon_\omega' - \epsilon_{\omega+1}' \\ Y_{\omega+1} &= Y_\omega + \Delta Y_\omega = r^2\epsilon_{\omega+1}' - \epsilon_{\omega+2}' \,. \end{aligned} \qquad (31)$$

Substituting into the foregoing expression for Y_ω the extrapolation-relation

$$\epsilon_\omega' = p_1 \cdot \epsilon_{\omega+1}' - p_2 \cdot \epsilon_{\omega+2}' \qquad (10'''')$$

gives two equations in $\epsilon_{\omega+1}'$ and $\epsilon_{\omega+2}'$, which are much more easily solved by pencil-and-paper arithmetic than by formulating their values completely.

The arithmetical tests for correctness of the work done are those for the smoothness operation as described in Part I, Section C, except that the second sum of deviations will not be negligible. The statistical tests, however, should be more exacting and approach the standards described in Part III, for the Formula B graduation of Part II.