EDUCATION AND EXAMINATION COMMITTEE

OF THE

SOCIETY OF ACTUARIES

COMMUTATION FUNCTIONS

by

Judy Feldman Anderson, FSA

600-99-99 **Printed in U.S.A.**

INTRODUCTION

Before the availability of personal computers and calculators able to perform repeated calculations at incredible speed, actuaries used a collection of functions, called commutation functions, as an easier way to calculate the actuarial present value of contingent payments. These functions are based on a deterministic model of survival and a constant, level rate of return. Given this model, they provided an efficient way of summarizing information for a variety of actuarial calculations.

With advances in technology, the limitations of commutation functions have become more evident. They are not well-suited to decrements with a select period, changing interest rates or irregular patterns of benefit increases over time. These features can be incorporated using basic principles and the rapid calculations provided by computers and programmable calculators. In addition, advances in technology allow actuaries to expand from deterministic to stochastic models and answer questions about the risk due to variability.

We are now in a period of transition. While commutation functions are no longer necessary to simplify calculation, they are still a common way of describing a variety of actuarial calculations and can be found in many computer programs, books and government regulations.

This study note will review commutation functions and how they are used to calculate the values, based on simple deterministic assumptions, of a variety of simple annuity and insurance functions.

THE BASICS FOR ANNUITIES

Using a deterministic model with a given survival function, we define l_x as the number of lives, out of some initial population of newborns, that survive to age x. The value of the survival function $s(x) = l_x/l_0$. It then follows that $p_x = l_{x+1}/l_x$ and ${}_n p_x = l_{x+n}/l_x$. With a constant interest rate i, the actuarial present value of an n-year pure endowment is $v^n \, {}_n p_x$.

Commutation functions were developed to eliminate the need to perform repeated calculations, exponentiation and simultaneously to limit the amount of data to be recorded.

The basic commutation function is $D_x = v^x \, l_x$. Dividing D_{x+n} by D_x gives the actuarial present value of an n-year pure endowment:

$${}_nE_x = A_{x:\overset{\,1}{\overline{n|}}} = \frac{D_{x+n}}{D_x} = \frac{v^{x+n}\, l_{x+n}}{v^x l_x} = v^n \, {}_n p_x$$

For a given mortality table, there were publications providing the values of D for all ages at a variety of fixed interest rates. Calculating an actuarial present value for a lump sum payment contingent on survival could then be reduced to one simple division of two previously recorded quantities.

For annuities of repeated level payments contingent on survival, the commutation function

$N_x = \sum_{t=x}^{\infty} D_t = \sum_{t=x}^{\infty} v^t l_t$ was introduced. When divided by the appropriate D, the values of annuity functions were easy to calculate:

$$\ddot{a}_x = \frac{N_x}{D_x} = \frac{\sum_{t=x}^{\infty} v^t l_t}{v^x l_x} = \sum_{n=0}^{\infty} v^n {}_n p_x$$

Similarly, an increasing annuity can be interpreted as the sum of a series of deferred life annuities. The commutation

function $S_x = \sum_{j=x}^{\infty} N_j = \sum_{j=x}^{\infty} \sum_{t=j}^{\infty} D_t = \sum_{j=x}^{\infty} \sum_{t=j}^{\infty} v^t l_t = \sum_{n=o}^{\infty} (n+1) v^{x+n} l_{x+n}$ can be used to simplify calculations for values of arithimetically increasing or decreasing annual annuities. For example:

$$(I\ddot{a})_x = \frac{S_x}{D_x} = \frac{\sum_{t=x}^{\infty} N_t}{D_x} = \frac{\sum_{t=x}^{\infty} \sum_{t=i}^{\infty} D_t}{D_x} = \frac{\sum_{j=x}^{\infty} \sum_{t=j}^{\infty} v^t l_t}{v^x l_x} = \sum_{j=x}^{\infty} \sum_{n=j-x}^{\infty} v^n {}_n p_x = \sum_{n=o}^{\infty} (n+1) v^n {}_n p_x$$

Other forms of annuities and various identities are described in Appendix A.

THE BASICS FOR LIFE INSURANCES

For life insurances, payable at the end of the year, we use $d_x = l_x - l_{x+1}$ and $q_x = 1 - p_x = d_x / l_x$. Then we define $C_x = v^{x+1} d_x$. For a one-year term insurance:

$$A^{\ 1}_{x:\overline{1|}} = \frac{C_x}{D_x} = \frac{v^{x+1} d_x}{v^x l_x} = v q_x$$

For terms longer than one year, the commutation function $M_x = \sum_{t=x}^{\infty} C_t = \sum_{t=x}^{\infty} v^{t+1} d_t$ and the relationship $d_{x+n} / l_x = {}_{n|}q_x$ are used. For example, a whole life insurance is:

$$A_x = \frac{M_x}{D_x} = \frac{\sum_{t=x}^{\infty} C_t}{D_x} = \frac{\sum_{t=x}^{\infty} v^{t+1} d_t}{v^x l_x} = \sum_{n=0}^{\infty} v^{n+1} {}_{n|}q_x$$

Another commutation function is

$R_x = \sum_{j=x}^{\infty} M_j = \sum_{j=x}^{\infty} \sum_{t=j}^{\infty} C_t = \sum_{j=x}^{\infty} \sum_{t=j}^{\infty} v^{t+1} d_t = \sum_{n=o}^{\infty} (n+1) v^{x+n} d_{x+n}$. This function can be used to simplify manual calculations of arithmetically increasing, or decreasing, insurances. For example:

$$(IA)_x = \frac{R_x}{D_x} = \frac{\sum_{j=x}^{\infty} M_j}{D_x} = \frac{\sum_{j=x}^{\infty} \sum_{t=j}^{\infty} C_t}{D_x} = \frac{\sum_{j=x}^{\infty} \sum_{t=j}^{\infty} v^{t+1} d_t}{v^x l_x} = \sum_{j=x}^{\infty} \sum_{n=j-x}^{\infty} v^{n+1} {}_{n|}q_x = \sum_{n=o}^{\infty} (n+1) v^{n+1} {}_{n|}q_x$$

Other forms of insurances and various identities are described in appendix B.

OTHER CONSIDERATIONS

1.Payments More Frequent than Annually

For payments occurring more frequently than annually, we make use of the approximation $\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$. The commutation function created for use in approximating the actuarial present value of m payments per year, contingent on survival, is $N_x^{(m)} = N_x - D_x \cdot \frac{m-1}{2m}$.

Values for these annuities can be calculated as follows:

$$\ddot{a}_x^{(m)} = \frac{N_x^{(m)}}{D_x} = \frac{N_x - D_x \frac{m-1}{2m}}{D_x} = \ddot{a}_x - \frac{m-1}{2m}$$

2.Benefit Premiums

To calculate benefit premiums, divide the actuarial present value of the benefits by the actuarial present value of a unit of premium. For example, using commutation functions, the calculation of a whole life annual benefit premium would be:

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{M_x / D_x}{N_x / D_x} = \frac{M_x}{N_x}$$

Other premium formulas are included in Appendix C.

3.Salary Scales

To include a salary scale in commutation functions, multiple tables were often published for various combinations of interest rate and salary scale. For commutation functions reflecting a level percentage salary scale assumption, a modified interest rate could be used. Letting i be the interest rate and s be the salary scale assumption, the modified interest rate would be $k = \frac{1+i}{1+s} - 1$.

4.Combining Periods with Different Interest Rates

For situations where different interest rates are applicable to different periods of time, separate sets of commutation functions, for each interest rate period, would be used in discrete blocks. This is identical to the situation where a salary scale is included only during the deferral period for an annuity payable annually on retirement. For example, consider the actuarial present value of an annuity which pays 100% of final pay using an interest rate of i and salary scale of s. The actuarial present value could be calculated as follows:

$$(final\ pay) \cdot {}_{65-x|}\ddot{a}_x = (current\ pay) \cdot \frac{D_{65@k\%}}{D_{x@k\%}} \cdot \frac{N_{65@i\%}}{D_{65@i\%}} = (current\ pay) \cdot \frac{(1+s)^{65-x}}{(1+i)^{65-x}} \cdot {}_{65-x}p_x \cdot \ddot{a}_{65@i\%}$$

$$where,\ k = \frac{1+i}{1+s} - 1$$

The key is to pair each commutation function with another at the same interest rate so that the ratios will reduce to a discount factor and survival probability for the appropriate period.

5.Combining Periods with Different Decrements

Different decrements may aply to different periods of time. For example, death, withdrawal and disability may all be applicable to the pre-retirement period but not post-retirement. Typically, for multiple decrements, an equivalent single decrement table would be formed and commutation functions calculated based on the adjusted single decrement table. As with changes in the interest rate, to reflect decrements effective over a limited period, the commutation functions from each set of assumptions could be paired to properly estimate the appropriate p_x's. For example, let $\hat{D}_x$ be the commutation function for decrements of mortality and withdrawal. The present value of a deferred life annuity payable at age 65 only if the individual does not withdraw before age 65 is:

$$_{65-x|}\ddot{a}_x = \frac{\hat{D}_{65}}{\hat{D}_x} \cdot \frac{N_{65}}{D_{65}}$$

SUMMARY

In summary, calculating the values of simple annuities and insurances is not difficult using the above described commutation functions stored in a table by age for a constant interest rate. However, this assumes that, for the entire period under consideration, there is a constant level interest rate, there is a fixed table of decrements, and the only items to be calculated are actuarial present values. These simplifying assumptions were necessary to facilitate manual calculations.

APPENDIX A
Annuities

Actuarial Present Value

Whole Life Annuity

- Due

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x = \frac{N_x}{D_x}$$

- Immediate

$$a_x = \sum_{k=1}^{\infty} v^k \, {}_kp_x = \frac{N_{x+1}}{D_x}$$

n-Year Temporary Life Annuity

- Due

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x = \frac{N_x - N_{x+n}}{D_x}$$

- Immediate

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_x = \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

n-Year Deferred Whole Life Annuity

- Due

$${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x = \frac{N_{x+n}}{D_x}$$

- Immediate

$${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_kp_x = \frac{N_{x+n+1}}{D_x}$$

n-Year Certain and Whole Life Annuity

- Due

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k \, {}_kp_x = \ddot{a}_{\overline{n}|} + \frac{N_{x+n}}{D_x}$$

- Immediate

$$a_{\overline{x:\overline{n}|}} = a_{\overline{n}|} + \sum_{k=n+1}^{\infty} v^k \, {}_kp_x = a_{\overline{n}|} + \frac{N_{x+n+1}}{D_x}$$

Additional Relationships:

$$\ddot{a}_{x:\overline{n|}} = \frac{N_x - N_{x+n}}{D_x} = \frac{D_x + N_{x+1} - N_{x+n}}{D_x} = 1 + a_{x:\overline{n-1|}}$$

$$_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x} = \frac{N_x - N_x + N_{x+n}}{D_x} = \ddot{a}_x - \ddot{a}_{x:\overline{n|}}$$

$$\ddot{a}_{\overline{\overline{x:n|}}} = \ddot{a}_{\overline{n|}} + \frac{N_{x+n}}{D_x} = \ddot{a}_{\overline{n|}} + \frac{N_x - N_x + N_{x+n}}{D_x} = \ddot{a}_{\overline{n|}} + \ddot{a}_x - \ddot{a}_{x:\overline{n|}}$$

APPENDIX B
Insurances Payable At End Of Year Of Death

Actuarial Present Value

- Whole life insurance

$$A_x = \sum_{t=0}^{\infty} v^{t+1}\ {}_{t|}q_x = \frac{M_x}{D_x}$$

- n-Year term insurance

$$A^{\ 1}_{x:\overline{n|}} = \sum_{t=0}^{n-1} v^{t+1}\ {}_{t|}q_x = \frac{M_x - M_{x+n}}{D_x}$$

- n-Year endowment insurance

$$A_{x:\overline{n|}} = \sum_{t=0}^{n-1} v^{t+1}\ {}_{t|}q_x + v^n\ {}_n p_x = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

- n-Year deferred whole life insurance

$${}_{n|}A_x = \sum_{t=n}^{\infty} v^{t+1}\ {}_{t|}q_x = \frac{M_{x+n}}{D_x}$$

Additional Relationships:

$$C_x = v^{x+1} d_x = v^{x+1}(l_x - l_{x+1}) = v D_x - D_{x+1}$$

$$M_x = \sum_{t=0}^{\infty} c_{x+t} = \sum_{t=0}^{\infty} v D_{x+t} - \sum_{t=0}^{\infty} D_{x+1+t} = v N_x - N_{x+1}$$

$$A_x = \frac{M_x}{D_x} = \frac{v N_x - N_{x+1}}{D_x} = v\ \ddot{a}_x - a_x$$

$$d\ \ddot{a}_x + A_x = d\ \ddot{a}_x + v\ \ddot{a}_x - a_x = \ddot{a}_x - a_x = 1$$

$$A^{\;1}_{x:\,\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} = \frac{v\,N_x - N_{x+1} - v\,N_{x+n} + N_{x+n+1}}{D_x} =$$

$$\frac{v(N_x - N_{x+n}) - (N_{x+1} - N_{x+n+1})}{D_x} = v\;\ddot{a}_{x:\,\overline{n}|} - a_{x:\,\overline{n}|}$$

$$A_{x:\,\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} = \frac{v\,N_x - N_{x+1} - v\,N_{x+n} + N_{x+n+1} + D_{x+n}}{D_x} =$$

$$\frac{v(N_x - N_{x+n}) - (N_{x+1} - N_{x+n})}{D_x} = v\;\ddot{a}_{x:\,\overline{n}|} - a_{x:\,\overline{n-1}|}$$

$$d\,\ddot{a}_{x:\,\overline{n}|} + A_{x:\,\overline{n}|} = d\,\ddot{a}_{x:\,\overline{n}|} + v\,\ddot{a}_{x:\,\overline{n}|} - a_{x:\,\overline{n-1}|} = \ddot{a}_{x:\,\overline{n}|} - a_{x:\,\overline{n-1}|} = 1$$

APPENDIX C
Annual Benefit Premiums

	Annual Benefit Premium
Whole life insurance	$P_x = \dfrac{A_x}{\ddot{a}_x} = \dfrac{M_x}{N_x}$
n-Year term insurance	$P^{\,1}_{x:\overline{n}\vert} = \dfrac{A^{\,1}_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}} = \dfrac{M_x - M_{x+n}}{N_x - N_{x+n}}$
n-Year endowment insurance	$P_{x:\overline{n}\vert} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}} = \dfrac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}$
h-Payment whole life insurance	${}_hP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\vert}} = \dfrac{M_x}{N_x - N_{x+h}}$
h-Payment, n-year endowment insurance	${}_hP_{x:\overline{n}\vert} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}} = \dfrac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+h}}$
n-Year pure endowment	$P_{x:\overline{n}\vert}^{\;\;\,1} = \dfrac{A_{x:\overline{n}\vert}^{\;\;\,1}}{\ddot{a}_{x:\overline{n}\vert}} = \dfrac{D_{x+n}}{N_x - N_{x+n}}$
n-Year deferred whole life annuity	$P({}_{n\vert}\ddot{a}_x) = \dfrac{A_{x:\overline{n}\vert}^{\;\;\,1}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\vert}} = \dfrac{(D_{x+n}/D_x)(N_{x+n}/D_{x+n})}{(N_x - N_{x+n})/D_x} = \dfrac{N_{x+n}}{N_x - N_{x+n}}$