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PREMIUMS AND RESERVES IN MULTIPLE DECREMENT THEORY

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INTRODUCTION

RELATIONSHIPS among premiums and reserves for life insurance are familiar to actuaries. The aim of this paper is to examine analogous relations for the case of a general insurance with benefits payable in respect to several forms of decrement. There may be a number of applications for such theory; a likely one is to the funding of pension benefits.

In Part I, relations are given for the total annual premium and the total reserve. In this part, and throughout the paper, a continuous model is used, and the general technique is to manipulate the differential equation for the function representing the reserve.

In each of the other three parts is presented a system of splitting the total premium and the total reserve into components related to the different benefits. In Part II are given relations for independent premium and reserve components, these being the ones usually considered in practical work. If there are m causes of decrement, and i refers to a particular cause, then the ith independent premium and reserve components are based on all m forces of decrement but, in general, depend only on the ith benefit.

In Part III, a system of *dependent* premium and reserve components is introduced. Here the ith premium does, in general, depend on all benefits, since it is considered that the total reserve, not merely the ith reserve component, will be available to offset the benefit payment when decrement occurs. Dependent premiums and reserves may be useful for cases where benefits are related to the total reserve, as may happen in pension plans with vesting of some or all of the total reserve. Under appropriate circumstances, corresponding independent and dependent components are equal, and conditions for such equality are discussed.

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In Part IV we consider components which we call Loewy premiums and reserves. Alfred Loewy (1873–1935) was a European mathematician who wrote on both pure and actuarial mathematics. His paper "Zur Theorie und Anwendung der Intensitäten in der Versicherungsmathematik" [1]* was influential in developing our general approach to this paper and, in addition, is the origin of the ideas in Part IV. Loewy premiums (reserves) are differences of the set of total premiums (reserves) obtained by incorporating successively, usually one at a time, the various forces of decrement and associated benefits. Such components may be of practical value when consideration is being given to adding a new benefit to an already existing insurance.

In the Appendix there is given a special multiple decrement table used for a pension plan example of the theory, and a summary of the three types of premium and reserve components for this example.

I. TOTAL PREMIUMS AND RESERVES

Outline

In this first part of the paper there is presented the notation and the multiple decrement theory that will be utilized as a basis for subsequent portions. The theory is developed in regard to a general insurance with annual premiums payable continuously and with benefits payable immediately on occurrence of the contingencies involved. Our immediate objective is to establish relationships in regard to the total premium and the total reserve for the general insurance. The starting point is a differential equation for the function representing the total reserve. Various integrations of this equation lead to various formulas for the total premium and corresponding formulas for the total reserve. In a final example, application of the theory is made to a special set of pension provisions.

Notation

We consider a general group of insured persons who are subject to m causes of decrement (1), (2), . . . , (m). For a person who enters this group at age x, the annual rate or force of decrement from cause (i), operating in the instant of attainment of age $x + t$, will be denoted by $\mu_{x+t}^{(i)}$. In general, the forces $\mu_{x+t}^{(i)}$, $i = 1, 2, \ldots, m$, will be on an aggregate rather than a select basis but, as far as the theory is concerned, it will usually be immaterial which basis is the case. It will be assumed that the forces are sufficiently well-behaved functions that the special integrals and functions we shall base on these forces will have determinate values. (The reader who is concerned about this matter is invited to read the full dissertation.)

* Numbers in brackets refer to the list of References at the end of the paper.

The total force of decrement at the instant of attaining age $x + t$, denoted by $\mu_{x+t}^{(T)}$, is defined by the relation

$$\mu_{x+t}^{(T)} = \sum_{i=1}^{m} \mu_{x+t}^{(i)} . \tag{1}$$

There will also be need for the complementary force $\mu_{x+t}^{(-i)}$, where

$$\mu_{x+t}^{(-i)} = \mu_{x+t}^{(T)} - \mu_{x+t}^{(i)} . \tag{2}$$

By now the reader will have observed that we have started with the forces as our initial concept and have yet to define the survivorship and decrement functions commonly used for multiple decrement tables. Our perverseness in disregarding the usual order of defining discrete rates of decrement and other discrete multiple decrement functions and then proceeding to the definition of forces of decrement [2], [3, p. 255] is not without reason. Our plan is to develop the theory on a continuous basis throughout and to bring in discrete functions only when it is necessary for practical calculation purposes. It seemed fitting, then, to begin with annual rates defined on a continuous basis [4, p. 43].

Proceeding now to survivorship and decrement functions, we define $l_{x+t}^{(T)}$ by the relation

$$\frac{dl_{x+t}^{(T)}}{dt} = -l_{x+t}^{(T)} \mu_{x+t}^{(T)} \tag{3}$$

and by assigning a value to $l_x^{(T)}$. There follow the usual relations

$$l_{x+t}^{(T)} = l_x^{(T)} e^{-\int_0^t \mu_{x+s}^{(T)} ds} \tag{4}$$

$$l_x^{(T)} - l_{x+t}^{(T)} = \int_0^t l_{x+s}^{(T)} \mu_{x+s}^{(T)} d s . \tag{5}$$

The decrement functions are defined by

$$d_{x+k}^{(i)} = \int_k^{k+1} l_{x+t}^{(T)} \mu_{x+t}^{(i)} dt \tag{6}$$

$$d_{x+k}^{(T)} = \sum_{i=1}^{m} d_{x+k}^{(i)} , \tag{7}$$

where $x + k$ ranges over the integral ages used in the multiple decrement table.

Related to the multiple decrement table so constructed there is a family of *associated single decrement tables* [5, § 9], [3, p. 257], one for each cause of decrement. For cause (i), the single decrement table has survivorship and decrement functions defined by

$$l_{x+t}^{[i]} = l_x^{[i]} e^{-\int_0^t \mu_{x+s}^{(i)} ds} \tag{8}$$

$$d_{x+k}^{[i]} = \int_k^{k+1} l_{x+t}^{[i]} \mu_{x+t}^{(i)} dt = l_{x+k}^{[i]} - l_{x+k+1}^{[i]} . \tag{9}$$

The square brackets in the superscript are being used to denote that only the force $\mu_{x+t}^{(i)}$ is considered to be operating. We will refer to this table as table $[i]$. If $l_x^{(T)}$ is chosen equal to

$$\prod_{i=1}^{m} l_x^{[i]} ,$$

it follows readily from equations (4) and (8) that

$$l_{x+t}^{(T)} = \prod_{i=1}^{m} l_{x+t}^{[i]} . \tag{10}$$

One may, of course, take any subset of the forces $\mu_{x+t}^{(i)}$, $i = 1, 2, \ldots, m$, to define a decrement table based on that subset. Of particular interest is the table based on the complementary force $\mu_{x+t}^{(-i)}$, as defined in equation (2). For this table, denoted by $[-i]$, the survivorship function is determined by the relation

$$l_{x+t}^{[-i]} = l_x^{[-i]} e^{-\int_0^t \mu_{x+s}^{(-i)} ds} . \tag{11}$$

For monetary functions it will be assumed that the force of interest is δ. There will be need for commutation functions such as

$$D_{x+t}^{(T)} , \quad D_{x+t}^{[i]} , \quad D_{x+t}^{[-i]} , \quad \bar{N}_{x+t}^{(T)} , \quad \bar{N}_{x+t}^{[i]} , \quad \bar{N}_{x+t}^{[-i]}$$

and annuity values such as $\bar{a}_{x:\overline{n}|}^{(T)}$, $\bar{s}_{x:\overline{n}|}^{(T)}$, $\bar{a}_{x:\overline{n}|}^{[i]}$ based on δ and the given forces of decrement. For actuarial readers it seems unnecessary to define these in detail; however, full definitions are available in the dissertation.

Having indicated the definitions and notations to be used for multiple decrement and related functions, we turn now to notation for the general n-year insurance to be considered in the following. It will be assumed that the insurance begins at age x, and that the insured is subject to the causes of decrement (1), (2), . . . , (m). In case of termination at age

$x + t$, $t < n$, by reason of cause (i), the benefit $B_{x+t}^{(i)}$ is payable. In the event of survival of the insured to the end of the n-year insurance term, a maturity value, denoted by $B_{x+n}^{(T)}$ is provided. Payments to the insuring organization are assumed to consist of level annual premiums payable momently, denoted by $\bar{P}^{(T)}$. Such a general insurance includes many of the common forms of insurances and also many pension plan benefits and their funding. For example, $\bar{P}^{(T)}$ may be the entry age normal cost for pension benefits $B_{x+t}^{(i)}$ appropriate to the various modes of termination.

For this general n-year insurance, the reserve at duration t will be denoted by ${}_t\bar{V}^{(T)}$ (right hand subscripts being omitted, as has been done already for $\bar{P}^{(T)}$). A systematic exploration of relations for $\bar{P}^{(T)}$ and ${}_t\bar{V}^{(T)}$ is our present objective. It will be found that there is a correspondence between the resulting formulas for $\bar{P}^{(T)}$ and ${}_t\bar{V}^{(T)}$. In general, we shall first establish the various formulas for $\bar{P}^{(T)}$ and then observe the corresponding formulas for ${}_t\bar{V}^{(T)}$.

The Premium-Reserve Equation

Let us suppose $l_x^{(T)}$ persons become covered at age x by the general insurance indicated in the previous section, and consider the total fund at duration t, namely $l_{x+t}^{(T)} \cdot {}_t\bar{V}^{(T)}$. For the rate of change in this total fund, we have

$$\frac{d\,(l_{x+t}^{(T)} \cdot {}_t\bar{V}^{(T)})}{dt} = l_{x+t}^{(T)}\bar{P}^{(T)} + \delta\, l_{x+t}^{(T)} \cdot {}_t\bar{V}^{(T)} - \sum_{i=1}^{m} l_{x+t}^{(T)}\mu_{x+t}^{(i)}B_{x+t}^{(i)}, \quad (1\,2)$$

where the first two terms represent rates of income, and the sum term, the rates of outgo. By differentiating the left member as a product, and rearranging we obtain

$$\frac{d\,{}_t\bar{V}^{(T)}}{dt} = \bar{P}^{(T)} + \delta\,{}_t\bar{V}^{(T)} - \sum_{i=1}^{m} B_{x+t}^{(i)}\mu_{x+t}^{(i)} + {}_t\bar{V}^{(T)}\mu_{x+t}^{(T)}, \qquad (1\,3)$$

which may be regarded as the basic equation for premiums and reserves for the general insurance.

The differential equation, with appropriate preparation, may be integrated in a variety of ways. By carrying through such an integration from 0 to n and assuming ${}_0\bar{V}^{(T)} = 0$ and ${}_n\bar{V}^{(T)}$ equals the given maturity value $B_{x+n}^{(T)}$, we obtain a formula for $\bar{P}^{(T)}$. Then, by similar integrations from 0 to t, and from t to n, the retrospective and prospective formulas for ${}_t\bar{V}^{(T)}$ are obtained. Not all integrations lead to explicit formulas for $\bar{P}^{(T)}$ and ${}_t\bar{V}^{(T)}$, but even where the formula is implicit there may be an interesting and possibly useful interpretation.

For a first integration, we rearrange formula (13) to the form

$$d_t\bar{V}^{(T)} - (\mu_{x+t}^{(T)} + \delta)\, {}_t\bar{V}^{(T)}\, dt = \bar{P}^{(T)}\, dt - \sum_{i=1}^{m} B_{x+t}^{(i)} \mu_{x+t}^{(i)}\, dt$$

and multiply through by $D_{x+t}^{(T)}$ which has as differential $-D_{x+t}^{(T)}(\mu_{x+t}^{(T)} + \delta)dt$. Then

$$d\,({}_t\bar{V}^{(T)} D_{x+t}^{(T)}) = \bar{P}^{(T)} D_{x+t}^{(T)} dt - \sum_{i=1}^{m} B_{x+t}^{(i)} D_{x+t}^{(T)} \mu_{x+t}^{(i)}\, dt,$$

from which it follows that

$$B_{x+n}^{(T)} D_{x+n}^{(T)} = \bar{P}^{(T)}\, (\bar{N}_x^{(T)} - \bar{N}_{x+n}^{(T)}) - \sum_{i=1}^{m} \int_0^n B_{x+t}^{(i)} D_{x+t}^{(T)} \mu_{x+t}^{(i)}\, dt \quad (14)$$

or

$$\bar{P}^{(T)} = \left\{ \sum_{i=1}^{m} \int_0^n B_{x+t}^{(i)} D_{x+t}^{(T)} \mu_{x+t}^{(i)}\, dt + B_{x+n}^{(T)} D_{x+n}^{(T)} \right\} \Big/ \, (\bar{N}_x^{(T)} - \bar{N}_{x+n}^{(T)})\,. \quad (15)$$

Formula (14), divided by $D_x^{(T)}$ and rearranged, would give the usual equation of value between premiums and benefits, while formula (15) is a usual relation for the determination of $\bar{P}^{(T)}$.

By rearranging formula (13) in the form

$$d_t\bar{V}^{(T)} - \delta\, {}_t\bar{V}^{(T)}\, dt = \bar{P}^{(T)}\, dt - \sum_{i=1}^{m} \mu_{x+t}^{(i)}\, (B_{x+t}^{(i)} - {}_t\bar{V}^{(T)})\, dt$$

and multiplying through by v^t, we obtain

$$d\,({}_t\bar{V}^{(T)} v^t) = \bar{P}^{(T)} v^t dt - \sum_{i=1}^{m} v^t \mu_{x+t}^{(i)}\, (B_{x+t}^{(i)} - {}_t\bar{V}^{(T)})\, dt\,. \quad (16)$$

Now, integrating from 0 to n, we get

$$B_{x+n}^{(T)} v^n = \bar{P}^{(T)} \bar{a}_{\overline{n}|} - \sum_{i=1}^{m} \int_0^n v^t \mu_{x+t}^{(i)}\, (B_{x+t}^{(i)} - {}_t\bar{V}^{(T)})\, dt,$$

which yields

$$\bar{P}^{(T)} = \left\{ \sum_{i=1}^{m} \int_0^n v^t \mu_{x+t}^{(i)}\, (B_{x+t}^{(i)} - {}_t\bar{V}^{(T)})\, dt + B_{x+n}^{(T)} v^n \right\} \Big/ \, \bar{a}_{\overline{n}|}\,. \quad (17)$$

In formula (17), the expression $\mu_{x+t}^{(i)}(B_{x+t}^{(i)} - {}_t\bar{V}^{(T)})dt$ represents, in respect to cause (i) of decrement, the momentary cost of insurance based on the net amount at risk, with the total reserve assumed available to offset the sum insured $B_{x+t}^{(i)}$. Formula (17) indicates that $\bar{P}^{(T)}$ is the uniform annual amount payable momently that is equivalent under interest to the momentary net costs of insurance in respect to all causes plus the maturity value $B_{x+n}^{(T)}$.

Other formulas for $\bar{P}^{(T)}$ may be obtained by rearranging formula (13) so that multiplication by $D_{x+t}^{[i]}$ or $D_{x+t}^{[-i]}$ will give a perfect differential in the left member. The details for these rather special formulas will be omitted.

Corresponding to each formula for $\bar{P}^{(T)}$ there will be related formulas for ${}_t\bar{V}^{(T)}$. These may be obtained by changing t to s, say, in formula (13), and then rearranging and applying such integrating factor as was used to obtain the premium formula. By integrating with respect to s from 0 to t one obtains a retrospective formula, and by integrating from t to n, a prospective formula. For example, corresponding to formula (15) for $\bar{P}^{(T)}$, one obtains

$$ {}_t\bar{V}^{(T)} = \left\{ \bar{P}^{(T)} (\bar{N}_x^{(T)} - \bar{N}_{x+t}^{(T)}) - \sum_{i=1}^{m} \int_0^t B_{x+s}^{(i)} D_{x+s}^{(T)} \mu_{x+s}^{(i)} d s \right\} \Big/ D_{x+t}^{(T)} \quad (18) $$

and

$$ {}_t\bar{V}^{(T)} = \left\{ \sum_{i=1}^{m} \int_t^n B_{x+s}^{(i)} D_{x+s}^{(T)} \mu_{x+s}^{(i)} d s + B_{x+n}^{(T)} D_{x+n}^{(T)} - \bar{P}^{(T)} (\bar{N}_{x+t}^{(T)} - \bar{N}_{x+n}^{(T)}) \right\} \Big/ D_{x+t}^{(T)}, \quad (19) $$

the usual retrospective and prospective formulas.

Examples of Total Premiums and Reserves

In this section we give two examples which make application of the preceding theory. The second example is a continuing one which will be used to illustrate additional portions of this paper.

Example 1. Suppose that (x) is subject to two causes of decrement and that he is covered by an n-year, continuous payment insurance which provides, in case he terminates due to cause (1), a level amount B; and in case of termination by cause (2) at age $x+t$, a known multiple λ_t of ${}_t\bar{V}^{(T)}$. It is further assumed that ${}_0\bar{V}^{(T)} = 0$, and that $B_{x+n}^{(T)}$ is some assigned maturity value. For this situation, equation (13) becomes

$$ \frac{d\,{}_t\bar{V}^{(T)}}{dt} = \bar{P}^{(T)} + \delta\,{}_t\bar{V}^{(T)} - B\mu_{x+t}^{(1)} - \lambda_t \cdot {}_t\bar{V}^{(T)} \mu_{x+t}^{(2)} + {}_t\bar{V}^{(T)} \mu_{x+t}^{(T)} . $$

This may be rearranged as

$$ d\,{}_t\bar{V}^{(T)} - [\delta + \mu_{x+t}^{(1)} + (1-\lambda_t)\,\mu_{x+t}^{(2)}]\,{}_t\bar{V}^{(T)} dt = \bar{P}^{(T)} dt - B\mu_{x+t}^{(1)} dt , $$

and as integrating factor we may use $D_{x+t}^{(\lambda T)}$ defined by

$$\frac{dD_{x+t}^{(\lambda T)}}{dt} = -\,[\delta + \mu_{x+t}^{(1)} + (1-\lambda_t)\,\mu_{x+t}^{(2)}]\,D_{x+t}^{(\lambda T)}, \quad D_x^{(\lambda T)} = D_x^{(T)}\,.$$

On integrating from 0 to n, and solving for $\bar{P}^{(T)}$, we obtain

$$\bar{P}^{(T)} = \left\{B\int_0^n D_{x+t}^{(\lambda T)}\,\mu_{x+t}^{(1)}\,dt + B_{x+n}^{(T)} D_{x+n}^{(\lambda T)}\right\} \Big/ \left(\bar{N}_x^{(\lambda T)} - \bar{N}_{x+n}^{(\lambda T)}\right),$$

where $\bar{N}_x^{(\lambda T)}$ is an integral of $D_{x+t}^{(\lambda T)}$. There would be corresponding formulas for ${}_t\bar{V}^{(T)}$.

Example 2. As a second application of the preceding multiple decrement theory, we compute the total premium and total reserves for a person entering at age 32 into a pension plan with the following benefits:

a) In the event of retirement on or after age 65, a life annuity of 1 per year for each year of service; for retirement between ages 60 and 65, the actuarial equivalent of the benefit available at age 65 is provided.

b) In case of disability after 15 years of service, an annuity of 2 per year for each year of service, payable to age 65, at which age the benefit is reduced to 1 per year for each year of service.

c) In the event of termination by death or withdrawal after 10 years of service, the total reserve is provided.

It will be assumed that all annuities are payable on a continuous basis, and that contributions (premiums) are payable continuously throughout service.

The causes of decrement, which will here be indicated by letter superscripts, are mortality, to be denoted by (d), and this operates throughout service; withdrawal, (w), which operates to age 60; disability (h), which operates to age 65; retirement, (r), which begins to operate at age 60 and continues until mandatory retirement in the year of age 68 to 69.

The annual rates of decrement for the given causes are those which form the basis of the UAW 1955 Tables [6]. The independent annual rates will be denoted by $q_x^{[\;]}$, with appropriate superscript. It may be noted that $q_x^{[d]}$ is the rate of mortality from the a–1949 Table [7]. Further, as indicated in the preceding paragraph, $q_x^{[w]} = 0$, for $x \geq 60$; $q_x^{[h]} = 0$ for $x \geq 65$; and $q_x^{[r]} = 0$ for $x < 60$, and $q_{68}^{[r]} = 1$. Interest is at the effective annual rate of 3 percent. Before indicating the construction of the necessary multiple decrement table and functions, we will examine the premium-reserve equation.

Since the entrant will receive no benefit in the event of death, withdrawal, or disability during the first 10 years, and since in this range $\mu^{(r)}_{32+t} = 0$, we have

$$\frac{d_t\overline{V}^{(T)}}{dt} = \overline{P}^{(T)} + (\delta + \mu^{(T)}_{32+t})\,{}_t\overline{V}^{(T)}\,, \qquad 0 \le t < 10\,.$$

After 10 years of service, the equation changes form because of benefits being payable in the event of death or withdrawal, thus

$$\frac{d_t\overline{V}^{(T)}}{dt} = \overline{P}^{(T)} + (\delta + \mu^{(T)}_{32+t})\,{}_t\overline{V}^{(T)} - (\mu^{(d)}_{32+t} + \mu^{(w)}_{32+t})\,{}_t\overline{V}^{(T)}\,, \qquad 10 \le t < 15\,.$$

For the period $15 \le t < 28$, there are disability benefits payable also, and the equation becomes

$$\frac{d_t\overline{V}^{(T)}}{dt} = \overline{P}^{(T)} + (\delta + \mu^{(T)}_{32+t})\,{}_t\overline{V}^{(T)} - (\mu^{(d)}_{32+t} + \mu^{(w)}_{32+t})\,{}_t\overline{V}^{(T)} - \mu^{(h)}_{32+t}B^{(h)}_{32+t}\,,$$

where

$$B^{(h)}_{32+t} = t\,[\bar{a}^{i}_{32+t} + \bar{a}^{i}_{32+t:\overline{33-t|}}]\,,$$

the annuity values being based on a modification of the 1944 Disabled Railway Employees Select Mortality Table [8, p. 18].

After age 60, the rate of withdrawal is assumed to be zero, and for $28 \le t < 33$, we have

$$\frac{d_t\overline{V}^{(T)}}{dt} = \overline{P}^{(T)} + (\delta + \mu^{(T)}_{32+t})\,{}_t\overline{V}^{(T)} - \mu^{(d)}_{32+t}\cdot{}_t\overline{V}^{(T)} - \mu^{(h)}_{32+t}B^{(h)}_{32+t} - \mu^{(r)}_{32+t}B^{(r)}_{32+t}\,,$$

where $B^{(r)}_{32+t} = t\overline{N}_{65}/D_{32+t}$, with $\overline{N}_{65}$ and D_{32+t} based on the a-1949 Table.

Finally, when $33 \le t < 37$, $\mu^{(h)}_{32+t}$ is zero, and the equation becomes

$$\frac{d_t\overline{V}^{(T)}}{dt} = \overline{P}^{(T)} + (\delta + \mu^{(T)}_{32+t})\,{}_t\overline{V}^{(T)} - \mu^{(d)}_{32+t}\cdot{}_t\overline{V}^{(T)} - \mu^{(r)}_{32+t}B^{(r)}_{32+t}\,,$$

where now $B^{(r)}_{32+t} = t\bar{a}_{32+t}$, with $\bar{a}_{32+t}$ based on the a-1949 Table.

These equations could be integrated, interval by interval, but the process can be visualized in one step by collecting the equations in the form

$$d_t\overline{V}^{(T)} - (\delta + \mu^{(T')}_{32+t})\,{}_t\overline{V}^{(T)}\,dt = (\overline{P}^{(T)} - \mu^{(h)}_{32+t}B^{(h)}_{32+t} - \mu^{(r)}_{32+t}B^{(r)}_{32+t})\,dt\,, \tag{20}$$

where

$$\mu_{32+t}^{(T')} = \begin{cases} \mu_{32+t}^{(d)} + \mu_{32+t}^{(w)} + \mu_{32+t}^{(h)}\,, & 0 \le t < 10\,, \\ \mu_{32+t}^{(h)} \,, & 10 \le t < 28\,, \\ \mu_{32+t}^{(h)} + \mu_{32+t}^{(r)} \,, & 28 \le t < 33\,, \\ \mu_{32+t}^{(r)} \,, & 33 \le t < 37 \end{cases}$$

and

$$B_{32+t}^{(h)} = 0\,,\ 0 \le t < 15\,, \quad \text{and} \quad \mu_{32+t}^{(h)} = 0\,,\ 33 \le t < 37$$
$$\text{and} \quad \mu_{32+t}^{(r)} = 0\,,\ 0 \le t < 28\,.$$

For this purpose a special multiple decrement table, with survivorship based on $\mu_{32+t}^{(T')}$ was developed, the functions for this table being denoted with a prime on the superscript. In this table, the survivorship function $l_{32+t}^{(T')}$, for $0 \le t < 10$, is based on the forces of death, withdrawal and disability; for $10 \le t < 28$, on the single force of disability; for $28 \le t < 33$, on the forces of disability and retirement; and in the final interval, on the force of retirement only. After $l_{32+t}^{(T')}$ has been so obtained, $l_{69}^{(T')}$ being zero, we proceed to get values of $D_{32+t}^{(T')}$ and

$$\bar{N}_{32+t}^{(T')} = \int_t^{37} D_{32+s}^{(T')} ds\,.$$

Then, by multiplying equation (20) by $D_{32+t}^{(T')}$, we have

$$d\,({}_t\bar{V}^{(T)} D_{32+t}^{(T')}) = \bar{P}^{(T)} D_{32+t}^{(T')} dt - B_{32+t}^{(h)} D_{32+t}^{(T')} \mu_{32+t}^{(h)} dt - B_{32+t}^{(r)} D_{32+t}^{(T')} \mu_{32+t}^{(r)} dt\,.$$

On integrating, and using ${}_0\bar{V}^{(T)} = D_{69}^{(T')} = 0$, and the above conditions on $B_{32+t}^{(h)}$, $\mu_{32+t}^{(h)}$ and $\mu_{32+t}^{(r)}$, we obtain

$$\bar{P}^{(T)} = \left\{ \int_{15}^{33} B_{32+t}^{(h)} D_{32+t}^{(T')} \mu_{32+t}^{(h)} dt + \int_{28}^{37} B_{32+t}^{(r)} D_{32+t}^{(T')} \mu_{32+t}^{(r)} dt \right\} \Big/ \bar{N}_{32}^{(T')}\,. \quad (21)$$

Similarly, for ${}_t\bar{V}^{(T)}$, we get

$${}_t\bar{V}^{(T)} = \left\{ \int_t^{33} B_{32+s}^{(h)} D_{32+s}^{(T')} \mu_{32+s}^{(h)} ds + \int_t^{37} B_{32+s}^{(r)} D_{32+s}^{(T')} \mu_{32+s}^{(r)} ds - \bar{P}^{(T)} \bar{N}_{32+t}^{(T')} \right\} \Big/ D_{32+t}^{(T')}. \quad (22)$$

To compute $l_{32+t}^{(T')}$, and the other necessary multiple decrement functions, we converted the independent rates $q_y^{[i]}$ into dependent rates, or probabilities, by the general relation

$$q_y^{(i)} \doteq q_y^{[i]}\left[1-\frac{1}{2}\sum_{j\neq i} q_y^{[j]}\right]. \tag{23}$$

(This may be established by setting

$$\begin{aligned} q_y^{(i)} &= \frac{1}{l_y^{(T)}}\int_0^1 l_{y+h}^{(T)}\mu_{y+h}^{(i)}dh \\ &= \frac{1}{l_y^{(T)}}\int_0^1 \left(\prod_{j\neq i} l_{y+h}^{[j]}\right) l_{y+h}^{[i]}\mu_{y+h}^{(i)}dh \end{aligned}$$

and assuming uniform distribution of decrement in the year of age for each of the tables $[i]$, so that $l_{y+h}^{[j]} = l_y^{[j]} - hd_{y+h}^{[j]}$, and $l_{y+h}^{[i]}\mu_{y+h}^{(i)}dh = d_y^{[i]}dh$.) In applying formula (23), the composition of $\mu_{32+t}^{(T')}$ for each interval must be observed. For example

$$q_{32+k}^{(h')} = \begin{cases} q_{32+k}^{[h]}\,[1-\frac{1}{2}\,(q_{32+k}^{[d]}+q_{32+k}^{[w]})\,]\,, & 0\leq k\leq 9\,, \\ q_{32+k}^{[h]} &, \quad 10\leq k\leq 27 \\ q_{32+k}^{[h]}\,[1-\frac{1}{2}\,q_{32+k}^{[r]}] &, \quad 28\leq k\leq 32\,, \end{cases}$$

where, under the uniform distribution assumption, all but the first relation is exact. In the final year of age, $q_{68}^{[r]} = 1$ and for this year it is assumed that $l_{68+h}^{[r]} = l_{68}^{[r]}(1-h)$.

To compute the value of integrals such as

$$\int_t^{33} B_{32+s}^{(h)}\mathrm{D}_{32+s}^{(T')}\mu_{32+s}^{(h)}d\,s\,,$$

appropriate commutation functions $\overline{\mathrm{C}}_{32+k}^{(h'B)}$, $\overline{\mathrm{M}}_{32+k}^{(h'B)}$ were defined by the relations

$$\overline{\mathrm{C}}_{32+k}^{(h'B)} = v^{32+k+1/2}B_{32+k+1/2}^{(h)}d_{32+k}^{(h')} \doteq \int_k^{k+1} B_{32+s}^{(h)}\,v^{32+s}l_{32+s}^{(T')}\mu_{32+s}^{(h)}d\,s\,,$$

$$\overline{\mathrm{M}}_{32+k}^{(h'B)} = \sum_{p=k}^{32}\overline{\mathrm{C}}_{32+p}^{(h'B)}\,.$$

Then, equations (21), (22), adapted for computation, are

$$\overline{\mathrm{P}}^{(T)} \doteq \{\overline{\mathrm{M}}_{47}^{(h'B)}+\overline{\mathrm{M}}_{60}^{(r'B)}\}\,/\,\overline{\mathrm{N}}_{32}^{(T')} \tag{24}$$

$${}_t\overline{\mathrm{V}}^{(T)} \doteq \{\overline{\mathrm{M}}_{32+t}^{(h'B)}+\overline{\mathrm{M}}_{32+t}^{(r'B)}-\overline{\mathrm{P}}^{(T)}\overline{\mathrm{N}}_{32+t}^{(T')}\}\,/\,\mathrm{D}_{32+t}^{(T')}. \tag{25}$$

The computed value of $\bar{P}^{(T)}$ is 5.92; selected values of ${}_t\bar{V}^{(T)}$ are given in Table 2 of the Appendix.

II. INDEPENDENT PREMIUMS AND RESERVES

Decomposition of the Total Premium and the Total Reserve

Having developed relations for the total premium and the total reserve for the n-year general insurance, we turn to the problem of splitting the total premium and total reserve into components corresponding to the individual causes of decrement. In practical work such component premiums and reserves are often determined by separate calculations for the individual benefits and then combined to give the total premium and total reserve. However, by looking at the matter as a problem of decomposition, one observes that for the distribution of the total quantities into components there exist several logical bases. In this part we shall discuss the component premiums and reserves that are usually assigned in respect to the various causes of decrement.

A component premium assigned to the ith cause will be denoted by $\bar{P}^{(i)}$ and the corresponding reserve by ${}_t\bar{V}^{(i)}$. Whatever the basis of assignment, we shall require that

$$\sum_{i=1}^{m} \bar{P}^{(i)} = \bar{P}^{(T)} \qquad (1)$$

$$\sum_{i=1}^{m} {}_t\bar{V}^{(i)} = {}_t\bar{V}^{(T)}, \qquad 0 \le t \le n . \qquad (2)$$

We shall assume that for $t = 0$ reserves are zero, and that the maturity value $B_{x+n}^{(T)}$ is distributed into components $B_{x+n}^{(i)}$, $i = 1, 2, \ldots, m$, and that these components will remain fixed throughout the discussion of the various ways of splitting $\bar{P}^{(T)}$ and ${}_t\bar{V}^{(T)}$. In practical problems there is usually some natural way of assigning the maturity components.

From the mathematical point of view, it is clear that conditions (1) and (2) could be satisfied by an indefinite number of sets of components $\bar{P}^{(i)}$ and ${}_t\bar{V}^{(i)}$. What we are interested in are those sets of components which have some direct and meaningful relationship to the m causes of decrement and the corresponding benefits.

In general, our starting point will be the premium-reserve equation (13) of Part I. For the component premiums and reserves to be discussed here and to be denoted by ${}^a\bar{P}^{(i)}$, ${}^a_t\bar{V}^{(i)}$, we take the system of equations

$$\frac{d\,{}^a_t\bar{V}^{(i)}}{dt} = {}^a\bar{P}^{(i)} + \delta\,{}^a_t\bar{V}^{(i)} - B_{x+t}^{(i)}\mu_{x+t}^{(i)} + {}^a_t\bar{V}^{(i)}\mu_{x+t}^{(T)}, \quad i = 1, 2, \ldots, m; \quad (3)$$

and assume that ${}^{a}_{0}\bar{V}^{(i)} = 0$, and that ${}^{a}_{n}\bar{V}^{(i)}$ has the fixed value $B^{(i)}_{x+n}$ previously mentioned. The equations (3), and the given maturity values, define a set of components which satisfy conditions (1) and (2).

To show that condition (1) is satisfied, we rearrange equations (3) to the form

$$d\,({}^{a}_{t}\bar{V}^{(i)}\,D^{(T)}_{x+t}) = {}^{a}\bar{P}^{(i)}\,D^{(T)}_{x+t}dt - B^{(i)}_{x+t}D^{(T)}_{x+t}\mu^{(i)}_{x+t}dt \tag{4}$$

and on integration obtain

$$B^{(i)}_{x+n}D^{(T)}_{x+n} = {}^{a}\bar{P}^{(i)}\,(\bar{N}^{(T)}_{x} - \bar{N}^{(T)}_{x+n}) - \int_0^n B^{(i)}_{x+t}D^{(T)}_{x+t}\mu^{(i)}_{x+t}dt\,, \qquad i = 1, 2, \ldots, m\,. \tag{5}$$

By addition of equations (5), there results

$$B^{(T)}_{x+n}D^{(T)}_{x+n} = \left(\sum_{i=1}^{m} {}^{a}\bar{P}^{(i)}\right)(\bar{N}^{(T)}_{x} - \bar{N}^{(T)}_{x+n}) - \sum_{i=1}^{m}\int_0^n B^{(i)}_{x+t}D^{(T)}_{x+t}\mu^{(i)}_{x+t}dt\,,$$

and then comparison with equation (14) of Part I yields

$$\sum_{i=1}^{m} {}^{a}\bar{P}^{(i)} = \bar{P}^{(T)}\,.$$

Condition (2) now follows easily since, by addition of equations (4),

$$d\left[\left(\sum_{i=1}^{m} {}^{a}_{t}\bar{V}^{(i)}\right)D^{(T)}_{x+t}\right] = \bar{P}^{(T)}D^{(T)}_{x+t}dt - \sum_{i=1}^{m} B^{(i)}_{x+t}D^{(T)}_{x+t}\mu^{(i)}_{x+t}dt = d\,({}_{t}\bar{V}^{(T)}D^{(T)}_{x+t})$$

and, further, the initial value of

$$\sum_{i=1}^{m} {}^{a}_{t}\bar{V}^{(i)}\,,$$

namely zero, equals the initial value of ${}_{t}\bar{V}^{(T)}$.

If the ith equation (3) is written as

$$d\,{}^{a}_{t}\bar{V}^{(i)} = {}^{a}\bar{P}^{(i)}dt + \delta\,{}^{a}_{t}\bar{V}^{(i)}dt + \mu^{(-i)}_{x+t}\cdot{}^{a}_{t}\bar{V}^{(i)}dt - \mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}^{a}_{t}\bar{V}^{(i)})\,dt\,, \tag{6}$$

we see that in the instant of attaining age $x + t$ the reserve decreases by the net insurance cost $\mu^{(i)}_{x+t}(B^{(i)}_{x+t} - {}^{a}_{t}\bar{V}^{(i)})dt$ in respect to the ith cause of decrement but increases by the reserve expected to be released in regard to the other causes, namely, ${}^{a}_{t}\bar{V}^{(i)}\mu^{(-i)}_{x+t}dt$. In effect, the ith reserve ${}^{a}_{t}\bar{V}^{(i)}$ is maintained for the sole purpose of providing the ith benefit and is not applicable to the other benefits. For this reason, we call ${}^{a}\bar{P}^{(i)}$ and ${}^{a}_{t}\bar{V}^{(i)}$, $i = 1, 2, \ldots, m$, the *independent* premiums and reserves corresponding to the m causes of decrement. It should be noted, however, from equation (6), that ${}^{a}\bar{P}^{(i)}$ and ${}^{a}_{t}\bar{V}^{(i)}$ are based on both $\mu^{(-i)}_{x+t}$ and $\mu^{(i)}_{x+t}$.

Formulas for the Independent Premiums and Reserves

The practical formula for the determination of ${}^{a}\bar{P}^{(i)}$ is obtained by solving the proper equation (5), and is

$$ {}^{a}\bar{P}^{(i)} = \left\{ \int_0^n B^{(i)}_{x+t} D^{(T)}_{x+t} \mu^{(i)}_{x+t} dt + B^{(i)}_{x+n} D^{(T)}_{x+n} \right\} \Big/ (\bar{N}^{(T)}_{x} - \bar{N}^{(T)}_{x+n}) \,. \quad (7) $$

The corresponding reserve formulas are:

$$ {}^{a}_{t}\bar{V}^{(i)} = \left\{ {}^{a}\bar{P}^{(i)} (\bar{N}^{(T)}_{x} - \bar{N}^{(T)}_{x+t}) - \int^{t} B^{(i)}_{x+s} D^{(T)}_{x+s} \mu^{(i)}_{x+s} ds \right\} \Big/ D^{(T)}_{x+t} , \quad (8) $$

and

$$ {}^{a}_{t}\bar{V}^{(i)} = \left\{ \int_t^n B^{(i)}_{x+s} D^{(T)}_{x+s} \mu^{(i)}_{x+s} ds + B^{(i)}_{x+n} D^{(T)}_{x+n} - {}^{a}\bar{P}^{(i)} (\bar{N}^{(T)}_{x+t} - \bar{N}^{(T)}_{x+n}) \right\} \Big/ D^{(T)}_{x+t} \,. \quad (9) $$

If equation (6) is multiplied by $D^{[i]}_{x+t}$, it can be put in the form

$$ d\,({}^{a}_{t}\bar{V}^{(i)} D^{[i]}_{x+t}) = {}^{a}\bar{P}^{(i)} D^{[i]}_{x+t} dt - B^{(i)}_{x+t} D^{[i]}_{x+t} \mu^{(i)}_{x+t} dt + {}^{a}_{t}\bar{V}^{(i)} D^{[i]}_{x+t} \mu^{(-i)}_{x+t} dt \,. $$

From the foregoing equation, we obtain

$$ {}^{a}\bar{P}^{(i)} = \bar{P}^{[i]} - \bar{P}^{*} \,, \quad (10) $$

where in this case

$$ \bar{P}^{[i]} = \left\{ \int_0^n B^{(i)}_{x+t} D^{[i]}_{x+t} \mu^{(i)}_{x+t} dt + B^{(i)}_{x+n} D^{[i]}_{x+n} \right\} \Big/ (\bar{N}^{[i]}_{x} - \bar{N}^{[i]}_{x+n}) $$

and

$$ \bar{P}^{*} = \left(\int_0^n {}^{a}_{t}\bar{V}^{(i)} D^{[i]}_{x+t} \mu^{(-i)}_{x+t} dt \right) \Big/ (\bar{N}^{[i]}_{x} - \bar{N}^{[i]}_{x+n}) \,. $$

Here $\bar{P}^{[i]}$ is the premium, based on the single decrement table $[i]$, that will provide $B^{(i)}_{x+t}$ in case of decrement at age $x + t$, and the maturity value $B^{(i)}_{x+n}$ in case of survival to age $x + n$. The equation (10) indicates that the multiple decrement premium ${}^{a}\bar{P}^{(i)}$ is less than the single decrement premium $\bar{P}^{[i]}$ by a uniform annual amount $\bar{P}^{*}$ equivalent, on the basis of table $[i]$, to the reserves ${}^{a}_{t}\bar{V}^{(i)}$ released by reason of the other causes of decrement. In exceptional circumstances, $\bar{P}^{*}$ might be negative.

The reserve formula corresponding to equation (10) is

$$ {}^{a}_{t}\bar{V}^{(i)} = {}_{t}\bar{V}^{[i]} - {}_{t}\bar{V}^{*} \,, \quad (11) $$

where ${}_t\overline{\mathrm{V}}^{[i]}$ is based on $\overline{\mathrm{P}}^{[i]}$, and

$${}_t\overline{\mathrm{V}}^{*} = \left\{ \overline{\mathrm{P}}^{*} \left(\overline{\mathrm{N}}_x^{[i]} - \overline{\mathrm{N}}_{x+t}^{[i]} \right) - \int_0^t {}^a_s\overline{\mathrm{V}}^{(i)} \mathrm{D}_{x+s}^{[i]} \mu_{x+s}^{(-i)} d s \right\} \Big/ \mathrm{D}_{x+t}^{[i]} .$$

By arranging equation (6) in the form

$$d\,{}^a_t\overline{\mathrm{V}}^{(i)} - \left(\delta + \mu_{x+t}^{(-i)}\right) {}^a_t\overline{\mathrm{V}}^{(i)} dt = {}^a\overline{\mathrm{P}}^{(i)} dt - \mu_{x+t}^{(i)} \left(B_{x+t}^{(i)} - {}^a_t\overline{\mathrm{V}}^{(i)}\right) dt$$

and using $\mathrm{D}_{x+t}^{[-i]}$ as integrating factor, one obtains

$$\begin{aligned} {}^a\overline{\mathrm{P}}^{(i)} = \left\{ \int_0^n \mathrm{D}_{x+t}^{[-i]} \mu_{x+t}^{(i)} \left(B_{x+t}^{(i)} - {}^a_t\overline{\mathrm{V}}^{(i)}\right) dt + B_{x+n}^{(i)} \mathrm{D}_{x+n}^{[-i]} \right\} & \\ \Big/ \left(\overline{\mathrm{N}}_x^{[-i]} - \overline{\mathrm{N}}_{x+n}^{[-i]} \right) . & \qquad (12) \end{aligned}$$

The formula shows that ${}^a\overline{\mathrm{P}}^{(i)}$ is the uniform annual amount which is equivalent, on the basis of table $[-i]$, to the momentary net costs $\mu_{x+t}^{(i)}(B_{x+t}^{(i)} - {}^a_t\overline{\mathrm{V}}^{(i)})dt$ and the maturity value $B_{x+n}^{(i)}$.

Examples of Independent Premiums and Reserves

As a first illustration, we shall indicate formulas for the independent premiums and reserves for the case of Example 2. It will be recalled that for the computation of $\overline{\mathrm{P}}^{(T)}$ and ${}_t\overline{\mathrm{V}}^{(T)}$ for that example, a special multiple decrement table (Table 1) was constructed. Once the values of ${}_t\overline{\mathrm{V}}^{(T)}$ had been determined, we proceeded to the computation of the independent premiums on the basis of the original UAW 1955 Tables. The formulas are easily verified, and are the following:

For retirement.

$$\begin{aligned} {}^a\overline{\mathrm{P}}^{(r)} \overline{\mathrm{N}}_{32}^{(T)} &= \int_{28}^{37} B_{32+t}^{(r)} \mathrm{D}_{32+t}^{(T)} \mu_{32+t}^{(r)} dt \\ &\doteq \sum_{k=28}^{36} v^{32+k+1/2} B_{32+k+1/2}^{(r)} d_{32+k}^{(r)} \\ &= \sum_{k=28}^{36} \overline{\mathrm{C}}_{32+k}^{(rB)} = \overline{\mathrm{M}}_{60}^{(rB)} . \end{aligned}$$

For disability,

$${}^a\overline{\mathrm{P}}^{(h)} \overline{\mathrm{N}}_{32}^{(T)} = \int_{15}^{33} B_{32+t}^{(h)} \mathrm{D}_{32+t}^{(T)} \mu_{32+t}^{(h)} dt \doteq \overline{\mathrm{M}}_{47}^{(hB)} .$$

For withdrawal,

$${}^a\overline{\mathrm{P}}^{(w)} \overline{\mathrm{N}}_{32}^{(T)} = \int_{10}^{28} {}_t\overline{\mathrm{V}}^{(T)} \mathrm{D}_{32+t}^{(T)} \mu_{32+t}^{(w)} dt \doteq \overline{\mathrm{M}}_{42}^{(wV)} .$$

For death,

$$ {}^{a}\bar{P}^{(d)}\bar{N}_{32}^{(T)} = \int_{10}^{37} {}_{t}\bar{V}^{(T)} D_{32+t}^{(T)} \mu_{32+t}^{(d)} dt \doteq \bar{M}_{42}^{(dV)} . $$

Values of the commutation functions were either obtained directly, or were adapted from the UAW 1955 Tables.

The results were:

$$ {}^{a}\bar{P}^{(r)} = 3.85 , \quad {}^{a}\bar{P}^{(h)} = 0.57 , \quad {}^{a}\bar{P}^{(w)} = 0.41 , \quad {}^{a}\bar{P}^{(d)} = 1.09 . $$

The total of these independent premiums is 5.92 which checks exactly with the value of $\bar{P}^{(T)}$ calculated on the basis of the special multiple decrement table described in Part I. As the number of significant figures in some stages of the numerical work was low, the exact check was something of a surprise.

Formulas for the reserves are readily adapted from the general formulas (8) and (9). For example,

$$ {}^{a}_{t}\bar{V}^{(h)} \doteq \{M(t) - {}^{a}\bar{P}^{(h)}\bar{N}_{32+t}^{(T)}\} / D_{32+t}^{(T)} , $$

where

$$ M(t) = \begin{cases} \bar{M}_{47}^{(hB)} , & 0 \le t < 15 , \\ \bar{M}_{32+t}^{(hB)} , & 15 \le t < 33 , \\ 0, & t \ge 33 . \end{cases} $$

Because all premiums have been assumed payable for the whole of service, small negative reserves develop in the later durations for those benefits which terminate before the final year of service. Sample reserves are given in Table 2 of the Appendix.

Example 3. At the beginning of this rather general example the only specialization of the general insurance we have been considering will be in regard to a particular cause of decrement, say (1), for which we assume that

$$ B_{x+t}^{(1)} = \alpha^{(1)} + \beta_t^{(1)} \cdot {}^{a}_{t}\bar{V}^{(1)} , \qquad 0 \le t < n , \quad (13) $$

where $\alpha^{(1)}$ and $\beta_t^{(1)}$ are known, preassigned coefficients, and, as usual, we assume a fixed maturity value $B_{x+n}^{(1)}$. Then, the first of equations (3) may be rearranged as

$$ d\,{}^{a}_{t}\bar{V}^{(1)} - (\delta + \mu_{x+t}^{(\tilde{T})})\,{}^{a}_{t}\bar{V}^{(1)} = {}^{a}\bar{P}^{(1)} dt - \alpha^{(1)} \mu_{x+t}^{(1)} dt , \qquad (14) $$

where

$$ \mu_{x+t}^{(\tilde{T})} = \mu_{x+t}^{(T)} - \beta_t^{(1)} \mu_{x+t}^{(1)} . \qquad (15) $$

Using superscript $(\tilde{T})$ to denote functions based on $\mu_{x+t}^{(\tilde{T})}$, we obtain from equation (14) that

$$^{a}\overline{\mathrm{P}}^{(1)} = \left\{ a^{(1)} \int_0^n \mathrm{D}_{x+t}^{(\tilde{T})} \mu_{x+t}^{(1)} dt + B_{x+n}^{(1)} \mathrm{D}_{x+n}^{(\tilde{T})} \right\} \Big/ \left(\overline{\mathrm{N}}_x^{(\tilde{T})} - \overline{\mathrm{N}}_{x+n}^{(\tilde{T})} \right) . \quad (16)$$

A number of specializations will now be indicated.

a) If $a^{(1)} = 0$, then

$$^{a}\overline{\mathrm{P}}^{(1)} = B_{x+n}^{(1)} / \bar{s}_{x:\overline{n}|}^{(\tilde{T})}$$

and, in this case

$$^{a}_{t}\overline{\mathrm{V}}^{(1)} = {}^{a}\overline{\mathrm{P}}^{(1)}\, \bar{s}_{x:\overline{t}|}^{(\tilde{T})} .$$

These two equations reveal that if $B_{x+t}^{(1)} = \beta_t^{(1)} \cdot {}^{a}_{t}\overline{\mathrm{V}}^{(1)}$, $0 \leq t < n$, and $\beta_t^{(1)}$ is continuous, then $^{a}\overline{\mathrm{P}}^{(1)}$ and $^{a}_{t}\overline{\mathrm{V}}^{(1)}$ $(t > 0)$ are greater than, equal to, or less than zero according as $B_{x+n}^{(1)}$ is greater than, equal to, or less than zero, *no matter how the values of $\beta_t^{(1)}$ are continuously assigned.*

b) If the insurance is a joint life endowment in respect to m lives $(x_1), (x_2), \ldots , (x_m)$, and if, when (x_1) is first to die, the benefit is $a^{(1)} = 1$, and $^{a}\overline{\mathrm{P}}^{(1)}$ is the corresponding independent premium, then the total force of decrement to be used in formula (16) is

$$\sum_{i=1}^{m} \mu_{x_i+t} ,$$

and,

$$^{a}\overline{\mathrm{P}}^{(1)} = \overline{\mathrm{P}}_{\overset{1}{x_1}x_2 \ldots x_m:\overline{n}|} + B_{x+n}^{(1)} / \bar{s}_{x_1x_2 \ldots x_m:\overline{n}|} .$$

The case where the benefit is $1 + \beta^{a}_{t}\overline{\mathrm{V}}^{(1)}$, β an integer, may also be handled readily if the mortality follows a Gompertz or Makeham law, but we shall omit the formulas.

c) If the insurance is a joint life term insurance in respect to m lives of equal age x, and if, when a specified member is the first to die, the benefit is $1 + \beta^{a}_{t}\overline{\mathrm{V}}^{(1)}$, β an integer, $0 \leq \beta \leq m$, then the total force of decrement to be used in formula (16) is $(m - \beta)\mu_{x+t}$ and

$$^{a}\overline{\mathrm{P}}^{(1)} = \overline{\mathrm{P}}_{\overset{1}{x}x \ldots (m-\beta):\overline{n}|} .$$

In particular,

$$^{a}\overline{\mathrm{P}}^{(1)} = \overline{\mathrm{P}}_{\overset{1}{x}x \ldots (m):\overline{n}|} \qquad \text{if} \qquad \beta = 0 ,$$

$$^{a}\overline{\mathrm{P}}^{(1)} = \overline{\mathrm{P}}_{\overset{1}{x}:\overline{n}|} \qquad \text{if} \qquad \beta = m - 1 .$$

In the case $\beta = m$, equation (16) becomes

$$^{a}\bar{P}^{(1)} = \left(\int_0^n v^t \mu_{x+t} dt\right) \Big/ \bar{a}_{\overline{n}|}.$$

The above formulas provide one form of generalization to the joint life case of the formulas developed in the single life case for insurances for face amount plus the reserve [9], [3, p. 120].

III. DEPENDENT PREMIUMS AND RESERVES

Definitions

In this part we examine a second way of decomposing the total premium and the total reserve into components corresponding to the various causes of decrements. The basic idea for this second decomposition is that the total reserve, not merely the component reserve, shall be available to offset the benefit payment whenever decrement from any cause occurs. Denoting the component premiums and reserves in this case by $^{b}\bar{P}^{(i)}$ and $^{b}_{t}\bar{V}^{(i)}$, $i = 1, 2, \ldots, m$, we take as starting point the system of equations

$$\frac{d\,^{b}_{t}\bar{V}^{(i)}}{dt} = {}^{b}\bar{P}^{(i)} + \delta\,^{b}_{t}\bar{V}^{(i)} - \mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}_{t}\bar{V}^{(T)})\,, \quad i = 1, 2, \ldots, m\,. \quad (1)$$

These clearly indicate that in case the ith decrement occurs at the instant of attaining age $x + t$, the total reserve ${}_{t}\bar{V}^{(T)}$ is used to offset the benefit payment $B^{(i)}_{x+t}$. Because the total reserve is a function of all the benefits, the resulting premiums $^{b}\bar{P}^{(i)}$, $i = 1, 2, \ldots, m$ are interdependent, and so also are the reserves, $^{b}_{t}\bar{V}^{(i)}$. For that reason we call $^{b}P^{(i)}$ and $^{b}_{t}\bar{V}^{(i)}$, $i = 1, 2, \ldots, m$, the *dependent* premiums and reserves corresponding to the m causes of decrement.

As for any splitting, we assume that for $t = 0$ reserves are zero, and that $^{b}_{n}\bar{V}^{(i)}$ has the fixed value $B^{(i)}_{x+n}$, $i = 1, 2, \ldots, m$, where

$$\sum_{i=1}^{m} B^{(i)}_{x+n} = B^{(T)}_{x+n}.$$

Then, to verify that condition II-(1) holds, we multiply equations (1) by v^t, and rearrange to the form

$$d\,(^{b}_{t}\bar{V}^{(i)}\,v^t) = {}^{b}\bar{P}^{(i)}\,v^t dt - v^t \mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}_{t}\bar{V}^{(T)})\,dt\,, \quad (2)$$

and on integration obtain

$$B^{(i)}_{x+n} v^n = {}^{b}\bar{P}^{(i)}\,\bar{a}_{\overline{n}|} - \int_0^n v^t \mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}_{t}\bar{V}^{(T)})\,dt\,, \quad i = 1, 2, \ldots, m\,. \quad (3)$$

Addition of equations (3) yields

$$B_{x+n}^{(T)} v^n = \left(\sum_{i=1}^{m} {}^b\overline{\mathrm{P}}^{(i)} \right) \bar{a}_{\overline{n}|} - \sum_{i=1}^{m} \int_0^n v^t \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\overline{\mathrm{V}}^{(T)})\, dt\,,$$

and comparison with equation I-(17) shows

$$\sum_{i=1}^{m} {}^b\overline{\mathrm{P}}^{(i)} = \overline{\mathrm{P}}^{(T)}\,.$$

Further, addition of equations (2), and comparison with equation I-(16), gives

$$d\left[\left(\sum_{i=1}^{m} {}_t^b\overline{\mathrm{V}}^{(i)} \right) v^t\right] = d\,({}_t\overline{\mathrm{V}}^{(T)} v^t)\,,$$

and since

$$\sum_{i=1}^{m} {}_0^b\overline{\mathrm{V}}^{(i)} = {}_0\overline{\mathrm{V}}^{(T)} = 0\,,$$

it follows that for any t, $0 \le t \le n$,

$$\sum_{i=1}^{m} {}_t^b\overline{\mathrm{V}}^{(i)} = {}_t\overline{\mathrm{V}}^{(T)}\,;$$

that is, condition II-(2) is satisfied by the dependent reserves.

Formulas for the Dependent Premiums and Reserves

Because of their involvement with all benefits it is generally more difficult to obtain explicit formulas for dependent premiums and reserves than for independent components. The most useful formulas for computational purposes are obtained from equations (3), which yield

$$ {}^b\overline{\mathrm{P}}^{(i)} = \left\{ \int_0^n v^t \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\overline{\mathrm{V}}^{(T)})\, dt + B_{x+n}^{(i)} v^n \right\} \Big/ \bar{a}_{\overline{n}|}\,, \qquad i = 1, 2, \ldots, m\,. \tag{4}$$

The corresponding reserve formulas are

$$ {}_t^b\overline{\mathrm{V}}^{(i)} = {}^b\overline{\mathrm{P}}^{(i)} \bar{s}_{\overline{t}|} - \int_0^t (1+i)^{t-s} \mu_{x+s}^{(i)} (B_{x+s}^{(i)} - {}_s\overline{\mathrm{V}}^{(T)})\, ds \tag{5}$$

and

$$ {}_t^b\overline{\mathrm{V}}^{(i)} = \int_t^n v^{s-t} \mu_{x+s}^{(i)} (B_{x+s}^{(i)} - {}_s\overline{\mathrm{V}}^{(T)})\, ds + B_{x+n}^{(i)} v^{n-t} - {}^b\overline{\mathrm{P}}^{(i)} \bar{a}_{\overline{n-t}|}\,. \tag{6}$$

To employ these formulas, one may first determine the values of ${}_t\bar{V}^{(T)}$ by procedures indicated in Part I, and then calculate the integrals by appropriate year by year approximations.

Another set of formulas may be obtained by using $D^{[i]}_{x+t}$ as an integrating factor for the ith equation (1) rearranged as

$$d\,{}^b_t\bar{V}^{(i)} - (\delta + \mu^{(i)}_{x+t})\,{}^b_t\bar{V}^{(i)}\,dt = {}^b\bar{P}^{(i)}\,dt - \mu^{(i)}_{x+t}B^{(i)}_{x+t}\,dt + {}^b_t\bar{V}^{(-i)}\,\mu^{(i)}_{x+t}\,dt\,,$$

where ${}^b_t\bar{V}^{(-i)} = {}_t\bar{V}^{(T)} - {}^b_t\bar{V}^{(i)}$. The resulting formula for ${}^bP^{(i)}$ is

$$ {}^b\bar{P}^{(i)} = \left\{\int_0^n B^{(i)}_{x+t}D^{[i]}_{x+t}\mu^{(i)}_{x+t}\,dt + B^{(i)}_{x+n}D^{[i]}_{x+n} - \int_0^n {}^b_t\bar{V}^{(-i)}\,D^{[i]}_{x+t}\mu^{(i)}_{x+t}\,dt\right\} \Big/ \left(\bar{N}^{[i]}_x - \bar{N}^{[i]}_{x+n}\right)\,, \tag{7}$$

which may be written in the form

$$ {}^b\bar{P}^{(i)} = \bar{P}^{[i]} - \bar{P}^{**}\,. \tag{8}$$

Thus ${}^b\bar{P}^{(i)}$ is equal to the premium $\bar{P}^{[i]}$, which on the basis of the single decrement table $[i]$ would provide the ith benefits, less the uniform amount $\bar{P}^{**}$ equivalent to the reserves ${}^b_t\bar{V}^{(-i)}$ that are made available on the happening of the ith decrement. In some circumstances it may occur that the reserves ${}^b_t\bar{V}^{(-i)}$ are negative, in which case ${}^b\bar{P}^{(i)}$ may exceed $\bar{P}^{[i]}$.

Finally, if the ith equation (1) is written as $d\,{}^b_t\bar{V}^{(i)} - (\delta + \mu^{(T)}_{x+t})\,{}^b_t\bar{V}^{(i)}dt = {}^b\bar{P}^{(i)}dt - \mu^{(i)}_{x+t}(B^{(i)}_{x+t} - {}_t\bar{V}^{(T)})dt - {}^b_t\bar{V}^{(i)}\mu^{(T)}_{x+t}dt$ and $D^{(T)}_{x+t}$ is used as integrating factor, we obtain

$$ {}^b\bar{P}^{(i)} = \left\{\int_0^n D^{(T)}_{x+t}\,[\mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}_t\bar{V}^{(T)}) + {}^b_t\bar{V}^{(i)}\,\mu^{(T)}_{x+t}]\,dt + B^{(i)}_{x+n}D^{(T)}_{x+n}\right\} \Big/ \left(\bar{N}^{(T)}_x - \bar{N}^{(T)}_{x+n}\right)\,. \tag{9}$$

This may be rearranged in the form

$$ {}^b\bar{P}^{(i)} = \left\{\int_0^n D^{(T)}_{x+t}\,[\mu^{(i)}_{x+t}\,(B^{(i)}_{x+t} - {}^b_t\bar{V}^{(-i)}) + {}^b_t\bar{V}^{(i)}\,\mu^{(-i)}_{x+t}]\,dt + B^{(i)}_{x+n}D^{(T)}_{x+n}\right\} \Big/ \left(\bar{N}^{(T)}_x - \bar{N}^{(T)}_{x+n}\right)\,. \tag{10}$$

Equation (10) indicates that for the case of dependent premiums and reserves the occurrence of the ith decrement makes ${}^b_t\bar{V}^{(-i)}$ available to offset the ith benefit and that, correspondingly, when any of the other decrements occur, ${}^b_t\bar{V}^{(i)}$ is provided. This reiterates the dependence of the premiums.

Comparison of Independent and Dependent Components

The following remarks are readily established:

a) If $B^{(i)}_{x+t} = 0,\ 0 \le t \le n,$ then ${}^a\mathrm{P}^{(i)} = 0$ and ${}^a_t\mathrm{V}^{(i)} = 0,\ 0 \le t \le n.$ In this situation,

$$ {}^b\overline{\mathrm{P}}^{(i)} = -\int_0^n {}_t\overline{\mathrm{V}}^{(T)}\, v^t \mu^{(i)}_{x+t} dt \Big/ \bar{a}_{\overline{n}|} $$

and represents an average gain occasioned by the happening of the ith decrement.

b) If $B^{(i)}_{x+t} = {}_t\overline{\mathrm{V}}^{(T)},\ 0 \le t < n,$ and $B^{(i)}_{x+n} = 0,$ then ${}^b\overline{\mathrm{P}}^{(i)} = 0$ and ${}^b_t\overline{\mathrm{V}}^{(i)} = 0,\ 0 \le t \le n.$ Here

$$ {}^a\overline{\mathrm{P}}^{(i)} = \int_0^n {}_t\overline{\mathrm{V}}^{(T)} \mathrm{D}^{(T)}_{x+t} \mu^{(i)}_{x+t} dt \Big/ (\overline{\mathrm{N}}^{(T)}_x - \overline{\mathrm{N}}^{(T)}_{x+n}), $$

and is relatively complicated.

c) If for all $j \ne i,\ B^{(j)}_{x+t} = 0,\ 0 \le t \le n,$ then ${}^a\mathrm{P}^{(i)} = \mathrm{P}^{(T)}$ and ${}^a_t\overline{\mathrm{V}}^{(i)} \equiv {}_t\overline{\mathrm{V}}^{(T)}.$

d) If for all $j \ne i, B^{(j)}_{x+t} = {}_t\overline{\mathrm{V}}^{(T)}, 0 \le t < n,$ and $B^{(j)}_{x+n} = 0,$ then ${}^b\overline{\mathrm{P}}^{(i)} = \mathrm{P}^{[i]} = \overline{\mathrm{P}}^{(T)}$ and ${}^b_t\overline{\mathrm{V}}^{(i)} \equiv {}_t\overline{\mathrm{V}}^{[i]} \equiv {}_t\overline{\mathrm{V}}^{(T)}.$

Under any practical circumstances, the independent premium ${}^a\mathrm{P}^{(i)}$ is nonnegative. However, the dependent premium ${}^b\mathrm{P}^{(i)}$ may be negative in cases where $B^{(i)}_{x+t} < {}_t\overline{\mathrm{V}}^{(T)}$ for a sufficient range of values of t. Negative reserves are possibilities in either the independent or dependent case.

As another comparison, one notes that ${}^a\mathrm{P}^{(i)}$ is unaffected by any changes in the benefits $B^{(j)}_{x+t}, j \ne i,$ which leave $B^{(i)}_{x+t}$ unaltered. The dependent premium ${}^b\overline{\mathrm{P}}^{(i)}$ is unaffected by changes in the benefits $B^{(j)}_{x+t}, j \ne i,$ which leave $B^{(i)}_{x+t} - {}_t\overline{\mathrm{V}}^{(T)}$ unaltered. In general, there is a distinct possibility, if $B^{(j)}_{x+t}$ is changed, that ${}^b\overline{\mathrm{P}}^{(i)}$ will change in value.

Conditions for Equality of Independent and Dependent Premiums and Reserves

The independent and dependent premiums, functioning in different fashions, can correctly, in general, be anticipated to be of different value. Similarly, the independent and dependent reserves will generally differ except for the preassigned maturity values. It is of interest to investigate the conditions under which the two schemes for splitting the total premium and the total reserve turn out to be the same. To do this we will seek conditions for the equality of the ith independent and dependent components, and one may then apply these conditions for $i = 1, 2, \ldots, m.$

From comparison of the defining relations

$$\frac{d\,{}^a_t\overline{V}^{(i)}}{dt} = {}^a\overline{P}^{(i)} + \delta\,{}^a_t\overline{V}^{(i)} - B^{(i)}_{x+t}\mu^{(i)}_{x+t} + {}^a_t\overline{V}^{(i)}\,\mu^{(T)}_{x+t} \qquad \text{II-(3)}$$

$$\frac{d\,{}^b_t\overline{V}^{(i)}}{dt} = {}^b\overline{P}^{(i)} + \delta\,{}^b_t\overline{V}^{(i)} - B^{(i)}_{x+t}\mu^{(i)}_{x+t} + {}_t\overline{V}^{(T)}\,\mu^{(i)}_{x+t}, \qquad \text{III-(1)}$$

it appears that if ${}^a_t\overline{V}^{(i)}\mu^{(T)}_{x+t} = {}_t\overline{V}^{(T)}\mu^{(i)}_{x+t}$, $0 \le t \le n$, and if, as we have assumed in our definitions, reserves are zero for $t = 0$, and ${}^a_n\overline{V}^{(i)}$ and ${}^b_n\overline{V}^{(i)}$ equal an assigned maturity value $B^{(i)}_{x+n}$, then the ith independent premium and reserve are respectively equal to the ith dependent premium and reserve. To establish this we multiply equations II-(3) and III-(1) by v^t and integrate to obtain

$${}^a\overline{P}^{(i)} = \left[\int_0^n v^t\,(B^{(i)}_{x+t}\mu^{(i)}_{x+t} - {}^a_t\overline{V}^{(i)}\,\mu^{(T)}_{x+t})\,dt + B^{(i)}_{x+n}v^n\right] \Big/ \bar{a}_{\overline{n}|}$$

$${}^b\overline{P}^{(i)} = \left[\int^n v^t\,(B^{(i)}_{x+t}\mu^{(i)}_{x+t} - {}_t\overline{V}^{(T)}\,\mu^{(i)}_{x+t})\,dt + B^{(i)}_{x+n}v^n\right] \Big/ \bar{a}_{\overline{n}|},$$

which shows that, under the above hypothesis, ${}^a\overline{P}^{(i)} = {}^b\overline{P}^{(i)}$. Using this fact, and the hypothesis again, we find, on subtracting III-(1) from II-(3), that

$$\frac{d\,({}^a_t\overline{V}^{(i)} - {}^b_t\overline{V}^{(i)})}{dt} = \delta\,({}^a_t\overline{V}^{(i)} - {}^b_t\overline{V}^{(i)}),$$

which shows

$$v^t\,({}^a_t\overline{V}^{(i)} - {}^b_t\overline{V}^{(i)}) = \text{a constant}.$$

By setting $t = 0$ or $t = n$, one sees the constant is zero, and hence the given condition is sufficient to establish both the equality of the premiums and the equality of the reserves.

The condition ${}^a_t\overline{V}^{(i)}\mu^{(T)}_{x+t} = {}_t\overline{V}^{(T)}\mu^{(i)}_{x+t}$, $0 \le t \le n$, is also necessary for the independent and dependent reserves to be equal for each intermediate time. For suppose ${}^a_t\overline{V}^{(i)} = {}^b_t\overline{V}^{(i)}$, $0 \le t \le n$. Then certainly their derivatives must be equal, and subtraction of III-(1) from II-(3) gives

$$0 = {}^a\overline{P}^{(i)} - {}^b\overline{P}^{(i)} + {}^a_t\overline{V}^{(i)}\,\mu^{(T)}_{x+t} - {}_t\overline{V}^{(T)}\,\mu^{(i)}_{x+t}. \qquad (11)$$

Since we have assumed reserves are zero for $t = 0$, it follows that ${}^a\overline{P}^{(i)} = {}^b\overline{P}^{(i)}$, and ${}^a_t\overline{V}^{(i)}\mu^{(T)}_{x+t} = {}_t\overline{V}^{(T)}\mu^{(i)}_{x+t}$ for all t in the range. The latter relation would also follow from equation (11) if we are given ${}^a\overline{P}^{(i)} = {}^b\overline{P}^{(i)}$.

We can summarize by stating:

Theorem 1. A necessary and sufficient condition that ${}^a_t\bar{V}^{(i)} = {}^b_t\bar{V}^{(i)}$, $0 \leq t \leq n$, is that ${}^a_t\bar{V}^{(i)}\mu^{(T)}_{x+t} = {}_t\bar{V}^{(T)}\mu^{(i)}_{x+t}$, $0 \leq t \leq n$, and each of these relations implies ${}^a\bar{P}^{(i)} = {}^b\bar{P}^{(i)}$.

One might anticipate that if ${}^a\bar{P}^{(i)} = {}^b\bar{P}^{(i)}$, then the corresponding reserves would be equal. It turns out, however, that this condition is not sufficient to establish the equality of the independent and dependent reserves throughout the n-year period, as we have been able to construct a counter example which will appear in the dissertation. The example is certainly artificial but it demonstrates that equality of the ith independent and dependent premiums does not imply equality of the corresponding reserves. The example also shows that even if ${}^a\bar{P}^{(i)} = {}^b\bar{P}^{(i)}$, $i = 1, 2, \ldots, m$, the reserve components may still differ.

A theorem that is useful in applications is the following:

Theorem 2. If $B^{(j)}_{x+t} = B_{x+t}$, $j = 1, 2, \ldots, m$, $0 \leq t < n$, and $\mu^{(i)}_{x+t} = c^{(i)}\mu^{(T)}_{x+t}$, $0 \leq t \leq n$, and, for the same $c^{(i)}$, $B^{(i)}_{x+n} = c^{(i)}B^{(T)}_{x+n}$, then ${}^a\bar{P}^{(i)} = {}^b\bar{P}^{(i)}$ and ${}^a_t\bar{V}^{(i)} = {}^b_t\bar{V}^{(i)}$, $0 \leq t \leq n$.

In other words, if the benefits payable on decrement may vary with t but not with cause of decrement, and if $\mu^{(i)}_{x+t}$ is a constant multiple of $\mu^{(T)}_{x+t}$ and $B^{(i)}_{x+n}$ is the same multiple of $B^{(T)}_{x+n}$, then the ith independent and dependent components are equal.

As a first step in proving Theorem 2, one notes on comparing equation II-(7) for ${}^a\bar{P}^{(i)}$ with equation I-(15) for $\bar{P}^{(T)}$ that, under the given conditions, ${}^a\bar{P}^{(i)} = c^{(i)}\bar{P}^{(T)}$. The relation ${}^a_t\bar{V}^{(i)} = c^{(i)}{}_t\bar{V}^{(T)}$ then follows from equations II-(8) and I-(18). Hence ${}^a_t\bar{V}^{(i)}\mu^{(T)}_{x+t} = {}_t\bar{V}^{(T)}\mu^{(i)}_{x+t}$, and Theorem 1 may be applied to complete the proof.

Examples of Dependent Premiums and Reserves

Some rather special difficulties appear when one considers dependent premiums and reserves for the pension benefits of Example 2. For that example, $q^{[r]}_{68} = 1$, and the corresponding force of retirement must increase indefinitely over the final year of age. For our calculations we assumed that

$$\mu^{[r]}_{68+t} = \frac{1}{1-t}, \qquad 0 \leq t < 1,$$

which implies $l^{(T')}_{68+t} = l^{(T')}_{68}(1 - t)$. Further, for formulas such as I-(17), III-(4), III-(6), maturity values must be assigned. These were immaterial for the calculation of the total and independent premiums and reserves for Example 2, because the survivorship function $l^{(T')}_{32+t}$ was assumed to be zero for $t = 37$. For dependent premiums, however, the maturity values are material and must be considered.

In our general theory, if $l_{x+n}^{(T)} \neq 0$, then ${}_t\overline{V}^{(T)}$ approaches the maturity value $B_{x+n}^{(T)}$ as $t \to n$. In order that premiums and reserves for cases where $l_{x+n}^{(T)}$ is zero may be consistent with the corresponding quantities when $l_{x+n}^{(T)}$ is different from zero but relatively small, it is necessary that, in all cases, the reserve approach the maturity value as $t \to n$. For Example 2, it may be shown that under the assumption in regard to $\mu_{68+t}^{(r)}$, $\lim_{t\to 37} {}_t\overline{V}^{(T)} = \lim_{t\to 37} B_{68+t}^{(r)} = 37\bar{a}_{69}$. Then for total maturity value $B_{69}^{(T)}$ we took $37\bar{a}_{69}$. A natural distribution into components is: $B_{69}^{(r)} = B_{69}^{(T)}$, $B_{69}^{(h)} = B_{69}^{(w)} = B_{69}^{(d)} = 0$.

The dependent premiums were then calculated by an approximate application of formula (4), namely

$$ {}^b\overline{P}^{(i)} \doteq \left\{ \sum_{k=0}^{n-1} [\, v^{k+1/2} (B_{x+k+1/2}^{(i)} - {}_{k+1/2}\overline{V}^{(T)}) \operatorname{colog}_e p_{x+k}^{[i]}] + B_{x+n}^{(i)} v^n \right\} \Big/ \bar{a}_{\overline{n}|}. $$

In the case of ${}^b\overline{P}^{(r)}$, a special integration was required for the final year. The resulting premiums were:

$$ {}^b\overline{P}^{(r)} = 6.11\,, \quad {}^b\overline{P}^{(h)} = 0.08\,, \quad {}^b\overline{P}^{(w)} = -0.23\,, \quad {}^b\overline{P}^{(d)} = -0.02\,. $$

The components for death and withdrawal are negative, since no benefit is paid during the first 10 years in respect to these causes of termination and ${}_t\overline{V}^{(T)}$ is payable thereafter. The dependent retirement premium is considerably higher than its independent counterpart, because it must provide the retirement benefits without aid of survivorship in respect to the other causes. In this instance, the total of the components exceeds $\overline{P}^{(T)}$ by 0.02, a not unreasonable discrepancy in view of the small number of significant figures available and the approximations employed.

Example 4. For this illustration, which is analogous to Example 3, we assume that in regard to a particular cause of decrement, say (1), that

$$ B_{x+t}^{(1)} = \alpha^{(1)} + \gamma_t^{(1)} \cdot {}_t\overline{V}^{(T)}, \qquad 0 \le t < n\,, \quad (12) $$

where $\alpha^{(1)}$ and $\gamma_t^{(1)}$ are known, preassigned coefficients, and further, there is a given maturity value $B_{x+n}^{(1)}$. Application of formula (4) to this case yields

$$ {}^b\overline{P}^{(1)} = \left\{ \alpha^{(1)} \int_0^n v^t \mu_{x+t}^{(1)} dt + \int_0^n v^t \mu_{x+t}^{(1)} (\gamma_t^{(1)} - 1)\, {}_t\overline{V}^{(T)} dt + B_{x+n}^{(1)} v^n \right\} \Big/ \bar{a}_{\overline{n}|}. \quad (13) $$

If $\gamma_t^{(1)} = 1$, then

$$^b\overline{\mathrm{P}}^{(1)} = \left\{ a^{(1)} \int_0^n v^t \mu_{x+t}^{(1)} dt + B_{x+n}^{(1)} v^n \right\} \Big/ \bar{a}_{\overline{n}|} \tag{14}$$

and, in particular, if $a^{(1)}$ is also zero,

$$^b\overline{\mathrm{P}}^{(1)} = B_{x+n}^{(1)} / \bar{s}_{\overline{n}|}. \tag{15}$$

Formula (14) is a form of generalization to a multiple decrement case of the formulas, [9], [3, p. 120], for insurances for face amount plus the reserve. Formula (15) indicates that if the benefit for a particular cause of decrement is ${}_t\overline{\mathrm{V}}^{(T)}$ for $0 \leq t < n$, then the dependent premium equals the sinking fund deposit required to accumulate the maturity value by the end of n years.

Example 5. If the insurance is a joint life endowment of sum insured 1 in respect to m lives (x_1), (x_2), . . . , (x_m), then Theorem 2 may be applicable. One such case is that in which the lives are of equal age and the total maturity value of 1 has been assigned in equal portions to the m lives. Then

$$^b\overline{\mathrm{P}}^{(1)} = {}^a\overline{\mathrm{P}}^{(1)} = \overline{\mathrm{P}}_{\overset{1}{x_1}x_2 \ldots x_m:\overline{n}|} + 1 / (m\, \bar{s}_{x_1x_2 \ldots x_m:\overline{n}|}) = \frac{1}{m} \overline{\mathrm{P}}_{x_1x_2 \ldots x_m:\overline{n}|}.$$

This is otherwise obvious from considerations of symmetry.

If the lives are of unequal age, but the same Gompertz law is applicable to each, and if the maturity value is distributed in proportion to the individual forces of mortality, then Theorem 2 is again applicable, and

$$^b\overline{\mathrm{P}}^{(1)} = {}^a\overline{\mathrm{P}}^{(1)} = \frac{\mu_{x_1}}{\mu_w} \overline{\mathrm{P}}_{w:\overline{n}|},$$

where w is the equivalent single age. For a whole life case, the condition in regard to distribution of the maturity value is unnecessary, it being assumed that the insurance continues for an indefinite term in accordance with the underlying Gompertz law.

IV. LOEWY PREMIUMS AND RESERVES

Introduction

One of the topics considered by Loewy in his paper [1] concerned the effect on the total premium and total reserve of adding an additional force of decrement and additional payment in the event of decrement from this new cause. Loewy compared the two premium-reserve equations of the same form as equation I-(13) but with and without the addition of the new force of decrement and payment, and obtained expressions for the change in the total premium and total reserve. If such a technique is repeated m times, starting with zero and introducing one force at a time, the m differences of the successive total premiums may be considered a splitting of the final total premium $\bar{P}^{(T)}$ into m components, each associated with a particular cause of decrement. Such components will be called Loewy premiums, and the corresponding reserve components will be called Loewy reserves.

The Loewy premium for a particular cause of decrement will depend not only on the particular benefit and force, but on the other benefits and forces used to calculate the preceding premiums. Thus, in contrast to the situation for independent and dependent premiums, the order in which the components are computed is material. To eliminate this last source of variation, we assume here that forces, benefits and premiums are in a preassigned order to remain fixed throughout the discussion.

Some new notation is needed. The total premium for the first k benefits will be denoted by $\bar{P}^{\underline{k}}$, and the corresponding total reserve by ${}_t\bar{V}^{\underline{k}}$. It is convenient to consider k to take on values 0, 1, 2, . . . , m where $\bar{P}^{\underline{0}} = {}_t\bar{V}^{\underline{0}} = 0$ and $\bar{P}^{\underline{m}} = \bar{P}^{(T)}$. The Loewy premium for the ith cause of decrement will be denoted by ${}^L\bar{P}^{(i)}$ and is defined by the relation ${}^L\bar{P}^{(i)} = \bar{P}^{\underline{i}} - \bar{P}^{\underline{i-1}}$. Similarly, we define the ith Loewy reserve by ${}^L_t\bar{V}^{(i)} = {}_t\bar{V}^{\underline{i}} - {}_t\bar{V}^{\underline{i-1}}$. As before, all initial reserves will be assumed to be zero, and maturity components will be as previously assigned. It is assumed that

$$ {}_n\bar{V}^{\underline{i}} = B^{\underline{i}}_{x+n} = \sum_{j=1}^{i} B^{(j)}_{x+n} , $$

so that ${}^L_n\bar{V}^{(i)} = B^{(i)}_{x+n}$.

Similar notations will be required for other functions appearing in the discussion. For example,

$$ \mu^{\underline{k}}_{x+t} = \sum_{j=1}^{k} \mu^{(j)}_{x+t} , $$

with $\mu^{\underline{0}}_{x+t} = 0$;

$$ D^{\underline{k}}_{x+t} = D^{\underline{k}}_{x} e^{-\int_0^t (\mu^{\underline{k}}_{x+s}+\delta)ds} ; \qquad \bar{N}^{\underline{k}}_{x+t} = \int_t^\infty D^{\underline{k}}_{x+s} d s . $$

Formulas for Loewy Premiums and Reserves

As for the other forms of premium components, several different formulas are available for the determination of Loewy premiums. There is, of course, the defining relation

$$ {}^L\bar{P}^{(i)} = \bar{P}^{\underline{i}} - \bar{P}^{\underline{i-1}} , \tag{1} $$

where, for instance, $\bar{P}^{\underline{i}}$ may be determined by application of formula I-(15), thus

$$\bar{P}^{\underline{i}} = \left\{ \sum_{j=1}^{i} \int_0^n B_{x+t}^{(j)} D_{x+t}^{\underline{i}} \mu_{x+t}^{(j)} dt + B_{x+n}^{i} D_{x+n}^{\underline{i}} \right\} \Big/ (\bar{N}_x^{\underline{i}} - \bar{N}_{x+n}^{\underline{i}}) , \quad (2)$$

and $\bar{P}^{\underline{i-1}}$ may be obtained similarly, or by summation of ${}^L\bar{P}^{(j)}$, $j = 1, 2, \ldots, i-1$.

For other forms, we follow Loewy and apply equation I-(13) for total premiums and reserves based on $(i-1)$ and i causes of decrement, respectively:

$$\frac{d\,{}_t\bar{V}^{\underline{i-1}}}{dt} = \bar{P}^{\underline{i-1}} + (\mu_{x+t}^{\underline{i-1}} + \delta)\,{}_t\bar{V}^{\underline{i-1}} - \sum_{j=1}^{i-1} B_{x+t}^{(j)} \mu_{x+t}^{(j)} \quad (3)$$

$$\frac{d\,{}_t\bar{V}^{\underline{i}}}{dt} = \bar{P}^{\underline{i}} + (\mu_{x+t}^{\underline{i}} + \delta)\,{}_t\bar{V}^{\underline{i}} - \sum_{j=1}^{i} B_{x+t}^{(j)} \mu_{x+t}^{(j)} . \quad (4)$$

Subtracting equation (3) from equation (4), we may write the resulting equation either as

$$\frac{d\,{}_t^L\bar{V}^{(i)}}{dt} = {}^L\bar{P}^{(i)} + (\mu_{x+t}^{\underline{i-1}} + \delta)\,{}_t^L\bar{V}^{(i)} - \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\bar{V}^{\underline{i}}) \quad (5)$$

or as

$$\frac{d\,{}_t^L\bar{V}^{(i)}}{dt} = {}^L\bar{P}^{(i)} + (\mu_{x+t}^{\underline{i}} + \delta)\,{}_t^L\bar{V}^{(i)} - \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\bar{V}^{\underline{i-1}}) . \quad (6)$$

If in equation (5) we transpose the term $(\mu_{x+t}^{\underline{i-1}} + \delta){}_t^L\bar{V}^{(i)}$, apply the integrating factor $D_{x+t}^{\underline{i-1}}$, and integrate from 0 to n, we find

$${}^L\bar{P}^{(i)} = \left\{ \int_0^n D_{x+t}^{\underline{i-1}} \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\bar{V}^{\underline{i}})\, dt + B_{x+n}^{(i)} D_{x+n}^{\underline{i-1}} \right\} \Big/ (\bar{N}_x^{\underline{i-1}} - \bar{N}_{x+n}^{\underline{i-1}}) . \quad (7)$$

Similarly, by appropriate steps applied to equation (6), we obtain

$${}^L\bar{P}^{(i)} = \left\{ \int_0^n D_{x+t}^{\underline{i}} \mu_{x+t}^{(i)} (B_{x+t}^{(i)} - {}_t\bar{V}^{\underline{i-1}})\, dt + B_{x+n}^{(i)} D_{x+n}^{\underline{i}} \right\} \Big/ (\bar{N}_x^{\underline{i}} - \bar{N}_{x+n}^{\underline{i}}) . \quad (8)$$

Either of equations (7) and (8) shows that the first Loewy premium, ${}^L\bar{P}^{(1)}$, equals $\bar{P}^{[1]}$, the single decrement premium for the first benefit. Equation (7) also shows that if $B_{x+t}^{(i)}$ equals the reserve ${}_t\bar{V}^{\underline{i}}$, then ${}^L\bar{P}^{(i)}$ is the annual premium which, under interest and the force of decrement μ_{x+t}^{i-1}, will accumulate to the maturity value $B_{x+n}^{(i)}$. Similarly, from equation (8), if $B_{x+t}^{(i)}$ equals the reserve ${}_t\bar{V}^{\underline{i-1}}$, then ${}^L\bar{P}^{(i)}$ is the annual premium which under interest and the force of decrement μ_{x+t}^{i} will accumulate to $B_{x+n}^{(i)}$. If, in these cases, $B_{x+n}^{(i)} = 0$, then ${}^L P^{(i)} = 0$ also.

It may be pointed out that if we change our viewpoint and require that the total maturity value $B_{x+n}^{(T)}$ shall be accumulated by means of the premium $\bar{P}^{\underline{0}}$, hitherto taken as zero, then $\bar{P}^{\underline{0}} = B_{x+n}^{(T)} v^n / \bar{a}_{\overline{n}|}$ and, in that case, from equation (8),

$$ {}^L\bar{P}^{(1)} = \left\{ \int_0^n D_{x+t}^{1} \mu_{x+t}^{(1)} (B_{x+t}^{(1)} - \bar{P}^{\underline{0}} \bar{s}_{\overline{t}|})\, dt \right\} \Big/ \bar{N}_x^{1} - (\bar{N}_{x+n}^{1}) , $$

which is related to Linton's analysis of the endowment premium [10], [3, p. 86].

Formulas for Loewy reserves are evident. As an example, the retrospective reserve formula, corresponding to equation (8) for ${}^L\bar{P}^{(i)}$, is

$$ {}^L_t\bar{V}^{(i)} = \left\{ {}^L\bar{P}^{(i)} (\bar{N}_x^{i} - \bar{N}_{x+t}^{i}) - \int_0^t D_{x+s}^{i} \mu_{x+s}^{(i)} (B_{x+s}^{(i)} - {}_s\bar{V}^{\underline{i-1}})\, ds \right\} \Big/ D_{x+t}^{i} . \tag{9} $$

Computation of Loewy Premiums and Reserves for Example 2

Computation procedures for Loewy premiums would be guided in all cases by the nature of the benefits involved and the tables available. This is illustrated by the computation of Loewy components for Example 2. For this example, the causes of decrement were assigned the order: retirement, first; disability, second; withdrawal, third; death, fourth.

The retirement component, ${}^L\bar{P}^{(r)}$, is simply the single decrement premium $\bar{P}^{[r]}$ which may be approximated by the formula

$$ \bar{P}^{[r]} \doteq \left\{ \sum_{k=28}^{36} B_{32+k+1/2}^{(r)} v^{32+k+1/2} d_{32+k}^{[r]} \right\} \Big/ \bar{N}_{32}^{[r]} . $$

To calculate ${}^L\bar{P}^{(h)}$, the second Loewy premium, the double decrement total premium $\bar{P}^{(r,\,h)}$ for both retirement and disability was computed and ${}^L\bar{P}^{(r)}$ subtracted from it. Since, from duration 10 onward, $l_{32+t}^{(T')}$ is based on only the two forces of disability and retirement, it was possible to employ the commutation functions $\overline{M}_{32+t}^{(h'B)}$, $\overline{M}_{32+t}^{(r'B)}$ previously used to compute $\bar{P}^{(T)}$ (*cf.* equation I-(24)). All that was necessary was to replace $l_{32+t}^{(T')}$, $0 \leq t < 10$, by values of $l_{32+t}^{(r,h)}$, based on $\mu_{32+t}^{(r)}$, $\mu_{32+t}^{(h)}$ and such that $l_{42}^{(r,h)} = l_{42}^{(T')}$, and to make the corresponding changes in the D and $\bar{N}$ functions. Then

$$^{L}\bar{P}^{(r,h)} = (\bar{M}_{47}^{(h'B)} + \bar{M}_{60}^{(r'B)}) / \bar{N}_{32}^{(r,h)} .$$

The withdrawal and death benefits, when payable, are equal to the total reserve. A joint Loewy premium was calculated for these benefits, by application of formula (7), which gives for this case

$$^{L}\bar{P}^{(w,d)} = \left\{ -\int_{0}^{10} D_{32+t}^{(r,h)} \mu_{32+t}^{(w,d)} \cdot {}_{t}\bar{V}^{(T)} dt \right\} \Big/ \bar{N}_{32}^{(r,h)}$$

$$\doteq \left\{ -\sum_{k=0}^{9} {}_{k+1/2}\bar{V}^{(T)} \bar{D}_{32+k}^{(r,h)} \operatorname{colog}_e (1 - q_{32+k}^{[w]}) (1 - q_{32+k}^{[d]}) \right\} \Big/ \bar{N}_{32}^{(r,h)} .$$

Finally, by multiplying the values of $D_{32+t}^{(r,h)}$ by the corresponding values of $l_{32+t}^{[w]}$, we obtained the function $D_{32+t}^{(r,h,w)}$. Then $^{L}\bar{P}^{(d)}$ was computed by the formula

$$^{L}\bar{P}^{(d)} \doteq \left\{ -\sum_{k=0}^{9} {}_{k+1/2}\bar{V}^{(T)} \bar{D}_{32+k}^{(r,h,w)} \operatorname{colog}_e (1 - q_{32+k}^{[d]}) \right\} \Big/ \bar{N}_{32}^{(r,h,w)} ,$$

and by subtraction from $^{L}\bar{P}^{(w,d)}$ the third component $^{L}\bar{P}^{(w)}$ was obtained.

The Loewy premiums so calculated were

$$^{L}\bar{P}^{(r)} = 6.11 , \quad ^{L}\bar{P}^{(h)} = 0.08 , \quad ^{L}\bar{P}^{(w)} = -0.24 , \quad ^{L}\bar{P}^{(d)} = -0.03 .$$

As a check, it may be seen that their sum is equal to 5.92, exactly the value found for $\bar{P}^{(T)}$. As mentioned in the discussion of the other types of premium components, perfect agreement with $\bar{P}^{(T)}$ was not expected.

Reserves were computed by corresponding formulas and are summarized in Table 2 of the Appendix. For this particular example and particular order used, the dependent and the Loewy components are very close in value. This would not necessarily be the case in general.

Relations for Annuity Values

By use of the Loewy technique, a number of relations for annuity and insurances values, based on $(m - 1)$ and on m forces of decrement, may be obtained. To establish these formulas, one may write appropriate differential equations and proceed as we did to obtain formulas (7) and (8). Alternatively, one may apply formulas (7) and (8) to the case where $B_{x+t}^{(j)} = 1$, $j = 1, 2, \ldots, m$, $0 \leq t < n$, and $B_{x+n}^{m-1} = B_{x+n}^{m} = 1$, so that $B_{+n}^{(m)} = 0$.

Then, from equation (7), we have

$$\bar{\mathrm{P}}^{\underline{m}} - \bar{\mathrm{P}}^{\underline{m-1}} = \left\{\int_0^n \mathrm{D}^{\underline{m-1}}_{x+t}\,\mu^{(m)}_{x+t}\,(1 - {}_t\bar{\mathrm{V}}^{\underline{m}})\,dt\right\} \Big/ (\bar{\mathrm{N}}^{\underline{m-1}}_{x} - \bar{\mathrm{N}}^{\underline{m-1}}_{x+n}) \,.$$

But, in the present case,

$$\bar{\mathrm{P}}^{\underline{m}} = 1/\bar{a}^{\underline{m}}_{x:\overline{n|}} - \delta\,, \qquad \bar{\mathrm{P}}^{\underline{m-1}} = 1/\bar{a}^{\underline{m-1}}_{x:\overline{n|}} - \delta\,,$$

and

$${}_t\bar{\mathrm{V}}^{\underline{m}} = 1 - \bar{a}^{\underline{m}}_{x+t:\overline{n-t|}}/\bar{a}^{\underline{m}}_{x:\overline{n|}}\,.$$

Substitution from these relations and simplification then yields

$$\bar{a}^{\underline{m-1}}_{x:\overline{n|}} - \bar{a}^{\underline{m}}_{x:\overline{n|}} = \left\{\int_0^n \mathrm{D}^{\underline{m-1}}_{x+t}\,\mu^{(m)}_{x+t}\,\bar{a}^{\underline{m}}_{x+t:\overline{n-t|}}\,dt\right\} \Big/ \mathrm{D}^{\underline{m-1}}_{x}\,. \tag{10}$$

Particular cases are:

$$\bar{a}_{\overline{n|}} - \bar{a}_{x:\overline{n|}} = \int_0^n v^t\,\mu_{x+t}\,\bar{a}_{x+t:\overline{n-t|}}\,dt\,, \tag{11}$$

$$\bar{a}_{x:\overline{n|}} - \bar{a}_{xy:\overline{n|}} = \left\{\int_0^n \mathrm{D}_{x+t}\,\mu_{y+t}\,\bar{a}_{x+t:y+t:\overline{n-t|}}\,dt\right\} \Big/ \mathrm{D}_x\,. \tag{12}$$

By similar arguments applied to equation (8), we obtain

$$\bar{a}^{\underline{m-1}}_{x:\overline{n|}} - \bar{a}^{\underline{m}}_{x:\overline{n|}} = \left\{\int_0^n \mathrm{D}^{\underline{m}}_{x+t}\,\mu^{(m)}_{x+t}\,\bar{a}^{\underline{m-1}}_{x+t:\overline{n-t|}}\,dt\right\} \Big/ \mathrm{D}^{\underline{m}}_{x}\,. \tag{13}$$

The particular cases, corresponding to formulas (11) and (12), are the familiar reversionary formulas:

$$\bar{a}_{\overline{n|}} - \bar{a}_{x:\overline{n|}} = \left\{\int_0^n \mathrm{D}_{x+t}\,\mu_{x+t}\,\bar{a}_{\overline{n-t|}}\,dt\right\} \Big/ \mathrm{D}_x \tag{14}$$

$$\bar{a}_{x:\overline{n|}} - \bar{a}_{xy:\overline{n|}} = \left\{\int_0^n \mathrm{D}_{x+t:y+t}\,\mu_{y+t}\,\bar{a}_{x+t:\overline{n-t|}}\,dt\right\} \Big/ \mathrm{D}_{xy}\,. \tag{15}$$

Greville [11, p. 288] has given a relation analogous to formula (13) for expectations of life.

Other Investigations

By working with the basic differential equations II-(3) and IV-(5) or IV-(6), one may obtain relations for the difference ${}^{a}\bar{\mathrm{P}}^{(i)} - {}^{L}\bar{\mathrm{P}}^{(i)}$. Similarly, from equations III-(1) and IV-(5) or IV-(6), one finds relations for ${}^{b}\bar{\mathrm{P}}^{(i)} - {}^{L}\bar{\mathrm{P}}^{(i)}$. The work is straightforward and the results complicated, and will not be presented in this condensation. Details may be found in the dissertation.

Another question investigated was how the ith independent premium is affected if an additional force of decrement $\mu_{x+t}^{(m+1)}$ and a corresponding benefit $B_{x+t}^{(m+1)}$ are added to the underlying general insurance arrangement. Let ${}^{a,m}_{\ t}\bar{V}^{(i)}$, ${}^{a,m+1}_{\ \ t}\bar{V}^{(i)}$ denote the ith independent reserve when m and $(m+1)$ forces, respectively, are operating. Then it may be shown that the introduction of the $(m+1)$th force produces a change in the ith independent premium of amount equal to

$$\left\{-\int_0^n D_{x+t}^{m}\mu_{x+t}^{(m+1)}\cdot {}^{a,m+1}_{\ \ t}\bar{V}^{(i)}\,dt\right\} \Big/ (\bar{N}_x^{m}-\bar{N}_{x+n}^{m})\,.$$

This change may be expressed alternatively as

$$\left\{-\int_0^n D_{x+t}^{m+1}\mu_{x+t}^{(m+1)}\cdot {}^{a,m}_{\ t}\bar{V}^{(i)}\,dt\right\} \Big/ (\bar{N}_x^{m+1}-\bar{N}_{x+n}^{m+1})\,.$$

The analogous question for the ith dependent premium was also studied, and it was found that the change in such premium resulting from the addition of the $(m+1)$th force and benefit is equal to

$$\left\{-\int_0^n v^t\mu_{x+t}^{(i)}\cdot {}^{L}_{t}\bar{V}^{(m+1)}\,dt\right\} \Big/ \bar{a}_{\overline{n}|}\,,$$

where ${}^{L}_{t}\bar{V}^{(m+1)}$ is the $(m+1)$th Loewy reserve for the augmented insurance.

Closely related to the Loewy analysis for premiums and reserves is a generalization of Lidstone's theory of variation of reserves due to changes in the annual rates of interest and mortality, including the famous equation of equilibrium. Expositions of this theory for life insurance reserves have been presented recently by Baillie [12], Simonsen [13] and Jordan [3]. In the dissertation, the analogous theory for the total reserve and the various types of component reserves has been indicated for the general insurance considered here.

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APPENDIX

TABLE 1

SPECIAL MULTIPLE DECREMENT TABLE FOR EXAMPLE 2*

Duration t	Survivors $l_{32+t}^{(T')}$	Retirements $d_{32+t}^{(r')}$	Disablements $d_{32+t}^{(h')}$	Withdrawals $d_{32+t}^{(w')}$	Deaths $d_{32+t}^{(d')}$
0	36,361		14	985	40
1	35,322		18	886	42
2	34,376		17	798	45
3	33,516		17	727	47
4	32,725		20	661	49
5	31,995		19	602	51
6	31,323		19	551	53
7	30,700		18	510	58
8	30,114		21	473	60
9	29,560		21	440	65
10	29,034		23		
11	29,011		23		
12	28,988		26		
13	28,962		29		
14	28,933		32		
15	28,901		35		
16	28,866		40		
17	28,826		46		
18	28,780		52		
19	28,728		57		
20	28,671		66		
21	28,605		74		
22	28,531		86		
23	28,445		102		
24	28,343		119		
25	28,224		141		
26	28,083		166		
27	27,917		193		
28	27,724	83	250		
29	27,391	88	318		
30	26,985	97	394		
31	26,494	111	480		
32	25,903	127	572		
33	25,204	10,082			
34	15,122	2,420			
35	12,702	2,286			
36	10,416	10,416			

* For the first ten years, the table is based on the annual (independent) rates of disability, withdrawal and mortality of the UAW 1955 Tables; thereafter it is based on the annual rates of disability and retirement. For details of construction, see Part I, Example 2.

TABLE 2

SUMMARY OF PREMIUMS AND RESERVES FOR EXAMPLE 2

TYPE OF COMPONENT	BENEFIT				
	Retirement	Disability	Withdrawal	Death	Total
	Premium Components				
Independent	3.85	0.57	0.41	1.09	5.92
Dependent	6.11	0.08	−0.23	− 0.02	5.94
Loewy	6.11	0.08	−0.24	− 0.03	5.92
	Reserve Components at End of 10 Years				
Independent	50.0	7.4	5.4	14.2	77.0
Dependent	71.1	1.2	4.3	0.4	77.0
Loewy	71.0	1.3	4.2	0.4	76.9
	Reserve Components at End of 20 Years				
Independent	127.1	16.0	−0.7	28.9	171.3
Dependent	166.6	1.6	3.1	0.3	171.6
Loewy	166.5	1.8	2.8	0.3	171.4
	Reserve Components at End of 30 Years				
Independent	258.3	11.3	−1.7	25.6	293.5
Dependent	295.1	− 2.7	1.5	0.1	294.0
Loewy	294.8	− 2.4	1.1	0.1	293.6

NOTE.—The variation in the total figures is due to various approximations employed and to the limited number of significant figures in some of the basic tables.

DISCUSSION OF PRECEDING PAPER

T. N. E. GREVILLE:

This paper brings out a number of interesting and useful concepts and relationships between actuarial functions which I am afraid may be missed by many readers because of the formidable-looking mathematical dress in which they are presented. This discussion has been written in the hope of bringing a few of the basic ideas down to a mathematical level where they can be understood by the great majority of actuaries.

EXPRESSION FOR NET LEVEL PREMIUM IN TERMS OF NET COSTS

Equation I-(17) of the paper expresses an interesting relationship which applies to single decrement as well as multiple decrement tables. I do not recall seeing any specific discussion of it before, although it appeared in Question 2(*a*) of Part 5 of the Joint Examinations in 1942. For simplicity, we take mortality as the only decrement and consider a level-premium life insurance contract (possibly with the amount of insurance varying from year to year). We shall also make the traditional assumption that death claims are payable at the end of the policy year of death. If P denotes the net level annual premium, ${}_tV$ the tth terminal reserve, and ${}_tF$ the amount of insurance during the tth policy year, this relationship becomes

$$P = \left[\sum_{t=0}^{n-1} v^{t+1} q_{x+t} ({}_{t+1}F - {}_{t+1}V) + v^n \, {}_nV \right] \Big/ \ddot{a}_{\overline{n}|} . \qquad (1)$$

This shows that the net level premium is the level annual payment equivalent on the basis of interest alone (without benefit of survivorship) to the payment at the beginning of each policy year of the cost of insurance for that year for the net amount at risk, together with payment at maturity (in the case of an endowment policy) of the amount then due. A similar relationship would hold for a single premium policy; in this case, the denominator $\ddot{a}_{\overline{n}|}$ would be omitted from the right member.

This formula would not be useful for the calculation of premiums but could be employed as a check on the computation of reserves or costs of insurance.

"INDEPENDENT" AND "DEPENDENT" PREMIUMS AND RESERVES

The concept of "independent" and "dependent" premiums and reserves arises immediately out of a consideration of multiple causes of decrement. We are considering here an insurance contract or other arrangement under which certain benefits are payable on the occurrence of any one of the various contingencies represented by decrements in the table. The benefits payable may vary by cause of decrement as well as duration (and may be zero in some cases). The payment to be made at maturity (if any) is arbitrarily assigned to one of the causes of decrement, or arbitrarily apportioned among them. The problem is to break down the net level premium and also the reserve at any duration into a number of components corresponding to the different causes of decrement. The authors point out that this can be done in a variety of ways and they call attention to two of these which seem the most obvious.

I had considerable difficulty at first in grasping the significance of the "independent" and "dependent" premiums and reserves and am not sure that these are the best names to call them. However, I have not been able to think of better ones. I am going to explain them in a slightly different way than the authors, thinking that this approach may possibly be a little clearer to some readers, as it was to me.

Under the "independent" method the net level premium corresponding to the ith decrement is the level payment which is just sufficient, with interest and benefit of survivorship (as against all decrements), to provide all benefits payable on the occurrence of the ith decrement and to make the payment at maturity (if any) assigned to the ith decrement, under the assumption that no other benefits are payable. In other words, there is a complete forfeiture when any of the other decrements occurs. Thus, in the case of a retirement plan providing for certain benefits on retirement, death, or withdrawal the "independent" retirement component of the net premium would be the level premium for the retirement benefits alone, computed on the assumption that no payment of any kind is made on death or withdrawal. The "independent" method is so called because the independent component corresponding to the ith decrement does not depend on what benefits are payable on the occurrence of other decrements.

The "dependent" method lends itself less readily to a simple explanation. The "dependent" premium component corresponding to the ith decrement is the level payment which, with interest (but without any benefit of survivorship), will just provide the additional amounts needed (*i.e.*, over and above the *total* reserve) to provide the benefits payable on the occurrence of the ith decrement, as well as any maturity value associated with the ith decrement. If the benefit payable when the ith decrement occurs is consistently less than the total reserve, the corresponding dependent component of the net level premium may be negative. If the benefits payable on the occurrence of decrements other than the ith are changed, this will generally change the total reserves, and will therefore change the ith dependent component of the premium.

In slightly different words suggested by formula (1) of this discussion, the dependent component of the net level premium is the level payment which, at interest alone, is exactly equivalent to the payment of the net costs of insurance (based on the total reserve) against the ith decrement as they occur, and payment at maturity of any maturity value associated with the ith decrement.

A SIMPLE EXAMPLE

These concepts became much clearer to me after applying them to a very simple example—that of a plain n-year endowment insurance policy. Two decrements are considered: death and maturity. The second decrement (maturity) has a force of decrement equal to zero at all ages, but a payment at maturity (of the face amount) is associated with it.

If the level net premium for this contract is analyzed by the "independent" method the respective components are, as might be expected, the level net premiums for n-year term insurance and n-year pure endowment. By the "dependent" method the maturity component is $1/\ddot{s}_{\overline{n}|}$, the sinking fund payment to accumulate the face amount in n years, and the death component is the difference between the sinking fund payment and the level net premium for the policy.

SOME UNEXPECTED ACTUARIAL RELATIONSHIPS

If the principle of equation I-(17) of the paper is applied to an n-year temporary life annuity due, we obtain the relation

$$\sum_{t=0}^{n-1} v^{t+1} q_{x+t} \ddot{a}_{x+t+1:\overline{n-t-1}|} = \ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}. \tag{2}$$

This equation can also be obtained, after some reduction, by applying equation (1) of this discussion to an annual premium n-year endowment insurance.

The interpretation of equation (2) is not immediately obvious. By contrast, it is fairly clear that

$$\sum_{t=0}^{n-1} v^{t+1} \, {}_tp_x \, q_{x+t} \ddot{a}_{\overline{n-t-1}|} = \ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}.$$

It is at first a bit surprising that these two summations should be equal, since clearly the summands are not, in general, equal. However, if we consider the progress of the reserve on the temporary life annuity, it is seen that this reserve suffers

(i) an annual increment from interest,

(ii) an annual decrement from disbursement of the annuity payment,

(iii) a further annual increment due to reserves released by death.

Inasmuch as the decrement (ii) exceeds the sum of the increments (i) and (iii), the reserve decreases, over the n-year period, from $\ddot{a}_{x:\overline{n}|}$ to zero. The annuity certain is affected also by the increment (i) and the decrement (ii), but the increment (iii) is lacking.

To compute the increment (iii), we note that the reserves at the beginning and at the end of the $(t+1)$th contract year are given by $a_{x+t:\overline{n-t-1}|}$ and $\ddot{a}_{x+t+1:\overline{n-t-1}|}$, respectively, and these are connected by the relation

$$(1+i)\, a_{x+t:\overline{n-t-1}|} = p_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}\,,$$

or

$$(1+i)\, a_{x+t:\overline{n-t-1}|} + q_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|} = \ddot{a}_{x+t+1:\overline{n-t-1}|}\,.$$

The second term of the left member therefore represents the increment due to reserves released by death. Since the excess of the present value of the annuity certain over that of the temporary life annuity must equal the discounted value of all such increments (iii), we obtain equation (2).

A similar analysis of an n-year forborne life annuity leads to the equation

$$\sum_{t=0}^{n-1} (1+i)^{n-t-1} q_{x+t}\,\ddot{s}_{x:\overline{t+1}|} = \ddot{s}_{x:\overline{n}|} - \ddot{s}_{\overline{n}|}\,,$$

which may be compared to the relation

$$\sum_{t=0}^{n-1} (1+i)^{n-t-1} q_{x+t}\,\ddot{s}_{\overline{t+1}|} / {}_{n-t-1}p_{x+t+1} = \ddot{s}_{x:\overline{n}|} - \ddot{s}_{\overline{n}|}\,.$$

Analysis along the same lines of the single premium for an n-year pure endowment gives, after some simplification,

$${}_nq_x = \sum_{t=0}^{n-1} q_{x+t} \cdot {}_{n-t-1}p_{x+t+1}\,, \tag{3}$$

which may be compared with

$${}_nq_x = \sum_{t=0}^{n-1} {}_tp_x\, q_{x+t}\,. \tag{4}$$

Equation (3) is easily verified algebraically, since $q_{x+t} = 1 - p_{x+t}$ and therefore the right member of equation (3) is equal to

$$\sum_{t=0}^{n-1} ({}_{n-t-1}p_{x+t+1} - {}_{n-t}p_{x+t}) = 1 - {}_np_x .$$

It can be interpreted in terms of a single premium n-year term insurance at a zero interest rate, but I do not see any verbalization in terms of probability relationships, such as is available for equation (4).

NOTATION

In the last five years there have appeared two official textbooks of the Society of Actuaries (those of Jordan and Spiegelman) using different notations for functions in multiple decrement tables and the associated single decrement tables. This paper presents still a third notation. As there appears to be increasing interest in the subject, it would be desirable, in my opinion, to reach agreement on a uniform system of notation.

HARWOOD ROSSER:

This paper sets a laudable precedent, which I, for one, hope will be followed, of digesting an academic dissertation, on a subject of great interest to actuaries, which would be read by very few in its original form. And who is better prepared, and more interested in doing this, than the writer of the dissertation and his faculty advisor? This is a function that a reviewer cannot hope to accomplish, for reasons of space limitation, if for no other. The authors are greatly to be complimented on reducing to more readable form what is probably, for most of us, a formidable treatise. They may have paraphrased the late Huey Long and taken as their slogan: "Every man his own popularizer."

Also, as one who has taught classes in life contingencies, I should like to see some sections of this paper added to the examination syllabus, at least on an optional basis, as by a reference in the Study Notes. For instance, their verbal interpretation of formula II-(10) offers the most satisfactory answer I have seen to a natural question that arises in the mind of the thoughtful Part 4 student. In fact, I have heard debates on it. (I refuse to admit whether or not I was one of the debaters.)

In this condensation, emphasis is placed upon the "decomposition" of a total premium, which it is supposed has already been calculated. However, except for the "dependent" approach, a process of adding one benefit at a time, or of building up the premiums and reserves, can be followed. For that matter, this is applicable also to the "dependent" case; but, as the name implies, the components will change whenever a new benefit is included. The authors hint that such a synthesis is possible, and no doubt this is more fully treated in the original work. My reason for mentioning this will appear later.

My greatest interest in this paper, however, is as a consulting actuary primarily concerned with pensions. The paper sheds a great deal of light on the actuarial theory involved in some of the variations thereof, especially as regards vesting. This is coming more and more to the forefront, as a result of labor union pressure.

To illustrate, we have on our books clients whose plans constitute interpretations of each of the four cases under *Comparison of Independent and Dependent Components* in Part III. These, with comments, are as follows:

a) There is no vesting. Hence a gain occurs on withdrawal. This gives rise to the negative dependent premium component.

b) This could be a self-insured plan with a death benefit equal to the reserve. Then there exists a certain analogy to Fassel-type coverage; that is, insurance for face amount or reserve, if greater. It is common to say that "mortality drops out" of the latter after a certain point, which means that it becomes a calculation involving interest only (*i.e.*, one less decremental force). The analogy would be complete if the Fassel plan used the reserve as the death benefit throughout. In a sense, as the algebra shows, the inclusion of the death benefit does not alter the total level cost. Presumably, the same result would be obtained by computing figures using one less decrement—*i.e.*, on a single-decrement table with the complementary force $\mu_{x+t}^{(-i)}$. This has computational advantages. For instance, if the plan takes account of mortality and withdrawal, then the annual cost would be calculated from single-decrement functions involving withdrawal only. If withdrawal rates and benefits are ignored, the cost calculations depend only on interest, as above.

c) This is a basic pension plan, where benefits are provided only upon survival to retirement age, although recognition is given in the calculations to termination by death, withdrawal, and perhaps disability. The premium and reserve equalities shown would be expected.

d) Here we have a fully vested plan. (To meet the conditions, it would probably have to be trusteed. Under an insured plan, the full reserve would usually not always be available at surrender. Deferment until retirement age of vested pensions for withdrawing employees might also be deemed to invalidate the conditions. So also might dividends.) This is a logical extension of the situation under (*b*). It reduces to a single-decrement table problem. In fact, if all retirements are at the same age (or if early retirements are handled by actuarial reduction of the pension—*i.e.*, by granting the reserve at early retirement), it reduces to a compound interest problem, as previously.

Some of these ideas, especially those in (*b*) above, foreshadow what is to come in Part IV, immediately following (8). This is, to a large extent, a clearer restatement of earlier remarks. In passing, we note that (7) and (8) suggest approximate methods of obtaining figures when correct calculations would be unduly laborious.

Many members of the Society have already encountered the Loewy technique, although not by that name, in Jenkins' paper in *RAIA* XXI: "Non-participating Premiums Considering Withdrawals." This has been required reading for Fellowship students for a number of years. His Table I shows the results of a direct application of the Loewy method.

Jenkins' Table III may be an example of decomposition of a total premium into "dependent" components, although this is less clear. The picture is somewhat confused by the expense element. It would be possible to regard expenses as a sort of benefit payable, not once, at the occurrence of a particular decrement, but yearly upon survival. However, it seems simpler to consider the net premium excluding expenses.

Of the three methods of decomposition into components, the dependent premiums and reserves, although more difficult to compute, make more sense to this reader. After all, upon termination the entire reserve is taken down, at least at the next valuation.

Also, this approach gives a truer picture of the cost allocation. Not uncommonly, when a consultant is working on a proposed pension plan, or the revision of an existing one, he is asked: "How much extra will it cost to add a death benefit, a disability benefit, perhaps some graded vesting on withdrawal, or some combination thereof?" A glance at the authors' Table 2 will furnish adequate commentary on the inappropriateness of using "independent" premiums. The Loewy method requires the client to answer in advance such questions as: "If you can't afford both, would you rather have the death benefit or the disability coverage?" The usual—and obvious—answer is: "That depends on their relative costs!"

Perhaps it is an incorrect inference, from the fact that it is dealt with last, that the authors regard the Loewy technique as the preferred one. For me, at least, it is not, for the reasons given. Computationwise, it has advantages over the "dependent" method; but I would normally regard it merely as an approximation to the latter (despite the authors' warning that it is not necessarily a good one). Even this I will not resort to if I can add "dependent" premiums to the long list of pension calculations we have adapted, with permanent instruction boards, for our IBM installation.

In any event, the actuarial profession is deeply indebted to these two men for presenting a new (and, in my opinion, preferable) method of premium decomposition (and synthesis), for so ably comparing it with those already in use, and for illuminating the whole subject of multiple decrement theory. Possibly they were wise to leave us to our own choice of method.

MARJORIE V. BUTCHER:

In their comprehensive and scholarly paper, Messrs. Bicknell and Nesbitt present a generalized approach to net premiums and reserves in multiple decrement theory. Their work is based upon repeated use of a set of basic differential equations which analyze by cause the expected approximate instantaneous increase in either the total policy reserve or a component of it. Upon application of an integrating factor, integration over the entire term yields an expression for the appropriate premium while integration over the first t or last $(n-t)$ years of the term yields the tth reserve.

For such differential equations several convenient integrating factors exist, including $D_{x+t}^{(T)}$, $D_{x+t}^{[i]}$, $D_{x+t}^{(-i)}$, D_{x+t}^{i}, $l_{x+t}^{(T)}$, $l_{x+t}^{[i]}$, $l_{x+t}^{(-i)}$ and v^t. Making a game out of obtaining premiums and reserves by such means, one may express quite an array of results. Those containing $D_{x+t}^{(T)}$ usually have the most familiar appearance, those with $D_{x+t}^{[i]}$ may be comparable with corresponding single decrement results, etc.; and among the factors, in each case one will offer computing advantages. I found it interesting to apply $l_{x+t}^{(T)}$ to I-(13), II-(3), III-(1) and IV-(6). There result, in turn,

$$\bar{P}^{(T)}(T_x^{(T)} - T_{x+n}^{(T)}) + \delta\int_0^n {}_t\bar{V}^{(T)} l_{x+t}^{(T)}\,dt = \sum_{i=1}^{m}\int_0^n B_{x+t}^{(i)} l_{x+t}^{(T)} \mu_{x+t}^{(i)}\,dt + B_{x+n}^{(T)} l_{x+n}^{(T)} \tag{1}$$

$$^{a}\bar{P}^{(i)}(T_x^{(T)} - T_{x+n}^{(T)}) + \delta\int_0^n {}_t^{a}\bar{V}^{(i)} l_{x+t}^{(T)}\,dt = \int_0^n B_{x+t}^{(i)} l_{x+t}^{(T)} \mu_{x+t}^{(i)}\,dt + B_{x+n}^{(i)} l_{x+n}^{(T)} \tag{2}$$

$$^{b}\bar{P}^{(i)}(T_x^{(T)} - T_{x+n}^{(T)}) + \delta\int_0^n {}_t^{b}\bar{V}^{(i)} l_{x+t}^{(T)}\,dt = \int_0^n (B_{x+t}^{(i)} - {}_t^{b}\bar{V}^{(-i)})\, l_{x+t}^{(T)} \mu_{x+t}^{(i)}\,dt + B_{x+n}^{(i)} l_{x+n}^{(T)} + \int_0^n {}_t^{b}\bar{V}^{(i)} l_{x+t}^{(T)} \mu_{x+t}^{(-i)}\,dt \tag{3}$$

$$
\begin{aligned}
{}^{L}\bar{\mathrm{P}}^{(i)}\left(\mathrm{T}_x^{(T)}-\mathrm{T}_{x+n}^{(T)}\right)+\delta\int_0^n {}_t^{L}\bar{\mathrm{V}}^{(i)} l_{x+t}^{(T)}\,dt
&=\int_0^n \left(B_{x+t}^{(i)}-{}_t\bar{\mathrm{V}}^{\underline{i-1}}\right) l_{x+t}^{(T)}\mu_{x+t}^{(i)}\,dt+B_{x+n}^{(i)} l_{x+n}^{(T)} \qquad (4)\\
&\quad+\sum_{j=i+1}^{m}\int_0^n {}_t^{L}\bar{\mathrm{V}}^{(i)} l_{x+t}^{(T)}\mu_{x+t}^{(j)}\,dt .
\end{aligned}
$$

According to survivorship group theory, (1) expresses the equivalence of the aggregate income and outgo of the whole policy, without discounting factors; the other equations do likewise relative to the ith benefit-decrement component. Incidentally, after comparing equations (3) and (4), one is not surprised when corresponding Loewy and dependent basis components in Example 2 of the paper, with the primary benefit ordered first, are nearly identical.

In some situations it would be convenient to allow the added generality of premiums varying with time. For this purpose one would alter condition II-(1) by requiring that the sum of all component premiums paid at any time equal the total premium paid at that time. One soon discovers that, even under simple schemes of premium variation, difficulties arise. For instance, if one wished each component premium to cease when its benefit period expired, thus introducing variation in only the total premium, in general at any time one would have different total premiums for independent, dependent and other bases of splitting. Besides, under the dependent basis, cessation of a negative dependent premium would actually increase the total premium. One type of varying premiums which fits the theory is a step function total premium which assumes various constant multiples of the initial premium from time to time, with each component premium in an invariant ratio to the total premium.

The Loewy formulas IV-(8) and (9) give a mathematical demonstration that if a surrender (withdrawal) benefit equals the full policy value resulting from the other benefits and their decrements, inclusion of the withdrawal benefit and decrement in calculations does not alter results. The Loewy theory is of value in changing or dropping a benefit and its related decrement, while the independent theory readily permits changing or dropping a benefit while retaining its decrement. Need for such alterations arises with respect to pensions and social security.

The authors take full advantage of the relative ease of thought and expression which the continuous basis affords. This writer appreciates their well-organized paper treating the essence of actuarial mathematics.

(AUTHORS' REVIEW OF DISCUSSION)

WILLIAM S. BICKNELL AND CECIL J. NESBITT:

A full reply to all the points raised by those who have discussed our paper would be somewhat difficult and lengthy. Our reply will be limited to some immediate remarks and to expressing our thanks to the discussers for the interest they have shown and the ideas they have contributed.

Dr. Greville presented an explanation and illustration of the concepts of "independent" and "dependent" premium and reserve components. To our minds, the essential point is that for the independent system each component premium and reserve is applied only toward its corresponding benefit while, for the dependent system, whenever a benefit becomes payable the total reserve for all benefits, and not the component reserve only, will be available to offset the payment. In some cases, the independent premium component corresponding to the ith cause of decrement may be affected by the benefits payable in respect to the other decrements, as happens, for instance, when the ith benefit is related to the total reserve.

Both Dr. Greville and Mr. Rosser consider the methods of the paper in relation to ordinary insurances and annuities. The methods are applicable on a discrete as well as continuous basis, and may be extended to cases not included in the paper—for instance, single premium situations. This is well illustrated in Dr. Greville's discussion. It may be noted from formula IV-(7) of the paper that if there are only two components, the first of which is associated with a zero force of decrement (that is, interest only), and the second with mortality, then the Loewy components will coincide with the dependent components. For example, formulas (2) and (3) of Dr. Greville's discussion may be obtained in this way by the Loewy method applied to the single premiums for a temporary annuity-due and a pure endowment.

We are indebted to Mr. Rosser for his reference to W. A. Jenkins' paper in *RAIA* XXI: "Non-participating Premiums Considering Withdrawals," as it suggests a good numerical illustration of the three methods. In regard to Jenkins' Table III, we would classify the decomposition as being according to the independent method. By the dependent method, one would have for the twenty year endowment premium the components

$$\sum_{t=0}^{19} v^t \mathrm{E}_{t+1} \Big/ \ddot{a}_{\overline{20}|}, \qquad \sum_{t=0}^{19} v^{t+1} q_{[x]+t}^{(w)} \left(s_{[x]+t}^{\mathrm{E}_{20}} - {}_{t+1}\mathrm{V} \right) \Big/ \ddot{a}_{\overline{20}|},$$

$$\sum_{t=0}^{19} v^{t+1} q_{[x]+t}^{(d)} \left[(1+i)^{1/2} - {}_{t+1}\mathrm{V} \right] \Big/ \ddot{a}_{\overline{20}|}, \qquad v^{20} / \ddot{a}_{\overline{20}|},$$

for expenses, surrender values, mortality and maturities, respectively, where ${}_{t+1}V$ denotes the asset share considering lapses as given in Jenkins' Table V, and E_{t+1}, $s^{E_{20}}_{[x]+t}$ denote the expenses and cash value for policy year $t+1$. The numerical results, with those of Jenkins' given for comparison, are shown in the accompanying table. It is to be noted that, as

DISTRIBUTION OF 20 YEAR ENDOWMENT GROSS ANNUAL PREMIUM PER $1,000 BY ELEMENTS OF COST

ELEMENT	INDEPENDENT COMPONENTS (AS GIVEN BY JENKINS)		DEPENDENT COMPONENTS	
	Amounts	Percentages	Amounts	Percentages
Expenses............	$ 9.20	22%	$ 7.07	17%
Surrender values.......	12.79	30	−.08	0
Mortality............	7.69	18	4.82	11
Maturities............	12.59	30	30.50	72
Total............	$42.27	100%	$42.31	100%

in the case of Example 2 in the paper, the dependent components give a much different analysis of the total premium than that provided by the independent components. For the purpose of gauging the effect of surrender values, the dependent method seems more appropriate than the independent method. As pointed out by Mr. Rosser, Jenkins' Table I gives a partial Loewy analysis, and it is interesting to note that, for the endowment premium, the Loewy component of $−.18 for withdrawal is close to the dependent component of $−.08.

Mrs. Butcher has caught our fascination with the variety of formulas available and has even gone on to add some of her own. She raises interesting questions about extending the methods to cases with premiums varying with time. If the insurance provides a somewhat loose aggregate of more or less unrelated benefits, and if premium terms are of various lengths, then the independent method may be the only one conveniently applicable. If, however, the insurance is a coherent set of interrelated benefits, and if the total premium and reserves may be obtained by direct means (that is, other than by computing the independent components and taking the sum), then the dependent approach may be useful. The Loewy method may be intermediate to the independent and dependent methods in respect to flexibility and adaptability.

These last remarks have bearing on Mr. Rosser's consideration of the practical implications of the dependent approach for pension computations. In case of complete vesting on termination before retirement, the dependent method is the simplest and the natural one to use. In cases where vesting is not complete, as in Examples 1 and 2 of the paper, but where the total reserves may still be computed directly (possibly on a discrete basis), the dependent method may retain advantage. It seems, however, that we have not made actuarial matters any simpler for the pension client. To the present choice of assumptions and of funding method we have added the possibility of choice as to how the funding method will be applied, namely on an independent, dependent or Loewy basis. We hasten to assure Mr. Rosser that we have no particular preference for the Loewy method, and that it appeared last in our paper because it is affected by the order of the components while the other two methods are not. However, we were intrigued by the dualism of formulas IV-(7) and IV-(8). In general, our interest was primarily in the theoretical relationships and only secondarily in applications. In this regard it is interesting that the four theoretical cases we used to compare independent and dependent components have practical pension fund counterparts.

The matter of notation has been brought up by Dr. Greville and also in a letter from Mr. Eugene Rasor. Our notation was based mainly on that used by Jordan, in the *Life Contingencies* text, with the exception that we used square brackets rather than a prime in the superscript to denote functions for the associated single decrement tables. The suggestion has been made that brackets be omitted as much as possible from the superscripts. We are somewhat against this suggestion as the superscript i may possibly be interpreted as an exponent or as indicating a disabled life function. We agree that some attempt to simplify and standardize multiple decrement notation should be made, but in view of the ramifications of pension funding functions it may be a difficult task.

Again, our thanks and appreciation to the discussers of the paper.