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ADDING OR INCREASING SUBSTANDARD EXTRAS ON POLICY CHANGES

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SINCE Mr. Hoskins' admirable papers on policy change practices some years ago[1] there has been little on this subject in our actuarial literature. I hope this brief note will be of interest to those involved in policy changes and that perhaps it will stimulate further discussions.

The theories and formulas in this note were developed during a review of our practices on adding or increasing substandard extra premiums on individual policies. This may occur on a policy change where the amount at risk is increased, requiring evidence of insurability; on reinstatement of a lapsed policy; or on converting an annual premium retirement annuity to a life insurance contract as of original date.

While I hope that the formulas developed here will be of practical use, the development is primarily along theoretical lines. Therefore, net functions will be used throughout. Furthermore, I will assume that the underwriters have determined the proper mortality classification after giving due consideration to the policy duration.[2]

It will simplify our approach if we consider conversion of an annual premium retirement annuity to a life policy where it has been determined that the policy is a substandard risk. Since no insurance has previously been in force we are concerned only with the extra mortality charges for the future.

There are, in general, two methods of charging for extra mortality:

1. An extra annual premium equal to the substandard extra annual premium which would have been charged as of the original date, plus a single premium charge on making the change equal to the excess of the substandard reserve over the standard reserve (or in the case of an increase in substandard mortality classification, the difference between the new and old substandard reserves).
2. No single premium charge but an annual extra based on the attained age and the future amount at risk.

These two methods are readily recognized as comparable to the two in general use for such changes as adding a supplementary benefit to a policy or changing to a higher premium plan.

[1] *TASA* XXV, 253; XXXV, 60.

[2] See "Reinstatement of Policies" by R. D. Murphy, *TASA* XVIII, 246.

While there may be some practical problems in the use of method 1 when adding a substandard extra premium (*e.g.*, negative reserves), the theory is routine and this discussion is therefore concerned only with method 2.

Using standard notation, with primed symbols to indicate substandard mortality functions, the net substandard extra premium to be added at the end of t years to an m-payment life policy issued at age x may be obtained by equating the reserve plus the value of future premiums to the value of future benefits thus:

$$(E + {}_m\mathrm{P}_x)\, \ddot{a}'_{x+t:\overline{m-t}|} + {}^m_t\mathrm{V}_x = \mathrm{A}'_{x+t}\,,$$

where E is the substandard extra to be added.

Solving for E and simplifying, we obtain the value of the extra premium as:

$$E = {}_{m-t}\mathrm{P}'_{x+t} - \frac{{}^m_t\mathrm{V}_x}{\ddot{a}'_{x+t:\overline{m-t}|}} - {}_m\mathrm{P}_x\,,$$

which might have been used as a starting point by general reasoning since it is obviously equal to the premium for a new limited payment life policy (substandard) to be full paid at the same age as the original policy, reduced both by the present reserve amortized over the balance of the premium paying period and by future standard premiums on the original basis.

By using the relationships:

$$\frac{1}{\ddot{a}} = \mathrm{P} + d$$

$${}^m_t\mathrm{V}_x = ({}_{m-t}\mathrm{P}_{x+t} - {}_m\mathrm{P}_x)\, \ddot{a}_{x+t:\overline{m-t}|}$$

the above formula becomes:

$$E = ({}_{m-t}\mathrm{P}'_{x+t} - {}_{m-t}\mathrm{P}_{x+t}) - {}^m_t\mathrm{V}_x\,(\mathrm{P}'_{x+t:\overline{m-t}|} - \mathrm{P}_{x+t:\overline{m-t}|})\,. \qquad (1)$$

This says that the extra premium is the attained age extra for a limited payment life policy for the full face amount, to be fully paid at the same age as the original policy, less the extra premium on an endowment policy with a face amount equal to the original limited payment life policy reserve, such endowment policy to run to the same age as the original policy.

A very similar formula may be derived for term insurance:

$$E^{\mathrm{Term}} = (\mathrm{P}'^{\,1}_{x+t:\overline{m-t}|} - \mathrm{P}^{\,1}_{x+t:\overline{m-t}|}) - {}_t\mathrm{V}^{\,1}_{x:\overline{m}|}\,(\mathrm{P}'_{x+t:\overline{m-t}|} - \mathrm{P}_{x+t:\overline{m-t}|})\,. \qquad (2)$$

For an ordinary life policy the formula becomes:

$$E^{OL} = (P'_{x+t} - P_{x+t})\ (1 - {}_tV_x)\,, \tag{3}$$

which is simply the attained age extra times the amount at risk.

Verbally an endowment policy extra is identical to that for ordinary life with the proviso that the age at maturity remains unchanged. The formula is not difficult to derive:

$$E^{End} = (P'_{x+t:\overline{m-t}|} - P_{x+t:\overline{m-t}|})\ (1 - {}_tV_{x:\overline{m}|})\,. \tag{4}$$

All of the above formulas assume that t is less than m. In the case of an endowment policy it is obviously inapplicable for $t \geq m$. For a limited payment life policy it is equally obvious that the extra to be added is the single premium extra if all premiums have been paid (if $t \geq m$). This is also evident where one annual premium remains ($m - t = 1$) and in fact formula (1) becomes just that:

$${}_1P'_{x+t} - {}_1P_{x+t}\,.$$

There are some interesting relationships in connection with formula (3). Consider the following two ordinary life policies:

Policy A: Taken out at age x with a unit face amount.

Policy B: Taken out at age $x + t$ with a face amount equal to the amount at risk on Policy A at age $x + t$.

In originally arriving at the extra to be added I reasoned that the amount at risk on and after age $x + t$ on Policy A would be the same as that under Policy B. If this were true, then the extra premium to provide for any extra mortality must be the same: thus formula (3).

The fact that the amounts at risk were the same seemed evident, but for proof it was only necessary to prove that the reserves on both policies would increase by the same amount in each year. This would be proved if the premium on Policy B were equal to the premium on Policy A plus interest in advance on the Policy A reserve, thus:

$$P_{x+t}(1 - {}_tV_x) = P_x + d\,{}_tV_x\,.$$

To prove that this equality is true, it may be reduced to an identity by using the following relationships:

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

$$A_{x+t} = 1 - d\,\ddot{a}_{x+t}\,.$$

It appears clear by inductive reasoning that the amount at risk in future years will be the same on both Policy A and Policy B.

From the standpoint of practical application, many complications arise. Since most of these must be solved on the basis of what is available in the particular company and to what extent theoretical accuracy must give way to simplicity of rules, the following brief comments should be sufficient.

The first problem is how to determine the amount at risk. We have been using the cash value instead of the reserve, partly because it is readily available. At least as important a reason for using the cash value touches on another phase of policy change theory; however, suffice it to mention here that we shifted from the use of the reserve to the use of the cash value wherever there was no contractual restriction, because the cash value is in effect an asset share and therefore represents the policyholder's true equity.

Formulas (3) and (4) are very simple in operation so that for ordinary life policies and endowments the theoretical formulas may be followed. For limited payment life plans, it may very well be desirable to sacrifice accuracy and use the limited payment attained age life extra premium times the amount at risk, thus:

$$({}_{m-t}P'_{x+t} - {}_{m-t}P_{x+t})\ (1 - {}^{m}_{t}V_x)\ .$$

Care must be taken here (as in endowments) to select the proper premium paying period.

It should be mentioned that since this note is concerned with policy changes, it has no direct connection with Mr. Shur's paper on the calculation of extra premiums.[3]

[3] *TSA* VI, 99.

MARJORIE V. BUTCHER:

Mr. Holcombe has written a very interesting paper presenting actuarial theory for the introduction or increase of substandard extra premiums at the time of a policy change. He has added meaning to his equations by giving careful verbal explanation of them. For the interpretation of his formula (3) by use of policies A and B, he stated that "for proof it was only necessary to prove that the reserves on both policies would increase by the same amount in each year." In the policy year $r+1$ of Policy B the reserve increases by $(1-{}_tV_x)({}_{r+1}V_{x+t}-{}_rV_{x+t})$, corresponding to an increase of ${}_{t+r+1}V_x-{}_{t+r}V_x$ under Policy A in the same year (its year $t+r+1$). From

$$ {}_hV_y = 1-\frac{\ddot{a}_{y+h}}{\ddot{a}_y}\,, \qquad \text{(i)} $$

the equality of the year to year increase in their reserves follows, namely

$$ (1-{}_tV_x)\,({}_{r+1}V_{x+t}-{}_rV_{x+t}) = {}_{t+r+1}V_x-{}_{t+r}V_x\,, \qquad \text{(ii)} $$

a somewhat amazing equation.

For the general m-payment n-year endowment policy, we have by Mr. Holcombe's methods

$$ E = {}_{m-t}P'_{x+t:\overline{n-t}|}-{}_{m-t}P_{x+t:\overline{n-t}|}-{}^m_tV_{x:\overline{n}|}\,(P'_{x+t:\overline{m-t}|}-P_{x+t:\overline{m-t}|})\,, \qquad \text{(iii)} $$

where $t<m$. This form specializes upon choice of m and n to any life or endowment policy of uniform amount with uniform premiums; for example, to the author's equations (1), (3) and (4). For this general policy I wished to investigate the error introduced by following the author's suggestion (made for limited payment life plans) that the form of E *correct* for ordinary life and m-year endowment plans might be *assumed* for other plans. We thus have to compare the assumed substandard extra, E^a, namely

$$ E^a = ({}_{m-t}P'_{x+t:\overline{n-t}|}-{}_{m-t}P_{x+t:\overline{n-t}|})\,(1-{}^m_tV_{x:\overline{n}|}) \qquad \text{(iv)} $$

with (iii). The magnitude of the error, E^a-E, may be reduced to the equivalent forms

$$ E^a-E = d\cdot{}^m_tV_{x:\overline{n}|}\left[\frac{\ddot{a}'_{x+m:\overline{n-m}|}}{\ddot{s}'_{x+t:\overline{m-t}|}}-\frac{\ddot{a}_{x+m:\overline{n-m}|}}{\ddot{s}_{x+t:\overline{m-t}|}}\right] \qquad \text{(v)} $$

$$ = d\cdot{}^m_tV_{x:\overline{n}|}\left[\frac{\ddot{a}'_{x+t:\overline{n-t}|}}{\ddot{a}'_{x+t:\overline{m-t}|}}-\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x+t:\overline{m-t}|}}\right]. \qquad \text{(va)} $$

Circumstances of individual policies and mortality tables would determine whether E^a is deficient or excessive. In the very special case $n - m = 1$, the bracketed quantity in (v) would be negative; thus, if reserves were positive, E^a would be deficient. Likewise, in the event $m - t = 1$, E^a is less than E as equation (v*a*) shows, and from (iii) and (iv) we find that

$$E = A'_{x+t:\overline{n-t}|} - A_{x+t:\overline{n-t}|}\,, \qquad \text{(iii}a\text{)}$$

$$E^a = E\,(1 - {}^{m}_{t}V_{x:\overline{n}|})\,. \qquad \text{(iv}a\text{)}$$

Whenever $n = m$ in (v), $E^a = E$; *i.e.*, as we already knew, E^a is a correct expression for the substandard extra for an m-year endowment and hence also for an ordinary life policy. For an m-payment life policy with positive reserves, it can be shown by work with the bracketed portion of (v) or (v*a*) that $E^a - E$ is positive (or zero, or negative) if

$$\frac{N'_{x+m}}{N_{x+m}} - \frac{N'_{x+t}}{N_{x+t}}$$

is also.

Similarly, one can obtain for term insurance the expression

$$E^a - E = {}_tV^{1}_{x:\overline{m}|}\,(P'^{\;1}_{x+t:\overline{m-t}|} - P^{\;1}_{x+t:\overline{m-t}|}) = {}_tV^{1}_{x:\overline{m}|}\left(\frac{1}{\ddot{s}'_{x+t:\overline{m-t}|}} - \frac{1}{\ddot{s}_{x+t:\overline{m-t}|}}\right). \qquad \text{(vi)}$$

In this instance, if reserves are positive and primed mortality is the greater, use of the simpler, assumed extra, E^a, is always nonconservative, as indicated by the last member of (vi).

The ideas of the paper carry over precisely if the Commissioners Reserve Valuation Method is used, merely by inserting Commissioners for net level reserves in each formula; *e.g.*, (1) becomes

$$E = {}_{m-t}P'_{x+t} - {}_{m-t}P_{x+t} - {}^{m}_{t}V^{C}_{x}\,(P'_{x+t:\overline{m-t}|} - P_{x+t:\overline{m-t}|})\,. \qquad \text{(vii)}$$

In fact, forms analogous to (vii) exist whenever one is using modified or adjusted premium reserves with level renewal premiums.

My appreciation is extended to the author for his revival of the topic of policy changes in actuarial literature through this fine note on substandard extras.

AUTHOR'S REVIEW OF DISCUSSION

Mrs. Butcher's discussion provides some valuable additions to my paper. There is just one of her formulas on which I should like to comment.

The magnitude of error which she discusses may be written directly from her equations (iii) and (iv) as:

$$E^a - E = -{}_t^m\mathrm{V}_{x:\overline{n|}}\left[\left({}_{m-t}\mathrm{P}'_{x+t:\overline{n-t|}} - {}_{m-t}\mathrm{P}_{x+t:\overline{n-t|}}\right) - \left(\mathrm{P}'_{x+t:\overline{m-t|}} - \mathrm{P}_{x+t:\overline{m-t|}}\right)\right].$$

Expressed verbally, the error is a negative amount equal to the product of the reserve at the time of change and the difference between the attained age extra for an $(m - t)$-payment $(n - t)$-year endowment and the attained age extra for an $(m - t)$-year endowment. Since, in general, both the reserve and the "difference" will be positive, the approximation understates the true cost. It becomes worse as an approximation as the reserve becomes larger and as the number of future premiums $(m - t)$ becomes smaller.

I wish to thank Mrs. Butcher for her interest in my paper and her discussion of it.