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A GENERAL METHOD OF CALCULATING EXPERIENCE NET EXTRA PREMIUMS BASED ON THE STANDARD NET AMOUNT AT RISK

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INTRODUCTION

THIS note presents a short method of calculating without approximation substandard annual extra premiums based on the standard net amount at risk. Realistic values of such extra premiums can be obtained if the net amount at risk is taken as the face amount less the standard valuation table reserve and the extra mortality is measured against a standard experience mortality table. The exact calculation of extra premiums on such a basis by the usual method is a tedious process involving lengthy summations. For this reason, values are seldom obtained at more than a few representative points.

The method proposed in this note, by means of specially constructed auxiliary commutation columns, eliminates the need for any summation and produces exact values of any extra premiums by relatively simple formulas. The method applies to all of the usual plans of insurance and can be used if the extra mortality is constant, a percentage of the standard table, or entirely independent of the standard table. Formulas are developed in the appendixes which are applicable if the net amount at risk is based on the cash value (computed by the standard nonforfeiture method) or if continuous functions are used.

One large company has used this method to obtain experience extra premiums which were used as a guide in establishing the level of substandard extra premium rates.

The saving in calculations offered by the proposed method is illustrated for an ordinary life plan by the following formulas for the extra premium π_x:

Usual Method $$\pi_x = \frac{v}{N_x^B} \sum_{t=0}^{\infty} D_{x+t}^B \left(q_{x+t}^B - q_{x+t}^A \right) \left(1 - {}_{t+1}V_x^C\right)$$

Proposed Method $$\pi_x = \frac{N_x^D}{N_x^B} \left[P_x^D - P_x^C \right]$$

where the superscript indicates that the actuarial function is to be determined on the particular mortality table as defined below:

A—standard experience table

B—substandard experience table

C—valuation table for standard business

D—special auxiliary mortality table

An approximate method of calculating extra premiums based on the net amount at risk is given by Bassett in his paper, "Extra Premiums Based on the Net Amount at Risk."[1] The method given there, while shorter than the one presented in this note, is applicable only if the extra mortality is constant and is measured against the valuation table.

APPLICATION OF THE METHOD

The first step is the construction of a special mortality table D by means of the following equations, starting with an arbitrary radix l_a^{D}, a being the lowest age needed:

$$d_x^{\mathrm{D}} = l_x^{\mathrm{D}} q_x^{\mathrm{C}} + l_x^{\mathrm{B}}(q_x^{\mathrm{B}} - q_x^{\mathrm{A}}) \tag{1}$$

$$l_{x+1}^{\mathrm{D}} = l_x^{\mathrm{D}} - d_x^{\mathrm{D}} . \tag{2}$$

Equations (1) and (2) are followed in a strictly algebraic sense and any negative values produced by these equations are used in succeeding steps. Thus, it is possible for values of l_x^{D} and d_x^{D} to be negative. Equation (1) can be simplified for calculation purposes if a simple relationship exists between tables A and B. For example, if the extra mortality is constant we have

$$q_x^{\mathrm{B}} = q_x^{\mathrm{A}} + k$$

$$d_x^{\mathrm{D}} = l_x^{\mathrm{D}} q_x^{\mathrm{C}} + k l_x^{\mathrm{B}},$$

while if the extra mortality is a percentage of the standard table, we have

$$q_x^{\mathrm{B}} = (1 + k) q_x^{\mathrm{A}}$$

$$d_x^{\mathrm{D}} = l_x^{\mathrm{D}} q_x^{\mathrm{C}} + \left(1 - \frac{1}{1+k}\right) d_x^{\mathrm{B}} .$$

Commutation functions for table D are constructed by the usual formulas and may be negative. Net premiums on table D are expressed in terms of commutation symbols in the usual manner.

[1] *TSA* II, 1.

With the necessary commutation functions for tables B, C and D available, π can now be obtained from

$$\pi = \frac{N_x^D - N_{x+n}^D}{N_x^B - N_{x+n}^B}\,[P^D - P^C] \tag{3}$$

which reproduces exactly, for any plan of insurance with a death benefit of one, values given by

$$\pi = \frac{v}{N_x^B - N_{x+n}^B} \sum_{t=0}^{m-1} D_{x+t}^B\,(q_{x+t}^B - q_{x+t}^A)\;(1 - {}_{t+1}V^C)\;, \tag{4}$$

where the following definitions apply to the particular plan of insurance being considered:

π = level annual extra premium based on the net amount at risk
x = age at issue
n = number of years in premium paying period
m = number of years of coverage
${}_{t+1}V$ = terminal reserve for year $t + 1$, net level premium method
P = net level annual premium

MATHEMATICAL PROOF

Equation (1) used in constructing table D can be transformed as follows:

$$l_x^D q_x^D = l_x^D q_x^C + l_x^B(q_x^B - q_x^A)$$

$$l_x^D(q_x^D - q_x^C) = l_x^B(q_x^B - q_x^A)\;. \tag{5}$$

Replacing x in (5) by $x + t$, multiplying both sides of the equation by v^{x+t} and substituting in (4), we have

$$\pi = \frac{v}{N_x^B - N_{x+n}^B} \sum_{t=0}^{m-1} D_{x+t}^D\,(q_{x+t}^D - q_{x+t}^C)\;(1 - {}_{t+1}V^C)\;. \tag{6}$$

Hence, if we can show that

$$v\sum_{t=0}^{m-1} D_{x+t}^D\,(q_{x+t}^D - q_{x+t}^C)\;(1 - {}_{t+1}V^C) = (N_x^D - N_{x+n}^D)\;(P^D - P^C)\;, \tag{7}$$

substitution in formula (6) will prove formula (3).

Proof of (7) is as follows:

$$l_{x+t+1}^D \cdot {}_{t+1}V^D = l_{x+t}^D \cdot {}_tV^D(1 + i) + l_{x+t}^D P^D(1 + i) - l_{x+t}^D q_{x+t}^D \tag{8}$$

$${}_{t+1}V^C = {}_tV^C(1 + i) + P^C(1 + i) - q_{x+t}^C(1 - {}_{t+1}V^C)\,. \tag{9}$$

Subtracting $q^D_{x+t} \cdot {}_{t+1}V^C$ from both sides of (9) and rearranging,

$$
\begin{aligned}
&{}_{t+1}V^C(1 - q^D_{x+t}) \\
&\qquad = {}_tV^C(1+i) + P^C(1+i) - q^C_{x+t} - (q^D_{x+t} - q^C_{x+t})\,{}_{t+1}V^C.
\end{aligned} \tag{10}
$$

Multiplying both sides of (10) by l^D_{x+t},

$$
\begin{aligned}
&l^D_{x+t+1} \cdot {}_{t+1}V^C = l^D_{x+t} \cdot {}_tV^C(1+i) \\
&\qquad + l^D_{x+t}P^C(1+i) - l^D_{x+t}q^C_{x+t} - l^D_{x+t}(q^D_{x+t} - q^C_{x+t})\,{}_{t+1}V^C.
\end{aligned} \tag{11}
$$

Subtracting (11) from (8),

$$
\begin{aligned}
&l^D_{x+t+1}[{}_{t+1}V^D - {}_{t+1}V^C] \\
&\qquad = l^D_{x+t}[{}_tV^D - {}_tV^C](1+i) + l^D_{x+t}[P^D - P^C](1+i) \\
&\qquad\qquad - l^D_{x+t}(q^D_{x+t} - q^C_{x+t})(1 - {}_{t+1}V^C).
\end{aligned} \tag{12}
$$

Multiplying both sides of (12) by v^{x+t+1} and summing over the life of the plan,

$$
\begin{aligned}
&\sum_{t=0}^{m-1} D^D_{x+t+1}({}_{t+1}V^D - {}_{t+1}V^C) = \sum_{t=0}^{m-1} D^D_{x+t}({}_tV^D - {}_tV^C) \\
&+ \sum_{t=0}^{n-1} D^D_{x+t}(P^D - P^C) - v\sum_{t=0}^{m-1} D^D_{x+t}(q^D_{x+t} - q^C_{x+t})(1 - {}_{t+1}V^C).
\end{aligned} \tag{13}
$$

(The range of the summation in the premium term is only over the premium paying period as P^D and P^C drop out of equations (8) and (9) for $t \geq n$.)

The first summation in (13) expanded is

$$
\begin{aligned}
&D^D_{x+1}({}_1V^D - {}_1V^C) + D^D_{x+2}({}_2V^D - {}_2V^C) \\
&\qquad + \ldots + D^D_{x+m-1}({}_{m-1}V^D - {}_{m-1}V^C) + D^D_{x+m}({}_mV^D - {}_mV^C).
\end{aligned} \tag{14}
$$

The second summation in (13) expanded is

$$
\begin{aligned}
&D^D_x({}_0V^D - {}_0V^C) + D^D_{x+1}({}_1V^D - {}_1V^C) \\
&\qquad + D^D_{x+2}({}_2V^D - {}_2V^C) + \ldots + D^D_{x+m-1}({}_{m-1}V^D - {}_{m-1}V^C).
\end{aligned} \tag{15}
$$

Since ${}_0V^D - {}_0V^C = 0$ and ${}_mV^D - {}_mV^C = 0$, it follows that (14) $-$ (15) $=$ 0, and (13) reduces to (7) which proves formula (3).

Equation (7) states that the present value of payments of the appropriate net amount at risk to the extra deaths is equal to the present value of the extra premiums.

APPENDIX 1—SPECIAL PLANS

Plans with Varying Death Benefits

For such plans we have

Usual Method

$$\pi = \frac{v}{N_x^B - N_{x+n}^B} \sum_{t=0}^{m-1} D_{x+t}^B (q_{x+t}^B - q_{x+t}^A)(F_{t+1} - {}_{t+1}V^C) \quad (16)$$

Proposed Method

$$\pi = \frac{N_x^D - N_{x+n}^D}{N_x^B - N_{x+n}^B} [P^D - P^C] \quad (3)$$

where F_{t+1} = death benefit in year $t + 1$

P = net level annual premium for the plan.

By introducing F_{t+1} into equations (8) and (9) and making the necessary changes in the succeeding equations, it can be easily demonstrated that formula (3) is applicable here.

Plans with Nonlevel Net Premiums

For these plans we have

Usual Method $$\pi = \frac{v}{N_x^B - N_{x+n}^B} \sum_{t=0}^{m-1} D_{x+t}^B (q_{x+t}^B - q_{x+t}^A)(1 - {}_{t+1}V^C) \quad (4)$$

Proposed Method

$$\pi = \frac{\sum_{t=0}^{n-1} (P_t^D - P_t^C) D_{x+t}^D}{N_x^B - N_{x+n}^B} \quad (17)$$

where ${}_{t+1}V^C$ = actual reserve on the plan (*not* the net level premium reserve)

P_t = net premium payable at time t.

Formula (17) is a more general form of formula (3). By making P^D and P^C in equations (8) and (9) functions of duration and modifying the succeeding equations accordingly, it can be shown that formulas (4) and (17) produce identical results.

APPENDIX 2—CASH VALUE BASIS

If the net amount at risk is based on the cash value (computed by the standard nonforfeiture method) we have

Usual Method

$$\pi = \frac{v}{N^B_x - N^B_{x+n}} \sum_{t=0}^{m-1} D^B_{x+t}\,(q^B_{x+t} - q^A_{x+t})\ (1 - {}_{t+1}CV^C) \qquad (18)$$

Proposed Method

$$\pi = \frac{N^D_x - N^D_{x+n}}{N^B_x - N^B_{x+n}}\,[P^D - P^C] + \frac{kN^D_x}{N^B_x - N^B_{x+n}}\,[\ddot{a}^C_{x:\overline{s}|} - \ddot{a}^D_{x:\overline{s}|}] \qquad (19)$$

where ${}_{t+1}CV^C = {}_{t+1}V^C - k\ddot{a}^C_{x+t+1:\overline{s-t-1}|}$

$k =$ constant for any given plan and age at issue.

$s =$ number of years at the end of which the cash value is first equal to the reserve.

Formula (19) can be derived as follows:

Substituting the above expression for ${}_{t+1}CV^C$ in (18),

$$\pi = \frac{v}{N^B_x - N^B_{x+n}} \sum_{t=0}^{m-1} D^B_{x+t}\,(q^B_{x+t} - q^A_{x+t})\ (1 - {}_{t+1}V^C)$$
$$+ \frac{vk}{N^B_x - N^B_{x+n}} \sum_{t=0}^{s-1} D^B_{x+t}\,(q^B_{x+t} - q^A_{x+t})\ \ddot{a}^C_{x+t+1:\overline{s-t-1}|}\,. \qquad (20)$$

The first term in (20) is equal to the right hand side of (4) and hence is equal to the right hand side of (3),

$$\frac{N^D_x - N^D_{x+n}}{N^B_x - N^B_{x+n}}\,[P^D - P^C]\ . \qquad (21)$$

The second term in (20) can be rewritten as

$$\frac{kD^B_x}{d\,(N^B_x - N^B_{x+n})}\left[\frac{v}{D^B_x} \sum_{t=0}^{s-1} D^B_{x+t}\,(q^B_{x+t} - q^A_{x+t})\ (1 - A^C_{x+t+1:\overline{s-t-1}|})\right]. \qquad (22)$$

The expression in brackets is a special case of the right hand side of (4) with $m = s$, $n = 1$, ${}_{t+1}V^{C} = A^{C}_{x+t+1:\overline{s-t-1}|}$, and is therefore equal to the right hand side of (3) with $P = A_{x:\overline{s}|}$ and $n = 1$. Therefore (22) reduces to

$$\frac{kD_x^B}{d\,(N_x^B - N_{x+n}^B)}\left[\frac{D_x^D}{D_x^B}\,(A^D_{x:\overline{s}|} - A^C_{x:\overline{s}|})\right] = \frac{kD_x^D}{d\,(N_x^B - N_{x+n}^B)}\,(A^D_{x:\overline{s}|} - A^C_{x:\overline{s}|})\ , \tag{23}$$

which can be written as

$$\frac{kD_x^D}{N_x^B - N_{x+n}^B}\,[\ddot{a}^C_{x:\overline{s}|} - \ddot{a}^D_{x:\overline{s}|}]\ . \tag{24}$$

Formula (19) can now be obtained by combining (21) and (24).

APPENDIX 3—CONTINUOUS FUNCTIONS

When continuous functions are used, we have

$$\bar{\pi} = \frac{\int_0^m v^{x+t}\, l^B_{x+t}\,(\mu^B_{x+t} - \mu^A_{x+t})\ (1 - {}_t\bar{V}^C)\, dt}{\bar{N}_x^B - \bar{N}_{x+n}^B}\,. \tag{25}$$

The method proposed in this note yields the following close approximation for $\bar{\pi}$:

$$\bar{\pi} \doteq \frac{\bar{N}_x^D - \bar{N}_{x+n}^D}{\bar{N}_x^B - \bar{N}_{x+n}^B}\,[\bar{P}^D - \bar{P}^C]\ , \tag{26}$$

where table D is the table defined in this note. Proof of (26) is as follows:

For any two mortality tables, C and D′, we have

$$\frac{d}{dt}\, l^{D'}_{x+t}\cdot {}_t\bar{V}^{D'} = l^{D'}_{x+t}\bar{P}^{D'} + \delta\, l^{D'}_{x+t}\cdot {}_t\bar{V}^{D'} - l^{D'}_{x+t}\mu^{D'}_{x+t} \tag{27}$$

$$\frac{d}{dt}\, {}_t\bar{V}^C = \bar{P}^C + \delta\, {}_t\bar{V}^C - \mu^C_{x+t}\,(1 - {}_t\bar{V}^C)\ . \tag{28}$$

Subtracting $\mu^{D'}_{x+t}\cdot {}_t\bar{V}$ from both sides of (28) and rearranging,

$$\frac{d}{dt}\, {}_t\bar{V}^C - \mu^{D'}_{x+t}\cdot {}_t\bar{V}^C = \bar{P}^C + \delta\, {}_t\bar{V}^C - (\mu^{D'}_{x+t} - \mu^C_{x+t})\ {}_t\bar{V}^C - \mu^C_{x+t}\,. \tag{29}$$

If both sides of (29) are multiplied by $l^{D'}_{x+t}$, the left hand side becomes

$$l^{D'}_{x+t}\,\frac{d}{dt}\, {}_t\bar{V}^C - {}_t\bar{V}^C\,(l^{D'}_{x+t}\mu^{D'}_{x+t}) = l^{D'}_{x+t}\,\frac{d}{dt}\, {}_t\bar{V}^C + {}_t\bar{V}^C\,\frac{d}{dt}\, l^{D'}_{x+t} = \frac{d}{dt}\, l^{D'}_{x+t}\cdot {}_t\bar{V}^C\ .$$

Therefore,

$$\frac{d}{dt}\, l^{D'}_{x+t}\cdot {}_t\bar{V}^{C} = l^{D'}_{x+t}\bar{P}^{C} + \delta l^{D'}_{x+t}\cdot {}_t\bar{V}^{C} - l^{D'}_{x+t}\,(\mu^{D'}_{x+t} - \mu^{C}_{x+t})\, {}_t\bar{V}^{C} - l^{D'}_{x+t}\mu^{C}_{x+t}\,. \tag{30}$$

Subtracting (30) from (27), we obtain

$$\frac{d}{dt}\,[l^{D'}_{x+t}\cdot {}_t\bar{V}^{D'} - l^{D'}_{x+t}\cdot {}_t\bar{V}^{C}] = l^{D'}_{x+t}\,(\bar{P}^{D'} - \bar{P}^{C}) + \delta l^{D'}_{x+t}\,({}_t\bar{V}^{D'} - {}_t\bar{V}^{C}) - l^{D'}_{x+t}\,(\mu^{D'}_{x+t} - \mu^{C}_{x+t})\,(1 - {}_t\bar{V}^{C})\,. \tag{31}$$

Multiplying (31) by v^{x+t} and integrating over the life of the plan, we obtain

$$\int_0^m v^{x+t}\frac{d}{dt}\,[l^{D'}_{x+t}\cdot {}_t\bar{V}^{D'} - l^{D'}_{x+t}\cdot {}_t\bar{V}^{C}]\,dt = (\bar{P}^{D'} - \bar{P}^{C})\int_0^n v^{x+t}l^{D'}_{x+t}dt + \delta\int_0^m v^{x+t}l^{D'}_{x+t}\,({}_t\bar{V}^{D'} - {}_t\bar{V}^{C})\,dt - \int_0^m v^{x+t}l^{D'}_{x+t}\,(\mu^{D'}_{x+t} - \mu^{C}_{x+t})\,(1 - {}_t\bar{V}^{C})\,dt\,. \tag{32}$$

(In the premium term the integral runs only over the premium paying period.)

The integral on the left of (32) can be integrated easily by parts, giving $\delta\int_0^m v^{x+t}l^{D'}_{x+t}({}_t\bar{V}^{D'} - {}_t\bar{V}^{C})dt$, which cancels the identical term on the right. Therefore, we have from (32)

$$(\bar{P}^{D'} - \bar{P}^{C})\,(\bar{N}^{D'}_{x} - \bar{N}^{D'}_{x+n}) = \int_0^m v^{x+t}l^{D'}_{x+t}\,(\mu^{D'}_{x+t} - \mu^{C}_{x+t})\,(1 - {}_t\bar{V}^{C})\,dt\,. \tag{33}$$

In order to make use of equation (33) in simplifying (25), we must first construct a special mortality table D′ such that

$$l^{D'}_{x+t}(\mu^{D'}_{x+t} - \mu^{C}_{x+t}) = l^{B}_{x+t}(\mu^{B}_{x+t} - \mu^{A}_{x+t}). \tag{34}$$

If such a table can be constructed, $\bar{\pi}$ will be given exactly by

$$\bar{\pi} = \frac{\bar{N}^{D'}_{x} - \bar{N}^{D'}_{x+n}}{\bar{N}^{B}_{x} - \bar{N}^{B}_{x+n}}\,[\bar{P}^{D'} - \bar{P}^{C}]\,. \tag{35}$$

Proof of this is analogous to the proof given in the traditional case.

However, the construction of table D′ would be somewhat formidable and therefore it is shown by the following arguments that table D as defined in this note is an excellent approximation to table D′.

Equation (34) is really the following differential equation:

$$\frac{d}{dt}\, l^{\mathrm{D}'}_{x+t} = -\, l^{\mathrm{D}'}_{x+t}\ {}^{\mathrm{C}}_{x+t} - l^{\mathrm{B}}_{x+t}\,(\mu^{\mathrm{B}}_{x+t} - \mu^{\mathrm{A}}_{x+t})\ . \tag{36}$$

Each of the tables A, B and C assumes a uniform distribution of deaths over each year of age; hence, for each of these tables, $\mu_{x+t} = q_x/(1 - tq_x)$ $(0 \leq t < 1)$. If this value of μ_{x+t} is substituted in (36) for each of the tables A, B and C, (36) can (with a little spade work) be integrated exactly between $t = 0$ and $t = 1$, giving the unfortunate result

$$l^{\mathrm{D}'}_{x+t} = l^{\mathrm{D}'}_x\ (1 - t\,q^{\mathrm{C}}_x) - \frac{l^{\mathrm{B}}_x\,(q^{\mathrm{B}}_x - q^{\mathrm{A}}_x)}{q^{\mathrm{C}}_x - q^{\mathrm{A}}_x}\,(1 - t\,q^{\mathrm{C}}_x)\ \log_e \frac{1 - t\,q^{\mathrm{A}}_x}{1 - t\,q^{\mathrm{C}}_x} \tag{37}$$

$$(0 \leq t \leq 1)\ .$$

However, by making use of the approximation

$$\log_e \frac{1 - t\,q^{\mathrm{A}}_x}{1 - t\,q^{\mathrm{C}}_x} = \log_e \left[1 + \frac{t\,(q^{\mathrm{C}}_x - q^{\mathrm{A}}_x)}{1 - t\,q^{\mathrm{C}}_x}\right] \doteq \frac{t\,(q^{\mathrm{C}}_x - q^{\mathrm{A}}_x)}{1 - t\,q^{\mathrm{C}}_x} \tag{38}$$

$$(0 \leq t \leq 1)$$

equation (37) becomes

$$l^{\mathrm{D}'}_{x+t} \doteq l^{\mathrm{D}'}_x - t[l^{\mathrm{D}'}_x q^{\mathrm{C}}_x + l^{\mathrm{B}}_x(q^{\mathrm{B}}_x - q^{\mathrm{A}}_x)] \quad (0 \leq t \leq 1) \tag{39}$$

which says that the deaths on table D′ are approximately uniformly distributed over each year of age and that

$$d^{\mathrm{D}'}_x \doteq l^{\mathrm{D}'}_x q^{\mathrm{C}}_x + l^{\mathrm{B}}_x(q^{\mathrm{B}}_x - q^{\mathrm{A}}_x)\ . \tag{40}$$

The approximate table D′ defined by (40) will be identical with table D defined by (1) if the radix $l^{\mathrm{D}'}_a$ is chosen equal to l^{D}_a. Hence table D may be substituted in (35) as an approximation for the exact table D′ defined by (37), thus proving (26). Tests indicate that the approximate values obtained from (26) are within one percent of the true values defined by (25).

Continuous function formulas for the special cases considered in appendixes 1 and 2 can be developed in an analogous manner.