II

Fundamentals of Pension Funding

Charles L. Trowbridge

I. Introduction

Among the tools of the pension actuary are a variety of techniques which for want of better terminology will here be called *funding methods.* By funding method is meant the budgeting scheme or the payment plan under which the benefits are to be financed. The choice of funding method in no way affects true over-all costs, which are a function of the benefits to be provided and certain other factors such as rates of mortality, interest, and employee withdrawal. The funding method is, however, the controlling factor in determining how much of the eventual cost is to be paid at any particular point of time. Funding method, as employed in this paper, should not be confused with funding *medium, i.e.,* the vehicle (such as Deposit Administration of Self-Administered Trust) by means of which the funding arrangements are carried out.

The funding methods commonly used in the pension field are perhaps fairly well understood by the actuaries who use them, but the actuarial literature on this subject is extremely sparse. The classic British papers on pensions devote themselves largely to the techniques of valuing complicated benefits. They put little or no emphasis on the possible variations in funding method, relying almost entirely on what is essentially individual level premium funding. Perhaps the best description of the various funding methods will be found in the "Bulletin on Section 23(p)" put out by the U.S. Treasury Department. Even this is only a very sketchy and superficial treatment, and the beginner in the pension field pretty much has to dig the ideas out for himself. This paper attempts, in some measure, to get at least the fundamentals of pension funding into actuarial literature.

Part II following introduces certain fundamental concepts, among them the "mature population" and "mature fund" concepts. By means of the "Equation of Maturity" a logical classification system for the various funding methods is devised. Assumptions and notation necessary for actuarial analysis are set forth.

Part III describes and classifies various methods which are thought to include most of those in common use among actuaries active in the pension field. The rather simple algebra is developed for each method of funding (under the rigid conditions of an initially stationary population) as a sort of theoretical base on which to build a more practical understanding.

Part IV looks into the characteristics of these methods under less idealistic conditions. Certain seeming inconsistencies which arise in practice are explained.

Part V introduces the rather treacherous subject of "adjustment for gains and losses," and describes various methods of making such adjustment.

II. Fundamental Concepts, Assumptions, Notation

Mature Population Concept

All actuaries are familiar with the "service" table derived from estimates of rates of death, withdrawal, and new hirings. The l_x^s column of this table represents approximately the age distribution of the employee group *after* the group reaches what we call a "stationary" condition.

Most employee groups today are immature; *i.e.*, they contain more younger members and fewer pensioners than the l_x^s column of the underlying service table would indicate. Yet most of us accept the idea that any employee group of sufficient size can be assumed (for want of better information) to approach a mature or stationary condition eventually. It seems logical, therefore, to employ this mature population concept in the classification of funding methods.

Equation of Maturity

It is apparent that a pension fund, like any other fund, grows or shrinks as income exceeds outgo, or vice versa. Contributions and interest make up income. Benefits paid are outgo. Thus if benefits (B) and contributions (C) are both assumed payable at the beginning of a year, and if the fund (F) is measured at the beginning of the year (prior to either contributions or benefits then due), the following relationship holds.

$$v\Delta F = C + dF - B \qquad (1)$$

where ΔF is the change in F over the year and d is the rate of discount.

It is the essence of the mature population concept that benefits (B) eventually become stationary. Moreover, it is characteristic of all of the funding methods described in this paper that at or after the time when the employee population becomes stationary, the contribution (C) and the fund (F) reach (or approach) a constant. ΔF therefore becomes zero and equation (1) becomes

$$C + dF = B \qquad (2)$$

where C, F, and B are all constants. Equation (2) can be thought of as an Equation of Maturity.

Note that this equation does not necessarily hold as soon as the population reaches maturity. Sufficient time must have elapsed so that C and F have reached their ultimate levels as well. In point of time the concept of a *mature fund* may therefore be one step beyond the idea of a *mature population.*

Classification of Funding Methods

In the Equation of Maturity, B and d are entirely independent of the funding method. Therefore, in the ultimate situation, the various funding methods differ only as to the relative sizes of F and C. At one extreme $F = 0$ and $C = B$; at the other $C = 0$ and $F = B/d$. Between these two extremes lie the funding methods commonly employed.

It is logical to classify these funding methods in ascending order of F (or descending order of C, which is the same thing). This scheme of classification will be used throughout this paper.

Assumptions

The actuarial analysis of the ultimate situation to which a given funding method leads is materially simplified if a mature population is assumed, not after many years, but right from the inauguration of the plan. The concept of an *initially* mature population (both as to active and retired lives) is therefore employed as a starting point and as a base on which to build. The unreality of the assumption that the employee population is stationary *from the beginning* is nonetheless recognized, and observations as to the more realistic situation follow in Parts IV and V.

Moreover, since this paper concerns itself only with fundamentals, complications arising from benefit increases, death benefits, etc., are avoided by assuming the simplest benefits possible. Unless otherwise indicated, the algebraic statements and demonstrations found in this paper are based on the following assumptions.

Assume a population, stationary from the moment the pension plan is established, such that the number attaining age x in a given year is l_x. It is immaterial to this discussion whether the table is of the single or multiple decrement type, so long as l_{x+1} represents the survivors one year hence of the group l_x. It is likewise immaterial whether l_x represents numbers of lives, or whether it be thought of as dollars of salary; *i.e.*, the l_x used in this paper can be thought of as meaning $s_x l_x$ in cases where a salary scale (a function of age only) is introduced.

Further assume a single retirement age r, and that the pension benefit for each life (or each \$1 of salary) reaching retirement age is \$1.00 payable annually in advance. Assume that the plan provides no death or withdrawal benefits of any description.

Notation

Let a be youngest age in the service table, so that the stationary population is supported by l_a new entrants yearly.

Let ω be limiting age of service table.

Let C_t represent the tth annual contribution to the pension plan, payable annually in advance. Superscripts to the left indicate the funding method under consideration. For example ${}^{EAN}C_1$ represents the first contribution under entry age normal, and ${}^{A}C_\infty$ represents the ultimate contribution if aggregate funding is used.

Let F_t represent the fund (or reserve) built up after t years (before contribution or benefits then due). Again superscripts indicate funding method.

III. Description and Classification of Funding Methods

Class I Funding

Under the scheme of classification previously described, Class I is logically assigned to what is commonly known as "pay as you go" funding. No contributions are made to the plan beyond those immediately necessary to meet benefit payments falling due. Contributions (${}^{P}C_t$) are exactly equal to benefits for all values of t, and ${}^{P}F_t$ is zero for all values of t.

Since the initially mature population previously described produces constant benefit payments, "pay-as-you-go" funding for such a group produces level contributions equal to

$$\sum_r^{\omega} l_x .$$

Class II Funding

If no funding whatsoever is contemplated for active lives, but if the present value of future pension benefits is contributed for each life as it reaches retirement, we have what has come to be known as "terminal" funding. Since this method produces higher eventual contributions and lower eventual reserves than any of the other common methods except Class I, terminal funding is assigned to Class II.

When terminal funding is applied to an initially mature population, all contributions except the first are equal and can be quantitatively expressed as $l_r\ddot{a}_r$. The principle of full funding for all retired lives requires, however, that the first contribution be considerably greater to fund the benefits of those already beyond retirement age at the time the plan is inaugurated. The initial contribution is in fact

$${}^{T}C_1 = \sum_r^{\omega} l_x \, \ddot{a}_x \text{ and exceeds}$$

the ultimate level contribution ${}^{T}C_\infty = l_r\ddot{a}_r$ by $\sum_{r+1}^{\omega} l_x \, \ddot{a}_x$.

This extra contribution in the first year arises because the plan was not always in existence but came into being after certain individuals had already retired. Here we find the first suggestion of "normal cost" and "accrued liability," two concepts frequently employed in the pension business.

Normal Cost is commonly understood to mean the level of contribution which a funding method would currently produce, were it not for a late start in paying for benefits. *Accrued Liability,* measured at any time, represents the difference between the then present value of future benefits and the present value of future normal costs. The portion of the accrued liability not offset by assets is called the *unfunded accrued liability.* The accrued liability, when measured at the establishment of the plan, is commonly referred to as the *initial accrued liability.*

Under Class II or terminal funding applied to an initially mature group we have seen that normal cost is represented by $l_r\ddot{a}_r$, and the initial accrued liability by

$\sum_{r+1}^{\omega} l_x \, \ddot{a}_x$. The accrued liability does not change with the passage of time if the group is mature from the beginning. Once the accrued liability has been paid off,

$${}^{T}C_\infty = l_r\ddot{a}_r$$

$${}^{T}F_\infty = \sum_{r+1}^{\omega} l_x \, \ddot{a}_x$$

and the fundamental Equation of Maturity can be checked out by the identity

$$l_r\ddot{a}_r + d\sum_{r+1}^{\omega} l_x\ddot{a}_x \equiv \sum_r^{\omega} l_x .$$

Note that ${}^{T}F_\infty$, the ultimate reserve built up, and the accrued liability are, as we might expect, algebraically identical.

Class III Funding

The so-called "unit credit" or "single premium" method of funding is the first method here considered that funds in any respect for employees not yet retired. Since this method builds up lower reserves than methods yet to be considered, it is here classified as Class III.

Unit credit funding is based on the principle that the pension to be provided at retirement age will be divided into as many "units" as there are active membership years, with one unit assigned to each year. The normal cost as to any individual pension in any year becomes the cost to fully fund on a single premium basis the unit assigned to that year. The accrued liability at any time is the present value of all units of pension assigned to prior years. Under this method of funding particularly the accrued liability is often referred to as the "past service" liability.

To the extent practicable the units assigned to various years are equal in amount. For any individual, therefore, the "normal" cost rises each year, since the value of a deferred annuity commencing at age r is an increasing function of attained age. For the group as a whole, however, the normal cost remains level under the rigid conditions previously imposed.

Algebraically the normal cost is

$$\frac{1}{r-a}\sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x \,.$$

The accrued liability is

$$\frac{1}{r-a}\sum_{a}^{r-1}(x-a)l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x \,.$$

Under this method of funding the initial accrued liability can be paid off in a variety of ways. A common method is to amortize the liability by means of an annuity certain over a period of n years, the accrued liability payment becoming $k\%$ of the initial accrued liability, where $k = 100/\ddot{a}_{\overline{n|}}$. A requirement in some plans using unit credit funding is that the accrued liability as to any individual will be funded by the time said individual retires. In any case, once the accrued liability is fully funded

$${}^{U}C_{\infty} = \frac{1}{r-a}\sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x$$

and

$${}^{U}F_{\infty} = \frac{1}{r-a}\sum_{a}^{r-1}(x-a)l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x .$$

Once again the ultimate fund and the accrued liability are equal under the rigid conditions imposed.

The algebraic identity

$$\frac{1}{r-a}\sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x +$$

$$d\left[\frac{1}{r-a}\sum_{a}^{r-1}(x-a)l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x\right] \equiv \sum_{r}^{\omega} l_x$$

is, of course, an expression of the Equation of Maturity applied to Class III funding. Note that it is also an algebraic statement that if the accrued liability is not paid off, but instead is amortized in perpetuity by paying interest alone, unit credit funding for an initially mature population degenerates into pay as you go.

Class IV Funding

Four of the better known funding methods are logically classed together, because we will see that once the ultimate condition has been reached these methods produce identical contributions and build up identical reserves.

1. Entry Age Normal Method

This method, as its title implies, visualizes the normal cost for any given employee as the level payment (or level percentage of pay) necessary to fund the benefit over the working lifetime of such employee. The normal cost for a unit benefit for any individual entering at age a is therefore

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}} .$$

The accrued liability as to any individual age x ($x < r$) is

$${}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}}\ddot{a}_{x:\overline{r-x|}} .$$

If we look at the group instead of the individual, we find the accrued liability is

$$\sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}}\sum_{a}^{r-1} l_x\ddot{a}_{x:\overline{r-x|}} .$$

When this last expression is written in the form

$$\sum_{r}^{\omega} l_x\ddot{a}_x + \sum_{a}^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}}\ddot{a}_{x:\overline{r-x|}}\right)$$

it is apparent that the initial accrued liability is simply viewed as the full net single premium for benefits for retired lives, plus the sum of the individual full net level premium reserves for each unit of benefit for active lives, where such reserves are calculated as of ages when accrued liability is being computed and as if funding began (and therefore net level premium was computed) at age a. The normal cost,

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}$$

for each active life, is of course

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x$$

for the group as a whole.

As in the unit credit method, the initial accrued liability can be funded in a variety of ways, commonly by level payments for a fixed number of years. There may be a requirement that accrued liability be funded with sufficient rapidity that benefits for all retired lives are completely funded. Once the accrued liability has been completely liquidated, ${}^{\text{EAN}}C_\infty$ is the normal cost

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x$$

and

$${}^{\text{EAN}}F_\infty = \sum_r^{\omega} l_x\ddot{a}_x + \sum_a^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}\right).$$

Once again the ultimate fund, under the rigid conditions imposed, becomes identical with the unchanging accrued liability. Once again an algebraic identity

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x +$$

$$d\left[\sum_r^{\omega} l_x\ddot{a}_x + \sum_a^{r-1} l_x\left({}_{r-x|}\ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\ddot{a}_{x:\overline{r-x}|}\right)\right] \equiv \sum_r^{\omega} l_x$$

proves out the Equation of Maturity, and at the same time shows us that if accrued liability payments are reduced to interest only, the contribution equals the benefits, and accordingly no funds are built up.

2. *Individual Level Premium Funding*

A second Class IV method funds the benefits as to any individual from date of entry (or date plan is established, if later) to retirement date as a level amount (or as a level percentage of pay). As to individuals who enter the group *after* the establishment of the plan, it is apparent that this method and entry age normal are identical. For the original staff, however, the individual level premium method of funding has the effect of funding the accrued liability (as to any individual) over his future working lifetime, or in exactly the same manner as the normal cost.

For an individual age x when the plan is inaugurated, individual level premium funding requires a payment of

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}$$

for each year that such individual remains in active service. But note that since

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}}$$

can be expressed as

$$\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} + \left(\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}} - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right),$$

the contribution under level premium funding can be viewed as the normal cost (*i.e.*, the cost for new entrants) plus an accrued liability payment of

$$\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}} - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}$$

Extending this concept to the entire population, we see that the initial contribution to the plan is simply

$$\begin{aligned} {}^{\text{ILP}}C_1 &= \sum_a^{r-1} l_x\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}} + \sum_r^{\omega} l_x\ddot{a}_x \\ &= \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x + \sum_a^{r-1}\left(\frac{{}_{r-x|}\ddot{a}_x}{\ddot{a}_{x:\overline{r-x}|}} - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right)l_x + \sum_r^{\omega} l_x\ddot{a}_x \end{aligned}$$

where the first term of the second form can be thought of as a normal cost, and the last two terms can be considered payment toward the accrued liability.

We find the situation t years after the inauguration of the plan to be as follows:

$$\begin{aligned} {}^{\text{ILP}}C_{t+1} &= \sum_{a+t+1}^{r-1} l_x\frac{{}_{r-x+t|}\ddot{a}_{x-t}}{\ddot{a}_{x-t:\overline{r-x+t}|}} + \sum_a^{a+t} l_x\frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} \\ &= \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\sum_a^{r-1} l_x + \sum_{a+t+1}^{r-1}\left(\frac{{}_{r-x+t|}\ddot{a}_{x-t}}{\ddot{a}_{x-t:\overline{r-x+t}|}} - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}}\right). \end{aligned}$$

The normal cost remains level but the accrued liability payment decreases each year as t increases, until after $r-a$ years accrued liability is all paid off.

It can thus be seen that individual level premium funding is really a special case of entry age normal, where accrued liability is funded over $r-a$ years by high initial but decreasing payments. The initial payment toward the accrued liability is especially high since, among other things, it completely funds for the initial pensioners. It can of course be demonstrated that the present value, as of date of plan, of these accrued liability payments is identical to the entry age normal initial accrued liability.

3. Aggregate Funding

The principle behind the aggregate method is that of equating present value of unfunded future benefits to present value of future contributions, where the contribution per active life (or per dollar of salary) per year is assumed constant. It may seem at first thought that the resulting contributions should remain level from year to year for an initially stable population, since the very principle implies spreading the value of total benefits levelly over future life years.

This supposition regarding the aggregate method is absolutely correct *provided* future new entrants are taken into account, both in valuing present value of future benefits and in calculating present value of future active life years. Demonstration I in the Appendix shows us that in the first year the so-computed aggregate contribution under our rigid conditions is exactly

$$\sum_r^{\omega} l_x$$

which we recognize as the pay-as-you-go payment.

Since the contribution just equals the benefits, no reserves build up and contributions continue to duplicate the level Class I contribution.

The common use of the aggregate method, however, ignores new entrants. The effect, of course, is to subtract $v/dl_a\ {}_{r-a}|\ddot{a}_a$ from the numerator and $v/dl_a\ddot{a}_{a\,\overline{r-a|}}$ from the denominator of equation (1) of Demonstration I. Since, where A, B, C, and D are positive constants, if

$$\frac{A}{B} > \frac{C}{D},$$

then

$$\frac{A}{B} > \frac{A+C}{B+D},$$

it follows that ${}^{A}C_1$ (new entrants disregarded) is greater than the level pay-as-you-go payment if

$$\frac{\sum_a^{r-1} l_x\ {}_{r-x}|\ddot{a}_x + \sum_r^{\omega} l_x\ddot{a}_x}{\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x|}}} > \frac{{}_{r-a}|\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}}.$$

This latter inequality is proven by the same algebraic principle.

The ignoring of new entrants therefore produces, in the first year, a contribution in excess of benefits, and starts the accumulation of a reserve.

In any year thereafter

$${}^{A}C_t = \frac{\sum_a^{r-1} l_x\ {}_{r-x}|\ddot{a}_x + \sum_r^{\omega} l_x\ddot{a}_x - {}^{A}F_{t-1}}{\sum_a^{r-1} l_x\ddot{a}_{x.\overline{r-x|}}} \sum_a^{r-1} l_x.$$

As F_t increases, ${}^{A}C_t$ decreases. It can be shown that as ${}^{A}C_t$ decreases, the increment to ${}^{A}F_t$,

$$\Delta^{A}F_t \equiv \left[d^{A}F_t + {}^{A}C_{t+1} - \sum_r^{\omega} l_x\right](1+i),$$

decreases. The fund continues to increase, but at a slower and slower rate, so long as $\Delta^{A}F_t$ is positive, *i.e.*, so long as

$${}^{A}C_t > \sum_r^{\omega} l_x - d^{A}F_t.$$

It is shown in Demonstration II that under this process ${}^{A}C_t$ approaches asymptotically its limit

$${}^{A}C_\infty = \frac{{}_{r-a}|\ddot{a}_a}{\ddot{a}_{a:\overline{r-a|}}} \sum_a^{r-1} l_x$$

which we recognize as the normal cost under other Class IV methods. Similarly ${}^{A}F_t$ approaches, but never reaches, a limit identical to ${}^{EAN}F_\infty$ and ${}^{ILP}F_\infty$ The aggregate method of funding can therefore be considered another special case of entry age normal, where the accrued liability is paid off rather rapidly at the beginning, but at a slower and slower rate, such that the accrued liability is completely paid off only at infinity.

If for instance the average temporary annuity y (see Demonstration II) is $100/k$, the first payment toward the accrued liability is $k\%$ of the accrued liability. Later payments are, however, $k\%$ of the decreasing *unfunded* accrued liability. Compare the foregoing with $k\%$ funding of the accrued liability under entry age normal, where the $k\%$ applies to the full accrued liability rather than to the unfunded portion only.[1]

It can be demonstrated that the initial contribution under the aggregate method is generally lower than that under individual level premium funding. A temporary annuity $\ddot{a}_{x:\overline{r-x}|}$ which decreases with advancing age is a sufficient, but not necessary, condition for ${}^{A}C_1 < {}^{ILP}C_1$. If, due to heavy withdrawal assumptions at young ages, $\ddot{a}_{x:\overline{r-x}|}$ increases through a significant portion of its range, there may be rare exceptions to the general relationship.

4. Attained Age Normal

There may be some confusion in respect to the "attained age normal" method, arising from certain Class III characteristics in what is essentially a Class IV method.

Total benefits are divided into past service and future service benefits exactly as under unit credit funding, and as under Class III funding there is complete freedom as to the manner in which the past service liability shall be paid off. For future service benefits, however, the aggregate method is adopted.

The first year contribution toward future service becomes

$$\frac{\dfrac{1}{r-a}\displaystyle\sum_a^{r-1}(r-x)l_x\,{}_{r-x|}\ddot{a}_x}{\displaystyle\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}\sum_a^{r-1} l_x.$$

Since this amount is somewhat higher than the Class III normal cost

$$\frac{1}{r-a}\sum_a^{r-1} l_x\,{}_{r-x|}\ddot{a}_x$$

(which is level under our initially mature population assumptions), it is apparent that future service contributions under attained age normal are of a decreasing nature.

Future service costs after the first year are commonly calculated in the form

$$\frac{\displaystyle\sum_a^{r-1} l_x\,{}_{r-x|}\ddot{a}_x + \sum_r^{\omega} l_x\ddot{a}_x - \begin{array}{c}\text{Unfunded past}\\ \text{service liability}\end{array} - {}^{AAN}F_{t-1}}{\displaystyle\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}\sum_a^{r-1} l_x.$$

We perhaps get a better idea of the essential characteristics of attained age normal, however, if we express the tth future service contribution in the identical form

$$\frac{\dfrac{1}{r-a}\displaystyle\sum_a^{r-1}(r-x)l_x\,{}_{r-x|}\ddot{a}_x - f_{t-1}}{\displaystyle\sum_a^{r-1} l_x\ddot{a}_{x:\overline{r-x}|}}\sum_a^{r-1} l_x,$$

where f_t is that portion of ${}^{AAN}F_t$ built up by the accumulated excess (with interest) of the attained age normal future service contribution over the unit credit one.

As f_t grows the attained age normal future contribution decreases. It can be shown that f_t approaches as a limit the amount by which Class IV accrued liability exceeds the Class III accrued liability, and that if the initial past service liability is completely liquidated ${}^{AAN}C_t$ and ${}^{AAN}F_t$ have as limits ${}^{EAN}C_\infty$ and ${}^{EAN}F_\infty$ respectively.

Attained age normal is therefore a true Class IV method. Its accrued liability is actually as great as under the other Class IV methods, but attained age normal looks at the accrued liability in two parts. The method imposes no restrictions as to how the "past service" part, equal in magnitude to the Class III accrued liability, shall be funded. The second portion is liquidated by the decreasing accrued liability payments, which are the excess of the future service contribution over the ultimate future service contribution. Similarity with the aggregate method is of course noted, but whereas under the aggregate method all accrued liability is liquidated by rigid decreasing payments, under attained age normal only a portion of the accrued liability is so funded and the funding as to the remaining accrued liability is unspecified.

Class V Funding

Beyond the various variations of Class IV funding previously discussed, there is nothing of a practical nature, but funding methods which produce higher eventual reserves and lower eventual contributions than any of the methods so far discussed are, of course, theoretically possible. Perhaps the simplest of these is *initial* funding, where an employee's benefits are fully funded as soon as he is hired.

Here normal cost is $l_a\ {}_{r-a|}\ddot{a}_a$, accrued liability and ${}^{1}F_\infty$ are both

$$\sum_{a+1}^{r-1} l_x\ {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x \ddot{a}_x,$$

and Equation of Maturity is expressed by

$$l_a\ {}_{r-a|}\ddot{a}_a + d\left[\sum_{a+1}^{r-1} l_x\ {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x \ddot{a}_x\right] \equiv \sum_{r}^{\omega} l_x$$

Class VI Funding

Even less practical than Class V, but included here only to illustrate the extreme in heavy funding, is what might be called *complete* funding. If by one means or another an accrued liability of $1/d\sum_{r}^{\omega} l_x$ is fully paid off, interest on the funds built up will exactly meet the benefit payments $\sum_{r}^{\omega} l_x$.

Illustration of Initially Mature Situation

It may be enlightening to illustrate the foregoing discussion of the operation of the various funding methods under the assumption of an initially mature population by means of a numerical example. Table I shows the l_x column of a hypothetical stationary population, made up of exactly 1,000 active and 150 retired lives, maintained by 100 new entrants each year all age 30. Each year 1% of the active lives retire and 9% die or withdraw. The combined rate of death and withdrawal is 16% at age 30, and approximately equivalent to the Standard Annuity Table q_x at retired ages.

TABLE I

x	l_x	x	l_x	x	l_x
30...........	100	55...........	14	80...........	5
31...........	84	56...........	13	81...........	4
32...........	71	57...........	13	82...........	4
33...........	60	58...........	12	83...........	4
34...........	51	59...........	12	84...........	3
35...........	44	60...........	11	85...........	3
36...........	40	61...........	11	86...........	2
37...........	36	62...........	11	87...........	2
38...........	34	63...........	10	88...........	1
39...........	32	64...........	10	89...........	1
40...........	30	65...........	10	90...........	1
41...........	28	66...........	10	91...........	1
42...........	27	67...........	9	92...........	1
43...........	26	68...........	9	93...........	1
44...........	25	69...........	9	94...........	1
45...........	24	70...........	8	95...........	1
46...........	23	71...........	8		
47...........	22	72...........	8		
48...........	21	73...........	7		
49...........	20	74...........	7		
50...........	19	75...........	7		
51...........	18	76...........	6		
52...........	17	77...........	6		
53...........	16	78...........	6		
54...........	15	79...........	5		

Table II illustrates the yearly contribution and the build up of funds under each of the several funding methods, assuming 2½% interest, a benefit of $420 annually, and the stationary population of Table I. Twenty years has been chosen as the period of amortization of the initial accrued liability for those funding methods permitting such treatment.

IV. Modifications for Initially Immature Fund

Let us at this point abandon one of the rigid assumptions previously imposed and look into the common situation where the group is *not* initially mature, but is to a greater or less extent immature. For the present we will continue to assume that all actuarial assumptions are realized, leaving the question of actuarial gains and losses to Part V. As we abandon the assumption that the group is initially mature (though we retain the concept that the population will eventually approach a stationary condition), we replace the l_x of the stationary population by the l'_x of the immature population. As we should expect, the identities expressing the Equation of Maturity do not hold after this substitution until such time as the population has become mature and the l'_x's approach the l_x's. Moreover, we find that the conclusions previously reached for the initially mature fund must be modified in several other respects.

Normal Costs No Longer Level

If the initial group is immature it follows that Class I funding will produce contributions which are initially very low, but which increase rather rapidly, eventually leveling off when maturity of the group is attained.

Class II funding requires contributions which tend to fluctuate rather widely as number of retirements vary from year to year. Moreover, beneath this erraticism of contributions is an underlying tendency for costs to increase, since as the group matures the number retiring each year tends to grow, even if the size of the staff as a whole remains stationary.

The normal cost for Class III or unit credit funding (for a given staff and benefits) remains constant *if* actuarial assumptions are realized, and *if* the average age of the active staff does not change. The average age here meant is not the simple arithmetic mean, but the age corresponding to the weighted average single premium deferred annuity, where the single premium at each age is weighted by units being funded at such age. If the group is initially immature, however, it is axiomatic that this average age will slowly increase and normal costs will slowly rise before eventually leveling off. This rise may be pronounced if the group is unusually young at the establishment of the plan.

The possibility of increasing normal costs, even if all actuarial assumptions are realized, is not eliminated under Class IV funding. The expected increase in average age of the active life group will not, in itself, produce increasing normal costs. Level normal costs do, however, depend upon the average age of new entrants into the plan. If this average entry age remains constant and other actuarial assumptions are realized, normal costs will remain constant (assuming staff and benefit levels do not change). Again this average entry age is not a simple arithmetic mean, but the age corresponding to the weighted average level premium where the level premium for each entry age is weighted by benefits for those entering at such age.

Accrued Liability No Longer Constant

We have previously seen that under the assumption of an initially mature population the accrued liability produced by any of the funding methods discussed does not change with the passage of time. It takes no mathematical demonstration to convince us that, if the population is initially immature, the accrued liability will rise as the population grows older.

As a corollary we find that the funds will grow beyond the initial accrued liability up to the level of the ultimate accrued liability (assuming initial accrued liability is completely funded). The excess of the ultimate over the initial accrued liability is built up by the early year excess of normal costs plus interest on the initial accrued liability over benefit payments.

Normal Cost Plus Interest on Accrued Liability No Longer Identical to Pay-As-You-Go

We found earlier that for an initially mature group a contribution equal to normal cost plus interest on the initial accrued liability was exactly equal to benefit payments; accordingly no funds were built up and Class I funding resulted. This was true regardless of whether normal costs and accrued liability were those of Class II, III, IV, V, or VI.

TABLE II

	Class I	Class II	Class III	Class IV				Class V	Class VI
	Pay As You Go	Terminal Funding	Unit Credit 20 Yr.	Entry Age Normal 20 Yr.	Individual Level Prem.	Aggregate	Attained Age Normal 20 Yr.	Initial Funding 20 Yr.	Complete Funding 20 Yr.
	Initial Accrued Liability								
	None	$502,104	$1,206,924	$1,471,873	$1,471,873	$1,471,873	$1,471,873	$1,706,173	$2,583,000
	Normal Cost								
	$63,000	$ 50,753	$33,563	$27,101	$27,101	$27,101	$27,101	$21,386	None
Beg. of Year	Contributions								
1......	$63,000	$552,857	$109,095	$119,214	$772,667	$183,109	$130,716	$128,163	$161,651
2......	"	50,753	"	"	164,606	170,060	128,367	"	"
3......	"	"	"	"	135,627	158,103	126,215	"	"
4......	"	"	"	"	116,377	147,145	124,242	"	"
5......	"	"	"	"	102,007	137,104	122,437	"	"
10......	"	"	"	"	61,631	98,178	115,427	"	"
15......	"	"	"	"	43,398	73,026	110,900	"	"
20......	"	"	"	"	34,348	56,775	107,975	"	"
21......	"	"	33,563	27,101	33,177	54,293	31,995	21,386	None
25......	"	"	"	"	29,855	46,274	30,552	"	"
30......	"	"	"	"	27,760	39,489	29,331	"	"
35......	"	"	"	"	27,101	35,105	28,542	"	"
40......	"	"	"	"	"	32,273	28,032	"	"
50......	"	"	"	"	"	29,260	27,489	"	"
Limit...	$63,000	50,753	33,563	27,101	27,101	27,101	27,101	21,386	None
End of Year	Funds								
1......	None	$502,104	$ 47,248	$ 57,620	$ 727,409	$ 123,112	$ 69,409	$ 66,792	$ 101,117
2......	"	"	95,677	116,680	849,740	235,926	138,145	135,253	204,762
3......	"	"	145,316	177,217	945,426	339,304	206,394	205,427	310,998
4......	"	"	196,197	239,267	1,023,774	434,036	274,327	277,354	419,890
5......	"	"	248,350	302,868	1,089,350	520,844	342,108	351,080	531,504
10......	"	"	529,335	645,536	1,298,517	857,380	683,671	748,294	1,132,853
15......	"	"	849,244	1,033,233	1,398,142	1,074,828	1,040,721	1,197,706	1,813,223
20......	"	"	1,206,924	1,471,873	1,444,780	1,215,329	1,425,696	1,706,173	2,583,000
21......	"	"	"	"	1,450,331	1,236,788	1,429,559	"	"
25......	"	"	"	"	1,464,588	1,306,112	1,442,038	"	"
30......	"	"	"	"	1,471,046	1,364,770	1,452,596	"	"
35......	"	"	"	"	1,471,873	1,402,671	1,459,419	"	"
40......	"	"	"	"	"	1,427,160	1,463,827	"	"
50......	"	"	"	"	"	1,453,208	1,468,516	"	"
Limit...	None	502,104	1,206,924	1,471,873	1,471,873	1,471,873	1,471,873	1,706,173	2,583,000

If the original group is immature the payment of normal cost plus interest only on the initial accrued liability differs from pay-as-you-go in two respects: (1) the contributions are more nearly level instead of sharply increasing, and (2) a fund is built up, at any time t being equal in amount to the excess of the accrued liability at time t over the initial accrued liability. Despite these differences the author prefers to consider these methods contemplating no amortization of the initial accrued liability as Class I methods.

General Relationship Between Normal Cost and Accrued Liability May Not Hold

In Part III we found that, if the population was initially mature, the funding method producing the higher normal cost produces the lower accrued liability, and vice versa. In the initially immature case this general relationship may not hold immediately. For example, the Class I pay-as-you-go payment may be initially lower than first year terminal funding normal cost, even though Class II funding produces an accrued liability and Class I has none. The explanation is, of course, that the paradoxical situation is temporary.

Not quite so obvious is the situation we find if the unit credit method of funding, applied to a given group, produces a lower initial accrued liability than entry age normal (a result one would expect) and yet turns up a lower initial normal cost as well. Such a result is due to the immaturity of the group, which we have seen invariably leads to normal costs which rise under unit credit funding. The lower normal cost under unit credit is a temporary feature only, and the present value of all normal costs is higher under Class III funding, even though the normal cost in early years may be lower.

Illustration of Initially Immature Situation

By changing the preceding illustration somewhat we can make a good numerical representation of the course of contributions and the build up of funds in an initially immature situation. Table III represents an immature population of 1,000 active lives, with no retired lives initially. If this group experiences death and withdrawal exactly in accordance with the service table illustrated in Table I, and if sufficient new entrants come in at age 30 each year to keep active staff up to 1,000, the initially immature group will slowly approach the stationary population shown in Table I.

Table IV illustrates the effect of the several funding methods under these conditions. Two "Class I" methods besides pay-as-you-go are here illustrated, both of which in this particular example build up greater reserves than Class II. Because Class V and VI are practically unimportant these methods have been excluded from the illustration.

TABLE III

x	l'_x	x	l'_x	x	l'_x
30...........	124	45...........	21	60...........	6
31...........	105	46...........	20	61...........	5
32...........	88	47...........	19	62...........	4
33...........	74	48...........	18	63...........	3
34...........	62	49...........	17	64...........	2
35...........	52	50...........	16	65 and up....	0
36...........	44	51...........	15		
37...........	38	52...........	14		
38...........	33	53...........	13		
39...........	29	54...........	12		
40...........	27	55...........	11		
41...........	25	56...........	10		
42...........	24	57...........	9		
43...........	23	58...........	8		
44...........	22	59...........	7		

V. Adjustment for Actuarial Gains and Losses

In Part III the operation of the various funding methods was described under conditions of (1) an initially mature population, and (2) experience strictly in accordance with the actuarial assumptions. The first of these ideal conditions was abandoned in the discussion of the initially immature population in Part IV. To complete the transition from the ideal to the realistic, we now abandon the second of the rigorous "ideal" conditions and look into the practical situation where the actuarial assumptions are never exactly realized.

Origin of Actuarial Gains and Losses

The calculation of the contribution for any given year under any funding method is always based on a set of assumptions or estimates. As the actual experience unfolds it is found that each of these estimates is in error to a greater or less extent, and that these errors give rise to what have come to be known as actuarial gains or losses. The reader can undoubtedly enumerate many of the sources of gains or losses, the net effect of all of which is the total actuarial gain or loss for any particular period. Some of these sources may be overlooked in thinking through pension valuation problems, however. As an aid to clear thinking, a partial list of sources of actuarial gains is therefore here included. In each case the converse represents a source of actuarial loss. Under certain plan provisions or particular funding media any of the following may have no effect (or even the opposite effect). In general, however, an element of actuarial gain arises if:

1. Rates of employee mortality are higher than assumed.
2. Rates of employee withdrawal (especially nonvested withdrawal) are higher than assumed.
3. Rate of interest earned is higher than assumed.
4. Benefits which cannot be determined exactly are overestimated. This could arise, for example, by assuming too steep a salary scale for benefits based on salary, or underestimating Social Security benefits under a "$100 less Social Security" plan.
5. Retirements occur at a higher age than assumed.
6. The value of the pension fund assets appreciates.
7. Errors of various types, overstating the liabilities, are corrected.
8. Provision for expenses of administration is overly adequate.

Determination of Amount of Actuarial Gain or Loss

It is seldom practical to determine the actuarial gain or loss for a given period by summing the various components. It is ordinarily not too difficult, however, to obtain the total gain or loss directly. The most convenient procedure for doing so depends somewhat on the method of funding.

An approach to the computation of gain or loss which has wide application is the comparison between (1) funds actually on hand at the end of the period, and (2) funds "expected" in accordance with the assumptions made. The latter is invariably the accrued liability at the end of the period, less the "expected" unamortized initial accrued liability, *i.e.*, unamortized initial liability at the beginning of the period, with interest to end of period, less payments within the period toward the initial accrued liability, with interest to end of period.

Under either unit credit (Class III) or entry age normal (Class IV) the desired result is obtained without difficulty by this general procedure.

Under Class II funding, where initial accrued liability is ordinarily paid off immediately, the gain or loss is measured by the excess of actual funds over present value of all benefits for retired lives.

Gains or losses under individual level premium funding *can* be obtained in exactly the generalized manner previously set forth. But since under this method payments toward the initial accrued liability are not immediately evident, and since "expected" funds are equal to the sum of the individual level premium reserves, it is more convenient to compare the actual funds with the level premium reserve. The calculation of the reserve item is somewhat arduous, and accordingly the adjustment for gains and losses is difficult under this method of funding, except under insured plans where no losses occur and dividends declared represent the gains.

The dollar amount of actuarial gains and losses involves some difficulty under the aggregate and attained age normal methods as well. The generalized procedure previously suggested is theoretically accurate, but is practically difficult because the calculations of the payments toward the initial accured liability consist of a year-by-year comparison of contributions made with the Class IV normal cost. The redeeming feature of both these methods, from a gain and loss viewpoint, is that adjustment for gain and loss can be easily made without previous determination of the absolute amount of such gain or loss.

TABLE IV

	Class I			Class II	Class III	Class IV			
	Pay As You Go	Unit Credit Int. Only	Entry Age Normal Int. Only	Terminal Funding	Unit Credit 20 Yr.	Entry Age Normal 20 Yr.	Individual Level Prem.	Aggregate	Attained Age Normal 20 Yr.
	Initial Accrued Liability								
	None	$ 431,924	$ 661,315	None	$ 431,924	$ 661,315	$ 661,315	$ 661,315	$ 661,315
	Ultimate Accrued Liability								
	None	$1,206,924	$1,471,873	$ 502,104	$1,206,924	$1,471,873	$1,471,873	$1,471,873	$1,471,873
	Initial Normal Cost								
	None	$ 26,371	$ 27,101	None	$ 26,371	$ 27,101	$ 27,101	$ 27,101	$ 27,101
	Ultimate Normal Cost								
	$ 63,000	$ 33,563	$ 27,101	$ 50,753	$ 33,563	$ 27,101	$ 27,101	$ 27,101	$ 27,101
Beg. of Year	Contributions								
1......	....	$ 36,906	$ 43,230	....	$ 53,402	$ 68,488	$ 126,488	$ 95,591	$ 77,889
2......	$ 840	37,902	"	$ 10,151	54,398	"	112,387	89,867	75,903
3......	2,100	38,771	"	15,226	55,267	"	101,472	84,685	74,106
4......	3,543	39,562	"	18,456	56,058	"	92,778	79,995	72,479
5......	5,326	40,234	"	23,070	56,731	"	85,061	75,728	70,999
10......	17,270	42,324	"	39,041	58,821	"	57,235	59,233	65,277
15......	30,006	43,437	"	42,295	59,933	"	42,032	43,331	61,484
20......	40,582	44,367	"	44,134	60,863	"	34,060	37,730	58,947
21......	42,356	44,543	"	44,409	34,008	27,101	33,002	36,858	31,521
25......	48,158	45,229	"	45,316	34,694	"	29,971	34,015	30,233
30......	54,443	45,468	"	55,829	34,934	"	27,900	31,568	29,125
35......	62,999	44,014	"	63,442	33,480	"	27,101	29,949	28,391
40......	65,559	43,612	"	50,369	33,077	"	"	28,930	27,929
50......	64,249	43,923	"	49,227	33,388	"	"	27,867	27,448
Limit...	63,000	44,098	43,230	50,753	33,563	27,101	27, 101	27,101	27,101
End of Year	Funds								
1......	None	$ 37,829	$ 44,311	None	$ 54,737	70,200	$ 129,651	$ 97,981	$ 79,836
2......	"	76,762	88,869	$ 9,543	111,002	141,293	247,228	191,683	158,772
3......	"	116,269	133,249	23,236	168,273	212,873	355,265	281,125	236,547
4......	"	156,094	177,260	39,103	226,307	284,763	455,613	366,515	313,120
5......	"	195,777	220,543	58,267	284,655	356,622	548,731	447,840	388,263
10......	"	380,564	417,303	178,161	569,997	707,342	918,561	794,067	737,424
15......	"	528,079	570,864	288,992	831,283	1,035,096	1,160,817	1,090,139	1,039,107
20......	"	638,136	682,428	364,714	1,070,060	1,343,743	1,315,868	1,251,691	1,302,039
21......	"	656,331	700,385	375,937	1,088,255	1,361,700	1,339,177	1,277,347	1,323,484
25......	"	719,556	760,785	410,149	1,151,480	1,422,100	1,413,822	1,362,729	1,395,201
30......	"	781,305	815,616	454,999	1,213,230	1,476,931	1,475,905	1,438,799	1,459,655
35......	"	803,511	834,594	528,172	1,235,435	1,495,910	1,495,910	1,471,378	1,484,796
40......	"	793,576	826,569	536,121	1,225,500	1,487,884	1,487,884	1,472,026	1,480,699
50......	"	770,124	806,286	501,002	1,202,048	1,467,601	1,467,601	1,460,955	1,464,588
Limit...	None	775,000	810,558	502,104	1,206,924	1,471,873	1,471,873	1,471,873	1,471,873

Technique of Gain or Loss Adjustment

The funding methods discussed in this paper employ one or either of two techniques in determining how much contributions will be adjusted to recognize previous actuarial gains or losses.

"Immediate" Method

The "immediate" method makes up any loss or offsets any gain, as soon as such gain or loss is evident, by addition to or deduction from the next contribution.

Pay-as-you-go funding invariably and automatically adjusts immediately for gain or loss. This is evident if we think of the contribution (actual benefit payments) as the sum of expected benefit payments plus adjustment for gain or loss. Class II or terminal funding also employs the immediate adjustment technique.

Fully insured plans (those employing conventional group annuities, group permanent, or individual insurance policies) in most cases apply any dividend against the next contribution. To the extent that the dividend immediately reflects actual experience, actuarial gain is recognized at once. The insurance company guarantees eliminate the possibility of actuarial loss.

Immediate adjustment is also theoretically possible under every other funding method considered, and is commonly used in several of them. Due to the difficulties of calculation it is seldom employed with the aggregate or attained age normal funding methods.

"Spread" Method

The "spread" method makes the gain or loss adjustment in easy stages, by spreading the adjustment into the future, such that the present value of future adjustments is equal to the dollar amount of the current gain or loss. Ordinarily the method of spreading follows the normal cost of the particular funding method employed, so that the adjustment for gain (or loss) becomes a deduction from (or addition to) future normal costs.

The spread adjustment method is the only convenient scheme under aggregate or attained age normal funding, has often been used with entry age normal, and is less commonly employed with unit credit funding. When gains and losses are spread under either of these last two methods the term "frozen initial liability" is frequently employed, to distinguish from the immediate adjustment forms of these same two methods.[2]

Mechanics of Spread Adjustment Under Class IV Methods

The entry age normal—frozen initial liability—method relies on the equation

Normal cost =

$$\frac{\text{Present Value All Benefits} - \text{Unamortized Initial Accrued Liability} - \text{Fund}}{\text{Weighted Average Temporary Annuity}}$$

which is an identity so long as "Fund" represents the funds which would have been built up if all actuarial assumptions were realized. By replacing expected fund by actual fund in the right side of this equation, we automatically compute normal cost adjusted for actuarial gain or loss. The adjustment becomes the dollar amount of such gain or loss divided by the weighted average temporary annuity. The same process, repeated in future valuations, respreads the unrecognized portion of the gain or loss over future life years of the then active group (at the same time spreading new gains or losses in the same fashion).

The amortization of the gain or loss by this method is identical to the amortization of the accrued liability under the aggregate method of funding. As might be expected the adjustment for the gain or loss of any period is never completed, but approaches zero as that period falls farther and farther into the past.

Under aggregate and attained age normal, the substitution of actual for expected fund has exactly the same effect as under entry age normal. Gains or losses are again spread in the decreasing asymptotic manner described above, and the adjustment is automatic, entailing no more work than if gain or loss did not exist.

Illustration. We can illustrate adjustment for gains and losses as far as it is discussed in this paper by going back to the illustration in Table IV. If at the end of the fourth year, for example, the fund suffers a loss of \$10,000 through depreciation of securities, the immediate adjustment method calls for an extra contribution of \$10,000 at the beginning of the 5th year. Table V following shows the amortization of this loss in future years when spread adjusting is employed.

VI. Conclusion

In the Bureau of Internal Revenue's "Bulletin on Section 23(p)" and Reg. 111, Sec. 29.23(p) may be found the Treasury rules as to the maximum contribution for which full tax deduction can be claimed. A brief statement of the maximum contribution under the various funding methods here discussed will conclude this discussion.

Class I and Class II funding are not specifically recognized. Presumably contributions would be fully deductible if within maximum contributions established for one of the recognized funding methods.

Unit credit and entry age normal funding are lumped together as "Clause (iii)" methods. Provided the actuarial assumptions are satisfactory, contributions under both are fully deductible up to normal cost plus 10% of the initial accrued liability.

TABLE V

Year	Extra Contribution Added to Normal Cost
5	$1,030
6	948
7	872
8	803
9	739
10	681
20	294
30	124
40	51
50	21

Aggregate and individual level premium funding make up the "Clause (ii)" methods. Contributions under the aggregate method are fully deductible if average temporary annuity does not drop below 5. As to individual level premium funding the Treasury sets forth various tests, one of which must be satisfied in order to obtain full deduction. The effect of these tests is to limit maximum deduction to that under the "normal cost plus 10%" rule established for entry age normal.

Attained age normal is described as a "special" method with both Clause (ii) and Clause (iii) characteristics. Contributions are fully deductible if the past service liability payment is no greater than 10%.

The Treasury specifically requires periodic adjustment for actuarial gains. The immediate adjustment technique results in lowest possible contributions and is of course entirely acceptable as to gains. Although the Treasury position on spread adjustment is not too clear, the Bulletin description of aggregate, attained age normal, and the frozen initial liability form of entry age normal seems to imply approval of spread adjustment.

Actuarial losses can evidently be made up no faster than 10% per year, since for tax purposes they are considered additions to the initial accrued liability. Spread funding as previously described will ordinarily keep extra contributions for actuarial loss within the 10% maximum, and appears to be an acceptable technique for losses as well as gains.

Appendix

Demonstration I

$$^{A}C_1 = \frac{\text{Present value benefits}}{\text{Present value future active life years}} \times \text{Current active lives}$$

(where both present values include future new entrants)

$$= \frac{\sum_a^{r-1} l_x\, {}_{r-x}|\ddot{a}_x + \sum_r^{\omega} l_x \ddot{a}_x + l_a\, {}_{r-a}|\ddot{a}_a (v + v^2 + \ldots.)}{\sum_a^{r-1} l_x \ddot{a}_{x:\overline{r-x}|} + l_a \ddot{a}_{a:\overline{r-a}|}(v + v^2 + \ldots.)} \sum_a^{r-1} l_x . \quad (1)$$

But $$\sum_a^{r-1} l_x\, {}_{r-x}|\ddot{a}_x \equiv v/d\, l_r \ddot{a}_r - v/d\, l_a\, {}_{r-a}|\ddot{a}_a \quad (2)$$

$$\sum_r^{\omega} l_x \ddot{a}_x \equiv 1/d \sum_r^{\omega} l_x - v/d\, l_r \ddot{a}_r \quad (3)$$

$$\sum_a^{r-1} l_x \ddot{a}_{x:\overline{r-x}|} \equiv 1/d \sum_a^{r-1} l_x - v/d\, l_a \ddot{a}_{a\cdot\overline{r-a}|} \quad (4)$$

$$v + v^2 + \ldots. \equiv v/d . \quad (5)$$

Substituting (2), (3), (4), and (5) in (1) $^{A}C_1 = \sum_r^{\omega} l_x$.

Demonstration II

Let

$$b = \sum_a^{r-1} l_x\, {}_{r-x}|\ddot{a}_x + \sum_r^{\omega} l_x \ddot{a}_x \equiv 1/d \sum_r^{\omega} l_x - v/d\, l_a\, {}_{r-a}|\ddot{a}_a$$

$$y = \frac{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|}}{\sum_{a}^{r-1} l_x} \equiv \frac{1/d \sum_{a}^{r-1} l_x - v/d l_a \ddot{a}_{a:\overline{r-a}|}}{\sum_{a}^{r-1} l_x}$$

$$p = \sum_{r}^{\omega} l_x .$$

Now

$$^{A}C_t = \frac{b - {}^{A}F_{t-1}}{y} \qquad (1)$$

and

$$^{A}F_t = ({}^{A}F_{t-1} + {}^{A}C_t - p)(1+i)$$

$$= \left[{}^{A}F_{t-1}\left(1 - \frac{1}{y}\right) + (b/y - p)\right](1+i) \qquad (2)$$

$$^{A}F_0 = 0$$

$$^{A}F_1 = (b/y - p)(1+i)$$

$$^{A}F_2 = (b/y - p)(1+i)\left[1 + (1+i)\left(1 - \frac{1}{y}\right)\right]$$

. . . .

$$^{A}F_t = (b/y - p)(1+i)[1 + s + s^2 + \ldots + s^{t-1}]$$

$$= (b/y - p)(1+i)\frac{1-s^t}{1-s} \qquad (3)$$

where

$$s = (1+i)(1 - 1/y).$$

Now

$$0 \le s \le 1$$

as

$$1 \le y \le 1/d \ ;$$

but

$$y = \frac{\sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|}}{\sum_{a}^{r-1} l_x} > 1 \text{, since } \ddot{a}_{x:\overline{r-x}|} > 1$$

and

$$y = \frac{1/d \sum_{a}^{r-1} l_x - v/d l_a \ddot{a}_{a:\overline{r-a}|}}{\sum_{a}^{r-1} l_x} < \frac{1}{d},$$

since $v/d l_a \ddot{a}_{a\,\overline{r-a}|} > 1$.

$$\therefore \underset{t \to \infty}{\mathrm{L}} s^t = 0$$

and

$$^{A}F_\infty = (b/y - p)\left(\frac{1+i}{1-s}\right) = \frac{b - py}{1 - dy}. \qquad (4)$$

From (1)

$$^{A}C_\infty = \frac{b - {}^{A}F_\infty}{y} = \frac{p - bd}{1 - dy}. \qquad (5)$$

Substituting the right hand forms of the definitions of y and b in (5) we obtain

$$^{A}C_\infty = \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} \sum_{a}^{r-1} l_x . \qquad (6)$$

From (4)

$$^{A}F_\infty = \frac{b - py}{1 - dy} = \frac{b(1-dy) - y(p - bd)}{1 - dy}$$

$$= b - y\frac{p - bd}{1 - dy} = b - {}^{A}C_\infty \cdot y$$

$$= \sum_{a}^{r-1} l_x \, {}_{r-x|}\ddot{a}_x + \sum_{r}^{\omega} l_x \ddot{a}_x - \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{r-a}|}} \sum_{a}^{r-1} l_x \ddot{a}_{x:\overline{r-x}|} . \qquad (7)$$

Discussions

Cecil J. Nesbitt

In this paper Mr. Trowbridge has done an excellent job of classifying and illustrating the various methods of pension funding. From time to time we have discussed similar ideas at Michigan but have never organized a complete analysis such as the author presents. We knew, for instance, that the contributions and fund under the aggregate method for a mature population would approach limits but we did not realize what is now fairly obvious, that those limits would be the contribution and fund for the entry age normal cost method.

Throughout the paper the author uses discrete functions which, of course, are convenient for the calculation of illustrations. For purposes of exploring the theory, continuous methods have some advantages. With a few changes in assumptions and notation it is easy to obtain continuous function formulas parallel to the discrete function formulas of the paper. For example, if we assume that the retirement benefit is \$1 per year payable momently from age r, and let ${}^{A}C_t$ equal the annual rate of contribution at time t under the aggregate method, and ${}^{A}F_t$ the fund at time t, then corresponding to formulas of Demonstration II, we have

$$b = \int_a^r l_x \cdot {}_{r-x|}\bar{a}_x dx + \int_r^{\omega} l_x \bar{a}_x dx \equiv \frac{T_r}{\delta} - \frac{1}{\delta} l_a \cdot {}_{r-a|}\bar{a}_a$$

$$y = \frac{\int_a^r l_x \bar{a}_{x:\overline{r-x|}} dx}{\int_a^r l_x dx} \equiv \frac{\frac{1}{\delta}[(T_a - T_r) - l_a \bar{a}_{a:\overline{r-a|}}]}{T_a - T_r}$$

$$p = T_r$$

$${}^{A}C_t = \frac{b - {}^{A}F_t}{y}$$

$$d({}^{A}F_t) = {}^{A}C_t dt + {}^{A}F_t \delta dt - p dt$$

or

$$\frac{d^{A}F_t}{dt} + {}^{A}F_t\left[\frac{1}{y} - \delta\right] = \left[\frac{b}{y} - p\right]$$

whence

$${}^{A}F_t = -\frac{b - py}{1 - \delta y} e^{-[(1/y) - \delta]t} + \frac{b - py}{1 - \delta y};$$

from which it may be shown that

$${}^{A}F_{\infty} = \frac{b - py}{1 - \delta y}.$$

Demonstration I is related to the "general average premium" concept discussed by Feraud in *Actuarial Technique and Financial Organization of Social Insurance,* page 28. By "general average premium" is meant that contribution which if paid in respect to all present and future participants would be sufficient to provide benefits for all present and future participants. If π^c denotes such a premium for a mature population whose members are to receive the momently benefit indicated above, and if π^c is payable momently, then

$$\pi^c \frac{1}{\delta}(T_a - T_r) = \frac{1}{\delta} T_r$$

or

$$\pi^c = \frac{T_r}{T_a - T_r}. \tag{1}$$

The premium π^c is independent of the interest rate and the method amounts to pay-as-you-go funding.

If the population is immature to the extent that it contains individuals up to age r only but is otherwise distributed according to the service table, and if the general average premium for this case is denoted by π^m, then

$$\pi^m \frac{1}{\delta}(T_a - T_r) = \frac{1}{\delta}(l_r \bar{a}_r)$$

or

$$\pi^m = \frac{l_r \bar{a}_r}{T_a - T_r}. \tag{2}$$

The justification for formula (2) is that at each moment dt in the future, benefits of value $l_r \bar{a}_r dt$ will be incurred and so the total present value of benefits for present and future participants will be $1/\delta(l_r \bar{a}_r)$. This is a terminal (or maturity) funding method but differs from the Class II funding illustration in Table IV in that the contribution remains level from year to year by reason of the assumed service table distribution below age r.

If the population is just commencing to be built up from l_a new entrants each year at age a, and π^n denotes the general average premium for this case, then

$$\pi^n \frac{1}{\delta}(l_a \bar{a}_{a:\overline{r-a|}}) = \frac{1}{\delta}(l_a \cdot {}_{r-a|}\bar{a}_a)$$

or

$$\pi^n = \frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a|}}}; \tag{3}$$

that is, π^n is the entry age normal cost.

For the current cost funding situation with general average premium π^c, the accrued liability, defined as the present value of benefits for all present and future participants less the present value of all future premiums from such participants, remains constant at 0. The accrued liability in regard to just present participants is

$$\int_a^r l_x \cdot {}_{r-x}|\bar{a}_x dx + \int_r^\infty l_x \bar{a}_x dx - \pi^c \int_a^r l_x \bar{a}_{x:\overline{r-x}|} dx$$

which may be reduced to

$$\frac{1}{\delta}(\pi^c - \pi^n)\; l_a \bar{a}_{a\,\overline{r-a}|}\,. \tag{4}$$

The accrued liability for future new entrants, with value of benefits expressed in terms of normal cost, may be written as

$$\frac{1}{\delta}(\pi^n l_a \bar{a}_{a:\overline{r-a}|}) - \frac{1}{\delta}(\pi^c l_a \bar{a}_{a.\overline{r-a}|})$$

which is the negative of (4). Thus the accrued liability for present participants is balanced by an anticipated gain in regard to future new entrants to give a total accrued liability of 0.

For the maturity funding method with general average premium π^m, the accrued liability for retired participants ultimately becomes

$$\int_r^\infty l_x \bar{a}_x dx = \frac{1}{\delta}(\mathrm{T}_r - l_r \bar{a}_r)$$

$$= (\pi^c - \pi^m)\frac{1}{\delta}(\mathrm{T}_a - \mathrm{T}_r)\,. \tag{5a}$$

The accrued liability for active participants from ages a to r is

$$\int_a^r l_x \cdot {}_{r-x}|\bar{a}_x dx - \pi^m \int_a^r l_x \bar{a}_{x:\overline{r-x}|} dx$$

which reduces to

$$\frac{1}{\delta}(\pi^m - \pi^n)\; l_a \bar{a}_{a:\overline{r-a}|}\,. \tag{5b}$$

The gain in respect to future new entrants is also (5*b*); hence the total accrued liability for retired, active and future participants becomes just (5*a*), the liability for the retired group.

For the entry age normal cost funding situation with general average premium π^n, benefits and premiums for future new entrants exactly balance and the accrued liability for this group is always 0, and the total accrued liability is the liability for present participants (active or retired). It is convenient, however, in computing the total liability to calculate benefit values for both present and future participants and proceed similarly for the calculation of premium values. Thus, by the time the active group has grown, according to the service table, to include persons up to age r the total accrued liability is

$$\frac{1}{\delta}[l_r \bar{a}_r - \pi^n(\mathrm{T}_a - \mathrm{T}_r)] = \frac{1}{\delta}(\pi^m - \pi^n)(\mathrm{T}_a - \mathrm{T}_r)\,. \tag{6}$$

At time t years later the total accrued liability is

$$\frac{1}{\delta}[l_{r+t}\bar{a}_{r+t} + (\mathrm{T}_r - \mathrm{T}_{r+t}) - \pi^n(\mathrm{T}_a - \mathrm{T}_r)] \tag{7}$$

and the ultimate total accrued liability is

$$\frac{1}{\delta}[\mathrm{T}_r - \pi^n(\mathrm{T}_a - \mathrm{T}_r)]$$

or

$$\frac{1}{\delta}(\pi^c - \pi^n)(\mathrm{T}_a - \mathrm{T}_r)\,. \tag{8}$$

In the foregoing discussion, gains in respect to future new entrants have been taken into account. It should not be inferred, however, that I favor the discounting of such gains in regard to actual pension plans; in fact, I usually take the opposite attitude. Whatever our attitude be toward that question, we should be willing to examine and understand the possibilities. The paper of Miles M. Dawson, "The Actuarial Basis of Compulsory Insurance," was a good step in that direction.

For some while Michigan students have been presented problems along the lines indicated in the above discussion. My thanks are due to the author for giving a more complete background for such problems.

W. Rulon Williamson

Mr. Trowbridge's Tables II and IV suggest that present management is naturally considerate of future management's solvency.

Two Federal programs of pensions have had press attention lately, one because the concern seems too largely missing, the other because of an election.

1. The Federal Civil Service Retirement System has actuarial reports based on Mr. Trowbridge's Class IV-1 financing method. Last month Chairman Ramspeck had two linked articles in the *Washington Post.* He said that there was $4 billion in the fund, against

accrued liabilities of $9 billion. The $4 billion assets fell short of the two requirements, liabilities on existing pensioners and the guaranteed return of contributions with interest to employees and their survivors. This left nothing toward the employer's liability to active working employees. Congressman Murray quoted the ratio of 12 to 1 for benefit payments and employee contribution for existing pensioners.

2. OASI is being presented as "the biggest pension plan," and "the biggest life insurance system."

On October 7 in Mississippi (press release of October 3) the Federal Security Administrator said the FSA serviced 157 million Americans for health, welfare and education. He said that 100 million of them had OASI wage records in Baltimore.

Also on October 3, Governor Stevenson said in Columbus, Ohio: "Today 65 million people have built up substantial equities in the Social Security system. When you and your wife reach the age of 65, your share in the retirement fund will amount to the equivalent of a $15,000 annuity." The product of 65 million and $15,000 is about a trillion dollars; of 100 million and $15,000, $1½ trillion, of 157 million and $15,000, $2¼ trillion. From age 18 (the end of "dependent childhood") to 65 (the age for "eligibility to OASI") is 47 years. Top pension now is $85. Abject poverty is said to begin below $2,000 a year. The monthly pension corresponding to $2,000 is $65. Using that as the average pension "expected" at 65, the yearly unit would be $1.40 per year of presumptive work. Using only the trillion figure, and Mr. Trowbridge's unit method, Class III, a no-interest base would show about a half-trillion accrued liability. But using 2% interest, U.S. Life Table White Males, 1939-1941, pure annuities, and some rather ancient age distributions, might cut the accrued liability to $150 billion. The present trust fund is about 10% of that. 2% interest on the accrued liability would take $3 billion. $280 billion of life insurance at the annual death rate of 6 per 1,000 would call for provision of $1½ billions. The current liability for one unit of deferred annuity could run $6½ billion. The annual load would reach $11 billion. Current tax collection is about $4 billion.

The Mississippi speech also said: "We are conducting the business of Social Security so efficiently that we have been able to expand the benefits. This month, with few exceptions, each check was larger than the previous one by at least $5 or 12%, whichever was more. We had a little trouble getting the bill passed by Congress. But it went through." The prospect that biennially at each Congressional election $20 billion additional accrued liability is to be accepted is an intriguing one.

The picture of my worried countenance in the first number of the new *Life Magazine*, beneath which was the claim that I would figure the lowest rates on Social Security, seems to have been prophetic.

Clark T. Foster

Mr. Trowbridge has been guilty of an understatement in the introduction to his valuable paper in describing the need for a text on pension funding methods. He points out that the beginner in the pension field, in his attempts to educate himself, must rely on the *Bulletin on Section 23(p)*, put out by the United States Treasury Department. Remembering the hopeless feeling I had when first studying that complex document, I have the feeling that anyone with no other means of learning about pension funding would remain a beginner all his life.

There are two points I would like to raise in connection with this paper; first, to introduce two additional funding classes, which might be referred to as 1½ and 2½, and second, to comment on several methods of combining two or more of the classes the author has described.

Class 1½ belongs somewhere between Class 1, pay-as-you-go method, and Class 2, terminal funding. It has been used in a number of cases, particularly in some of the negotiated steel plans, as a means of leveling the cost in the first few years when the terminal funding cost is often quite high because of the large number of employees immediately eligible to retire. The present value of future pension benefits is paid into the fund in installments over a period of up to five years after each employee reaches retirement rather than in a lump sum at the time of retirement.

Class 2½ lies between the terminal funding of Class 2 and the full funding of Class 3 or 4. In terminal funding, no contributions are made until an employee retires. Under Class 3 or Class 4 funding, a contribution is normally made each year for each employee covered under the plan. Under Class 2½ funding, contributions are made only for employees who have reached a certain age or have completed a certain period of service, despite the fact that they are considered as being covered under the plan before satisfying such age or service requirements. The advantage of this method lies in the elimination of administrative records and actuarial calculations on the young or short-service employees who are most likely to terminate employment. It is particularly convenient to fund benefits only for employees age 35 or over if the plan provides benefits at age 65 based only on years of service up to a maximum of 30 years. Similarly, it is convenient to fund only for employees over age 40 if the maximum benefits are granted after 25 years of service.

In some cases, it is preferable to fund only for employees who have completed, say, two or three years of service. Upon an employee's completion of such a period of service, records are established, and his total estimated benefit is funded over his remaining years of employment.

Under any such program of funding, the annual cost for each employee for whom benefits are being funded is greater than if funding had started at employment, but since a large group of employees is not having any benefits funded, the reserves at any time are less than they would otherwise be. The pattern of contributions from year to year depends on the maturity of the group and its age distribution. In a new organization with a relatively low average age, the cost is likely to increase sharply as more and more employees pass the age at which funding commences.

This Class 2½ funding becomes identical with Class 2, or terminal funding, if the age at which benefits are funded is the retirement age. Similarly, this Class 2½ funding becomes identical with the Class 3 or Class 4 full funding methods if benefits are funded for employees as soon as they are eligible for coverage under the plan.

There are a number of ways by which the various funding methods are frequently combined. It is common, for instance, in a plan providing normal benefits in accordance with a percentage formula, but in which benefits are subject to a certain minimum, to fund the percentage accruals on a Class 3 unit credit method and to fund any additional benefit requited by the minimum on a Class 2 terminal funding basis at retirement. Similarly, in a plan allowing retirement at any time after age 65 with additional benefits accruing as a result of service after 65, it is convenient to assume that each employee will retire at 65 and fund such benefits on a Class 3 or a Class 4 program, funding any additional benefits resulting from service after age 65 on a terminal funding arrangement at the end of each year of service after age 65. The cost of such additional benefits is normally offset by the savings resulting from payments that would otherwise have been made to the employee during his period of postponement.

Another frequently used combination of methods is to establish a past service liability on the Class 3 unit credit method and to fund the future service benefits on a Class 4 individual level premium method. Occasionally, this combination might be further complicated by the use of Class 2 terminal funding for the purchase of disability benefits.

Just as the funding methods themselves may be combined, the various methods of handling actuarial gains and losses may also be combined. Frequently, future service gains are immediately used as a credit against a plan's normal cost, whereas past service gains are temporarily ignored, serving to shorten the period over which the past service liability is funded. This arrangement is possible as long as a corporation's total contributions for any year fall within the Internal Revenue Bureau's specified maximum. Occasionally, certain types of gains from either past or future service are taken immediately while others are spread over a period of years. For example, a loss resulting from salary increases in a plan involving an assumed salary scale may be spread over the period to an employee's retirement, while all other gains or losses are immediately recognized. Alternatively, the loss from salary changes might be allocated between past and future service, with the future service loss recognized at once and the past service loss spread over the past service funding period.

Robert F. Link

One can visualize the population of a group as an organism which passes through a period of growth, a period of maturity, and finally senescence. This is a rather idealistic description since the characteristics of growth, maturity and senescence are usually obscured by such extraneous factors as ups and downs of the economic cycle, changes in the characteristics of the particular industry, etc. The theory of most pension funding methods is most easily examined on the assumption that one has a stable group which can be expected to remain stable for a number of future years. However, this idea is realized so infrequently in practice that the examination and comparison of funding methods on the basis of a stable population may create or promote misconceptions rather than otherwise. Thus, any mathematical theory of pension funding must take as its laboratory a group population which is assumed to be subject to change in its composition as to ages, salaries and so forth.

Mr. Trowbridge is to be congratulated for a paper which (within the limited range of my own reading) appears to be the first attempt to state the definitions, axioms, and theorems of a true science of pension funding methods. His concept of the immature group gets the science of pension funding off immediately on the right foot. His descriptions of various funding methods and their operation in the context of a simple group population should probably be required reading for students of pension funding; they might well be adopted as the foundation for any future developments along these lines.

I am sure that Mr. Trowbridge will not be offended if I suggest that his paper has barely scratched the surface of a great body of potential scientific knowledge of various funding methods. Further points to be examined (and which have been intentionally avoided by Mr. Trowbridge) are such matters as:

a) The effect of differences between the actuarial assumptions and the true experience of the particular group;

b) Extension of his theories to cover the more realistic situation of multiple entry ages;

c) Examination of various methods of estimating pension costs with respect to their appropriateness in predicting future costs;

d) Miscellaneous matters such as the choice of correct and meaningful turnover rates, salary scales, etc.;

e) Special problems arising from the introduction of unusual benefits or employee contributions (in particular those arising from superimposition of an alternate benefit formula on an existing scale of benefits).

In describing the trend of normal costs under various classes of funding, Mr. Trowbridge has tended to give further documentation to what I believe to be an overworked thesis. This thesis is that rising costs under plans which are funded by the unit credit cost method are due mainly to the increase in the average premium age of the group. Occasionally an employer has requested that we explain the reason for rising costs under a deferred annuity plan and predict the trend of these costs for the future. We have found that of the total rise in cost only a small part was usually attributable to increasing average premium ages. The rest was due to such factors as:

a) The tendency, as a group progresses toward maturity, for a higher percentage of the total lives in the group to find themselves in the group of eligible employees;

b) Generally rising salaries (which have an intensified effect under an approximately integrated unit plan);

c) Amendments of the group annuity contract with respect to the rate basis of purchase.

The attempt to analyze trends of cost in terms of an initially immature group population leads to a rather interesting result. One tends to think of the asymptotic approach from the l'_x distribution of Mr. Trowbridge's immature population to the l_x distribution of his stationary population as a smooth progression of uniform direction. Actually this asymptotic approach looks more like a decreasing sine wave. This is somewhat evident from Mr. Trowbridge's illustrations; the terminal funding amounts shown in Table 4 rise until the 35th year, drop again to the 50th year, and reach an ultimate level higher than that of the 50th year. It can easily be seen that if lives leave a group only by retirement (there being no deaths or withdrawals at all) the population would tend to repeat itself on a cycle of $r - a$ years. The decrements have a damping effect on this tendency. This wave motion makes it just a little trickier to draw conclusions from numerical illustrations.

The Equation of Maturity can also be written in such a way as to exclude the liability for retired lives (this is equivalent to paying the benefit in a lump sum at retirement age). In this alternate form, the equation looks like this:

$$C + dF = vR\ddot{a}_r$$

where R is the total annual income for new retirements.

For certain purposes, there seems to be some merit in extending Mr. Trowbridge's notation to embrace two variables, entry age and attained age. This leads to a set of select functions which can be identified by the subscript x, y (x representing attained age and y representing entry age). If one assumes a constant percentage distribution of entry ages for each generation of new entrants, one should ultimately come up with a stationary population expressed by a distribution consisting of the values of $l_{x,y}$. Mr. Trowbridge's algebraic identities based on the Equation of Equilibrium will still apply for the unit credit cost method and the entry age normal cost method, since these equations can be expressed for the double variable case in terms of the sums of various items for each entry age.

A little caution is needed, however, when approaching the problem of ultimate cost under an aggregate funding method. The Equation of Maturity can be written with the ultimate cost per life expressed as an unknown (using the alternate form of the Equation of Maturity), as follows:

$$v\sum_{y} l_{r,y}B_y\ddot{a}_r = {}^{\mathrm{A}}\mathrm{CPL}_\infty\sum_{x,y} l_{x,y}$$

$$+d\left[\sum_{x,y} l_{x,y}B_y \cdot {}_{r-x}|\ddot{a}_{x,y} - {}^{\mathrm{A}}\mathrm{CPL}_\infty\sum_{x,y} l_{x,y}\ddot{a}_{x,y\,\overline{r-x}|}\right]$$

where CPL is the ultimate normal cost per life and B_y is the annual rate of retirement income for an entrant at age y. If we solve this equation, the normal cost per life turns out to be

$${}^{\mathrm{A}}\mathrm{CPL}_\infty = \frac{\sum_{y} l_{y,y}B_y \cdot {}_{r-y}|\ddot{a}_{y,y}}{\sum_{y} l_{y,y}\ddot{a}_{y,y\,\overline{r-y}|}}$$

and the normal cost turns out to be

$${}^{\mathrm{A}}C_\infty = {}^{\mathrm{A}}\mathrm{CPL}_\infty\sum_{x,y} l_{x,y}\,.$$

Put in words, the ultimate normal cost under any aggregate funding method is a constant amount paid each year for each active life, such constant amount to be the same as the amount which, if paid over the future lifetime of each of the entrants of one calendar year, would provide the benefits for these entrants. This result, in retrospect, is not particularly surprising. This normal cost is not precisely the same as that under individual entry age funding, since the latter would be expressed as follows:

$${}^{\mathrm{EAN}}C = \sum_{x,y} l_{x,y}B_y\frac{{}_{r-y}|\ddot{a}_{y,y}}{\ddot{a}_{y,y\,\overline{r-y}|}}\,.$$

The difference between ${}^{\mathrm{A}}C_\infty$ and ${}^{\mathrm{EAN}}C$ is that ${}^{\mathrm{A}}C_\infty$ "socializes" the cost as between entrants who, in ${}^{\mathrm{EAN}}C$, have different level premium costs. There is an actual numerical difference between the two, which is probably unimportant in most cases. The difference ought to be recognized, however, in any theoretical discussion. The two are identical when $B_y \cdot {}_{r-y}|\ddot{a}_{y,y} \div \ddot{a}_{y,y:\overline{r-y}|}$ is constant for all values of y.

It may be felt by some that this type of analysis of funding methods is of little practical value. As a young actuary struggling with practical questions of funding, I believe that many practical questions which have caused me great difficulty in the past can be answered by this paper and its sequelae.

Hilary L. Seal

The author has divided methods of funding pension plans into six classes, one of these being further subdivided into four different methods. He has thus specified nine different funding methods. However, by introducing the concept of alternative "immediate" or "spread" adjustments on account of the gains or losses that can occur in seven of these methods, he has effectively provided us with *sixteen* different ways of funding a pension plan. How many of these would be acceptable to the Treasury for tax deduction purposes?

Judging from the opinions expressed in their *Bulletin* of June 1945 the "spread" method of adjusting for *gains* would not be acceptable if the funding was based on (i) unit credit or (ii) individual level premium methods. On the other hand, *losses* could be made subject to some degree of "spread" by using the Treasury's "Special 10% base" in conjunction with the unit credit method. Further, the aggregate method as described in the *Bulletin* automatically uses the "spread" method of adjustment, though it is likely that the "immediate" type of adjustment could be adopted for *gains*. Naturally, the methods classified by the author as V and VI would not be acceptable for tax deduction.

The net result of these tax considerations is to reduce the author's sixteen funding methods to the nine employed in practice, namely:

(1) Pay-as-you-go
(2) Terminal
(3) Unit credit with immediate gains adjustment
(4) Entry age normal with immediate gains adjustment
(5) Entry age normal with spread gains adjustment (frozen initial liability)
(6) Individual level premium with immediate gains adjustment
(7) Aggregate with spread gains adjustment
(8) Attained age with immediate gains adjustment
(9) Attained age with spread gains adjustment

A comparison of the last seven of these from the income tax viewpoint was made by the speaker in the recently published *Proceedings of the Conference of Actuaries in Public Practice*, Vol. ii, 1952. It will be observed from that paper that, contrary to Mr. Trowbridge's opinion, the Treasury's published views on spreading gains are clear and, with one exception, reasonably consistent.

I mention, in conclusion, that methods (3), (8) and (9) above suffer from a serious limitation: they can be applied only where the benefits for each employee may be regarded as accruing each year on a level, or gradually changing, scale. The methods are difficult to apply, for example, to plans with a lump-sum death benefit or where the pensionable earnings base is the average of the five years' earnings preceding retirement.

Chalmers L. Weaver

This discussion is offered to supplement this excellent paper with the answer to a question that occurs on reading the paper. The author demonstrates that if the tabular assumptions as to mortality and interest prevail in the mature state the entry age normal and aggregate methods lead to identical ultimate funds and annual contributions. The question is how these two methods compare at maturity when the population mortality and interest earnings are not tabular. It is to be demonstrated that if the tabular assumptions as to mortality and interest are conservative the aggregate method produces a larger fund and smaller annual contribution at maturity than does the entry age normal method. The converse is true if the tabular assumptions as to mortality and interest are liberal.

The equations at maturity in this more general state are symbolically the same as those in the paper. The difference is that now the annuities in all the formulas given involve tabular assumptions as to mortality and interest that may differ from the mortality indicated by the l's of the population and the actual interest rate earned. The latter are combined with these annuities in the formulas. These more general conditions hold in the symbolic definitions of b, y, and p given in the paper, and in

$$T = \frac{{}_{r-a|}\ddot{a}_a}{\ddot{a}_{a:\overline{a-y}|}}\sum_a^{r-1} l_x .$$

We have, as in the paper,

$${}^{\mathrm{EAN}}F_\infty = b - Ty$$

$${}^{\mathrm{EAN}}C_\infty = p - d\,{}^{\mathrm{EAN}}F_\infty = p - db + dTy$$

$${}^{\mathrm{A}}F_\infty = b - {}^{\mathrm{A}}C_\infty y = \frac{b - py}{1 - dy}$$

$${}^{\mathrm{A}}C_\infty = \frac{b - {}^{\mathrm{A}}F_\infty}{y} = \frac{p - db}{1 - dy} = p - d\,{}^{\mathrm{A}}F_\infty$$

$$= p - db + d\,{}^{\mathrm{A}}C_\infty y .$$

From these we obtain

$${}^{\mathrm{A}}F_\infty = {}^{\mathrm{EAN}}F_\infty + (T - {}^{\mathrm{A}}C_\infty)y$$

$${}^{\mathrm{A}}C_\infty = {}^{\mathrm{EAN}}C_\infty - d(T - {}^{\mathrm{A}}C_\infty)y .$$

Note that if tabular assumptions are realized

$${}^{\mathrm{EAN}}C_\infty = T, \quad \text{and then} \quad {}^{\mathrm{EAN}}C_\infty = {}^{\mathrm{A}}C_\infty = T .$$

If tabular assumptions are conservative, it is obvious that under the entry age normal method

$${}^{\mathrm{EAN}}C_\infty = T - vG = T - (1 - d)G ,$$

where G is the gain that would turn up at the end of the year if T were the contribution.

$$vG = T(1 - dy) + db - p .$$

Then

$${}^{\mathrm{A}}C_\infty = T - (1 - d)G - dTy + d\,{}^{\mathrm{A}}C_\infty y$$

$${}^{\mathrm{A}}C_\infty = T - \frac{1 - d}{1 - dy}G$$

$${}^{\mathrm{A}}F_\infty = {}^{\mathrm{EAN}}F_\infty + \frac{1 - d}{1 - dy}Gy .$$

Since in practice $dy < 1$ we have
if $G > 0$

$${}^{\mathrm{A}}F_\infty > {}^{\mathrm{EAN}}F_\infty$$

$${}^{\mathrm{A}}C_\infty < {}^{\mathrm{EAN}}C_\infty ,$$

and if $G < 0$

$${}^{\mathrm{A}}F_\infty < {}^{\mathrm{EAN}}F_\infty$$

$${}^{\mathrm{A}}C_\infty > {}^{\mathrm{EAN}}C_\infty .$$

The aggregate method has a fundamental weakness. If the tabular assumptions are modestly on the conservative side the fund will grow to a materially higher limit than the funds that would be held by the entry age normal method. The converse is also true. The method is very sensitive to the tabular assumptions and should not be used unless there is frequent readjustment of these assumptions to realistic values. Most actuaries would rather use modestly conservative tabular assumptions, check these less frequently, and use a less sensitive method in the meantime.

Conclusion.—If tabular assumptions are conservative, then under any funding method the immediate adjustment of gains would still leave a fund that would be adequate. The spread method of adjustment would build up additional funds. If tabular assumptions are liberal, then under any funding method the immediate adjustment of losses would still leave a fund that would be inadequate. The spread method of adjustment would draw down the fund to a lower level.

A numerical illustration has been prepared. The fund and annual contribution at maturity have been computed for *active* lives for a population with entry age 35 and retirement age 65. The population has CSO mortality, and turnover of 5% at ages under 50 graded to no turnover at ages 60 and over. The earned rate of interest is set at three values, 2%, 2½% and 3%. The tabular assumptions are CSO mortality, no turnover, and 2½% interest. The figures for the unit credit method are also included.

		Unit Credit	Entry Age Normal	Aggregate
2% Interest	*F*	$932,000	$1,114,000	$1,607,000
	C	49,400	45,800	36,100
2½% Interest	*F*	$932,000	$1,114,000	$1,768,000
	C	44,600	40,200	24,200
3% Interest	*F*	$932,000	$1,114,000	$1,961,000
	C	39,800	34,600	9,900

George E. Immerwahr

Mr. Trowbridge's paper answers a long-standing need in actuarial literature for a description and analysis of the pension funding methods commonly used in the United States.

In discussing the Treasury rules relating to limitations applying to deductions for pension contributions, Mr. Trowbridge states that the Treasury position on spread adjustment for gains is not too clear, but that approval of spread adjustment is implied by its description of the aggregate, the attained age normal, and the frozen initial liability methods in the June 1945 *Bulletin on Section 23 (p)* of the Internal Revenue Code. The Treasury regulations on the matter, as revised in November 1948, state that "in determining the costs and limitations an adjustment shall be made on account of any experience more favorable than that assumed in the basis of limitations for prior years, and, unless such adjustments are consistently made every year by reducing the limitations otherwise determined by any decrease in liability or cost arising from experience in the next preceding taxable year more favorable than the assumed experience on which the costs and limitations were based, the adjustment shall be made by some other method approved by the Commissioner." [3], These regulations would imply the acceptability of adjustments made by the methods set forth in the 1945 *Bulletin,* since that bulletin has not been revoked, or by various other methods satisfactory to the Commissioner. While no amplification of the 1945 *Bulletin* has been published, it would appear, from the types of adjustment regularly employed by a number of consulting actuaries and insurance companies, that the following practices would be found satisfactory.

1. For plans where "spread adjustment" is implicit in the funding method, *e.g.*, in the aggregate method or frozen initial liability method as described in Mr. Trowbridge's paper, full contributions determined in accordance with the method would be deductible provided the following conditions (designed to prevent initial overfunding) are met:

a) adequate allowance is made for withdrawals and mortality, and all other assumptions are reasonable; and

b) no substantial proportion of the contributions is paid on behalf of employees who are unlikely to receive any benefits.

Condition (*b*) can perhaps best be satisfied for the typical plan by eliminating from coverage (at least for the purpose of computing contributions) those employees who fall below a specified age, such as 30, or those with less than a given number of years of service, such as 3 for salaried employees or 5 for hourly-paid employees, or those who fail to meet some appropriate combination requirement of age and years of service. Allowance for withdrawals may sometimes be omitted from condition (*a*) where no salary scale is assumed and where broad enough elimination from coverage is made under condition (*b*).

2. For plans funded by the entry age normal (nonfrozen) method or the unit credit method, a method may be used under which (rather than requiring the full immediate adjustment each year described in Part VIII of the 1945 *Bulletin*) the amount of deductible contribution in any year is based upon the revised costs of the plan as they would have been currently determined less any excess of (*a*) amounts of contributions actually taken as deductions in past years, over (*b*) the amounts which would have been deducted based on such currently redetermined cost. An illustration of the application of this method to a typical group annuity is shown in the accompanying table; the application of the method to a self-insured plan funded by the entry age normal method would be somewhat different but would follow from the same principle.

ILLUSTRATION OF MAXIMUM DEDUCTIBLE CONTRIBUTIONS UNDER CONVENTIONAL NONCONTRIBUTORY GROUP ANNUITY PLAN

(Minor interest adjustments ignored)

	YEAR OF PLAN		
	First	Second	Third
1. Initial past service cost as determined at inception of plan	$100,000	....	....
2. Reductions in initial past service cost due to withdrawals in year (other than deaths or ill-health terminations), whether past service annuities had been purchased for withdrawing members or not	6,000	$ 7,000	$ 2,000
3. Initial past service cost as redetermined at beginning of year	100,000	94,000	87,000
4. Gross current service costs in year	15,000	16,000	18,000
5. Cost credits allowed against current service contributions in year, arising from			
a) Cancellation of past service annuities already purchased	0	2,000	500
b) Cancellation of current service annuities already purchased	0	500	900
c) Total	0	2,500	1,400
6. Total contributions paid in year	30,000	30,000	30,000
7. Portion of contributions deductible (and deducted) under method referred to in this discussion			
a) Gross current service costs less current service cost credits (4 less 5*b*)	15,000	15,500	17,100
b) 10% of redetermined initial past service cost (10% of 3)	10,000	9,400	8,700
c) Cumulative deductions for previous years (sum of 7*g* for all previous years)	....	25,000	49,300
d) Sum of 7*a* for all previous years	....	15,000	30,500
e) Current year's 7*b* multiplied by number of previous years	6	9,400	17,400
f) Adjustment = sum of previous years' actual deductions less deduction on redetermined basis (7*c* less sum of 7*d* and 7*e*)	....	600	1,400
g) Deductible contribution for current year (7*a* plus 7*b* less 7*f*)	25,000	24,300	24,400

William M. Rae

It is somewhat difficult to visualize the relationship between the various, apparently unrelated, funding methods. Mr. Trowbridge has very ably demonstrated their relationship by the algebra pertaining to a mature population. I have also found the following general reasoning approach to be quite helpful. It is a prospective approach.

Every funding method calls for determining the group to be valued. This is usually all pensioners and present employees, or all except those not yet meeting certain minimum age and service requirements.

Having determined the group to be valued, every funding method can be viewed as calling for the calculation of the present value of all future benefits for the valuation group *as a closed group*. This present value, less funds on hand, is then split into two parts, (*a*) unfunded accrued liability and (*b*) present value of future normal costs. The split is dictated by the funding method chosen. Each part is then amortized over a period of years in the manner dictated by the particular funding method. Under some methods (*e.g.*, aggregate method, individual level premium method) the amortization scheme is the same for both parts. The different amortization schemes of the various funding methods cause the different incidence of annual cost between the methods.

The amortization scheme, in dollars, for (*b*) above can be level *as to an individual* or increasing as to an individual (*e.g.*, entry age normal with salary scale method, unit credit method), but will decrease in the aggregate as the closed valuation group is assumed to retire, die or withdraw.

The valuation process in subsequent years can be viewed in exactly the same fashion, subject to whatever adjustment for gains and losses is called for by the particular funding method. In subsequent years we will again be valuing a closed group, but the composition of the closed group will be different from that of the preceding year. New lives will have been added. These, broadly speaking, counterbalance the exits of the previous year. As a consequence the total normal cost will not actually decrease from year to year as might be inferred from the preceding paragraph.

It is theoretically possible to value an open group rather than a closed group, making assumptions as to new entrants in future years. Mr. Trowbridge does this in his Demonstration I. In practice it is rarely, if ever, done for private pension plans.

Frank L. Griffin, Jr.

The author is to be complimented on a clear exposition of the nature of various methods of budgeting pension costs. While the paper deals with matters largely theoretical, and therefore does not lend itself to a discussion from the standpoint of the strictly practical problems faced by consulting actuaries, nonetheless his general approach, omitting the mathematical symbolism, is sometimes found useful by consultants in dispelling for their clients the "technical mysteries" of different methods. Furthermore, an extension of the principles set forth in the paper makes possible an appraisal of the results obtained by using various methods in an actuarial valuation of costs, considering both the nature of the employee group and the purpose to be served by the particular valuation.

For his classification of funding methods, the author has made use of the "Equation of Maturity," $C + dF = B$, in which only the size of the ultimate contribution (C) and fund (F) will vary according to the method. Using the so-called mature population concept, he determines the ultimate C and F, by means of which the funding methods are classified in a logical order—namely, in ascending order of F (descending order of C). Omitting Classes V and VI, which were included in the paper for theoretical reasons only, the remaining classes are: (I) pay as you go, (II) terminal funding, and two classes (III and IV) of funding in advance of retirement. The separation of funding in advance of retirement into Classes III and IV was necessitated on the basis of the ultimate contribution required, a point on which further comment will be made later.

One or two comments relative to the mature population concept which forms the basis of the author's presentation may be in order. A "mature age distribution" and a "stationary population" are not one and the same for purposes of a pension forecast, since the size of an employee group may remain stationary indefinitely without its having reached a mature age distribution. The difference, of course, can be brought about by a varying number of new hires each year or by hirings at many different ages, rather than a uniform number of hires each year at the youngest age of the service table which is the unrealistic assumption inherent in the conventional maturity concept.

In the case of a well established organization, the assumption of a constant work force moving toward maturity in its age distribution is probably as defensible as any other approach. However, the "mature age distribution" which the group might be considered to reach would not be of the form usually assumed, namely, proportionate at all ages to the l_x^s column of the service table. The latter would be true, as indicated in the preceding paragraph, only if all new entrants came into service at the youngest age of the service table. If, for example, a constant number of annual new entrants were distributed in fixed ratios at each age from 20 to 40, the ultimate "mature age distribution" would be in proportion to the l_x^s column only at ages 40 and over. Below age 40, the distribution would be in proportion to

$$l_x^s \sum_{y=a}^{y=x} (H_y \div l_y^s)$$

where H_y is the percentage of total hirings at age y.

In practice, none of the "ideal" conditions of a mature population (either initially or in the ultimate) will ever be found. Notwithstanding this fact, the concept may serve a useful purpose as a limiting value in a pension projection. For example, if the actuary wishes to compare the results of a valuation by any particular cost method, with a projection of pension payouts considering future new entrants, the reasonableness of his results for a "going concern" or the relative trend of costs by different methods may be made apparent.

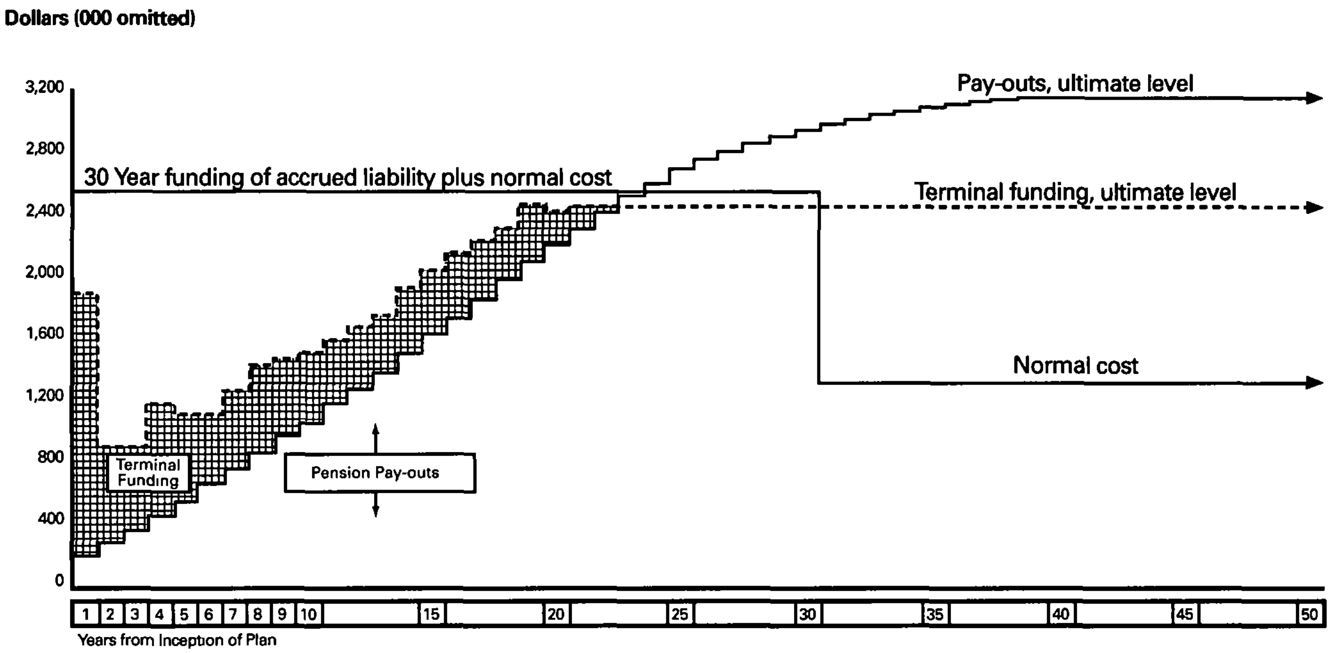
Since the actuary is confronted, not with a mature group, but with a group of unknown future age distribution and size, practical considerations usually dictate that any valuation he makes (Class III or IV) be limited to the group of employees existing on the date of valuation, without allowance on any empirical basis for any new entrants of the future. Depending on the actuarial cost method, the resulting costs may or may not reasonably approximate the long range requirements, even in a case where it is thought that the work force will remain constant in the future; and a projection of payouts (Class I) or terminal funding requirements (Class II), taking into account future new entrants on a reasonable basis for maintaining the work force, may help to establish the relative merits or deficiencies of different Class III or IV valuation methods for a "going concern."

Obviously, if we were in a position to predict the new entrants of the future with any accuracy, the projected requirements by Class I or II would be exactly equivalent financially to the contributions developed, in turn, by the initial and successive future valuations of the plan, by any valuation method selected. Therefore, the result obtained by a particular method in a single valuation, measured against a long range projection of disbursements, affords an indication of the reliability of such result in relation to future requirements, or, what is the same thing, the relative trends which contributions determined by different valuation methods will follow in future years.

The accompanying chart, prepared for a large organization, sets out the projected payouts and terminal funding requirements against the indicated annual contributions determined from an initial valuation by the entry age normal method. In this chart, the discounted value of payouts into perpetuity, considering new entrants, is practically identical to the discounted value of contributions into perpetuity, *if such contributions determined in respect of the present employee group only were to remain at their originally determined level.* The propriety of the valuation method for a continuing plan and a "going concern" is thus reasonably well established. In contrast, if the "unit purchase" method had been employed in this case, the indicated level of contribution (initially determined amount) would have been much less, leading to the conclusion that contributions by such a method would increase in the future if the group were to maintain its size.

CHART 1
PROJECTION OF PENSION CONTRIBUTIONS, PAYOUTS, AND TERMINAL FUNDING REQUIREMENTS

Assuming: (1) Constant Work Force Supported by New Hires with Identical Entry Ages as the Original Group
(2) Mortality, Disability, Withdrawal, and Interest as Assumed in Valuation and
(3) Indefinite Continuation of the Plan without Change



Other actuarial assumptions being the same, the entry age normal method always produces a higher accrued or past service liability than the unit purchase method. The relative size of the normal (or current service) costs, however, will depend on the existing age distribution on date of valuation. Examples of comparative figures by the two methods, derived from other cases, are as follows:

	Past Service Liability	Normal (Current Service) Cost
Case A: Entry age normal..	$37,908,000	$2,461,000
Unit purchase......	21,895,000	2,401,000
Case B: Entry age normal..	$ 820,000	$ 76,000
Unit purchase......	601,000	71,000

The wide difference in the results of initial valuation by the two methods, when it is a certainty that all methods must produce the same capitalized value of contributions, points up the absurdity of trying to compare such results without recognizing the different purposes they are intended to serve. The difference in purpose, implicit in the author's separate treatment of Class III and IV methods, may be stated briefly as follows. From one viewpoint (that of a going concern and a continuing plan), the funding requirements developed by the valuation should take into account not only the past but also the future requirements on a basis which will tend to equalize long range trends in the age distribution. The entry age normal method does this to the maximum extent possible for a group assumed to be stationary in size. From another viewpoint (that of establishing liquidation values under a terminating plan, *i.e.*, accrued liabilities without regard to the future), the requirements developed by the valuation will take into account only the past. The unit purchase method is the only one which provides this particular answer, and it will do no more.

The structure of Class IV methods, adapting them to the requirements of a "going concern," is therefore such as to strike a balance at a given moment of time between (*a*) the existing funds and anticipated future income, and (*b*) anticipated future disbursements. The Class III method (unit purchase) omits all consideration of future income, and of future disbursements arising from pension credits for service after the valuation date.

Of the Class IV methods, the entry age normal undoubtedly has more to commend it in the usual case than the others which the author has mentioned. It may be of interest, therefore, to illustrate the manner in which (and the conditions under which) this particular method will afford a reasonable representation of the long range requirements for an organization expecting to continue in business indefinitely. The following valuation results are presented on the basis of a conventional valuation, and the present values of both benefits and normal costs are with respect to present employees only, without allowance for any future new entrants.

1. Present Value of Future Benefit Payments	$42,000,000
2. Present Value of Future Normal Costs	$16,000,000
3. Balance = Gross Accrued Liability	$26,000,000
4. Funds Accumulated	$ 7,000,000
5. Balance = Unfunded Accrued Liability	$19,000,000
6. Normal cost (in addition to any payments toward unfunded accrued liability), used in determination of Item (2)	$ 1,300,000

Put in the form of a balance sheet, the asset items would be items (4), (2) and (5), usually in that order, and the balancing liabilities would be represented by item (1).

To complete the illustration, *if future new entrants were introduced in such a manner as to maintain a constant normal cost in future years,* the balance between the asset and liability figures would not be disturbed. For example, assuming new hires sufficient to maintain a constant work force and at the same entry ages as the group being replaced each year (one of several possible assumptions), the normal cost developed by the initial valuation would be paid in perpetuity, and the benefits ultimately payable to new entrants of the future would (on the actuarial assumptions) be exactly met by their normal costs. This being the case, items (1) and (2) of the above table would be increased by exactly the same amount, leaving all other figures unaffected. Thus, one of the virtues of the entry age normal method is that, even though future new entrants are not specifically considered in a valuation, the result will be as good an approximation to the long range level of costs for a well established continuing organization as it is possible to furnish.

Because the concept fits naturally into the author's classification of funding methods by means of an ultimate "Equation of Maturity," considerable stress has been given in the preceding paragraphs to the "going concern." This concept implies the use of a cost method which, for a constant work force, would develop an initial normal cost as consistent as possible with the normal costs developed in the ultimate situation. My remarks do not in any way bear upon the aptness of the experience assumptions selected by the actuary, nor upon widely divergent philosophies as to the timing of pension contributions over a long period of time. Obviously, there may be situations where even Class I or Class II funding can be considered appropriate, at least for temporary periods. Class III represents a big step toward funding in advance of retirement, and the fact that this method is not designed on a "going concern" basis does not destroy its usefulness for purposes other than the establishment of termination values. Class IV represents the practical ultimate in advance of funding.

Mr. Trowbridge's paper provides a logical system for the classification of actuarial cost methods and its clear and concise presentation will undoubtedly be found helpful by many students of the subject.

Author's Review of Discussion

Charles L. Trowbridge

Mr. Link's analysis puts certain limitations on the ultimate identity of the aggregate and entry age normal methods. He grants the conclusion reached by the paper where a single entry age is assumed, but proves that if the stationary population arises from entrants at several ages, ${}^{A}C_\infty$ and ${}^{EAN}C_\infty$ are no longer identical, although the numerical difference may be unimportant.

The aggregate method, in effect, views the multi-entry age stationary group as if it all entered at the youngest possible entry age, with later entrants treated like negative withdrawals. The resulting normal cost is, as Mr. Link says, a "socialization" of the individual normal costs at exact entry ages.

Class	Normal Cost	Accrued Liability (Ultimate Fund)
	Algebraically	
1½ (or II minus)	$\frac{\ddot{a}_r}{\ddot{a}_{r\,\overline{5}\rceil}}\sum_{r}^{r+4} l_x$	$\sum_{r+1}^{\omega} l_x\ddot{a}_x - \frac{\ddot{a}_r}{\ddot{a}_{r\,\overline{5}\rceil}}\sum_{r+1}^{r+4} l_x\ddot{a}_{x:\overline{r-a-5}\rceil}$
2½ (or III minus).......	$\frac{1}{r-a-5}\sum_{a+5}^{r-1} l_x \cdot {}_{r-x}\vert\ddot{a}_x$	$\frac{1}{r-a-5}\sum_{a+5}^{r-1}(x-a-5)l_x \cdot {}_{r-x}\vert\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x$
2½ (or IV minus).......	$\frac{{}_{r-a-5}\vert\ddot{a}_{a+5}}{\ddot{a}_{a+5:\overline{r-a-5}\rceil}}\sum_{a+5}^{r-1} l_x$	$\sum_{a+5}^{r-1} l_x \cdot {}_{r-x}\vert\ddot{a}_x + \sum_{r}^{\omega} l_x\ddot{a}_x - \frac{{}_{r-a-5}\vert\ddot{a}_{a+5}}{\ddot{a}_{a+5\,\overline{r-a-5}\rceil}}\sum_{a+5}^{r-1} l_x\ddot{a}_{x\cdot\overline{r-x}\rceil}$
	Numerically	
1½ (or II minus)	\$53,206	\$ 401,542
2½ (or III minus).......	\$35,410	\$1,131,208
2½ (or IV minus).......	\$30,978	\$1,312,888

Mr. Link is also quite correct that the initially immature group approaches the ultimate mature state asymptotically from both sides. Under the conditions stated for Table IV, the group passes from badly immature to somewhat overmature, then back to slightly immature, etc. This came as somewhat of a surprise to the author, as it did to Mr. Link.

Mr. Foster's Classes 1½ and 2½ can be presented algebraically and numerically in the same fashion as the other methods have been analyzed in the paper. The following table presents the formulas for normal cost and ultimate fund for the initially stationary population, assuming that in Class 1½ the funding period beyond retirement is uniformly five years, and assuming in Class 2½ a "waiting period" of five years.

Class 2½ naturally breaks into two subclasses, depending on whether Class III or Class IV funding becomes subject to the waiting period. Perhaps Mr. Foster would permit me to call these III minus and IV minus. The numerical illustration can be considered an addition to Table II, and to the limiting situation in Table IV.

For practical work involving Class 2½ it is not uncommon to offset the cheapening of the funding method due to exclusion of certain lives from the funding by overconservative assumptions with respect to the included lives. This tends to "remove the minus" and bring the funding closer to Classes III or IV.

Mr. Rae analyzes the various funding methods by means of the concept of an "ever changing closed group." This analysis is particularly appealing because it follows exactly the kind of group actually employed in practical valuations. The "open group" approach to which he refers is of considerable theoretical interest, even though the necessity for assumptions as to future new entrants eliminates it for most practical work.

Dr. Nesbitt's (and Mr. Feraud's) "general average premium" is of course a result of the "open group" approach. The general average premium π^c in the stationary population assumed is equivalent to the pay-as-you-go contribution, which is in turn equivalent to π^n, or what the paper refers to as Class IV normal cost, plus interest on the Class IV accrued liability. If we now shift our frame of reference and think of π^c instead of π^n as the "normal cost," the corresponding "accrued liability" becomes 0. It is under these latter definitions that the anticipated gains from future new entrants offset the shortage of funding in respect to the initial group. Dr. Nesbitt states that in general he does not advocate the discounting of such gains; I assume this means that he ordinarily recommends that the accrued liability (in the sense used in the paper) should eventually be funded.

Mr. Griffin views Class III funding as essentially retrospective, looking back at benefits accrued. On the other hand he thinks of the Class IV methods as fundamentally prospective, and points out that under certain conditions the initial normal cost is representative of the ultimate cost. These conditions involve among others an unchanging average entry age.

Suppose, however, that present hiring policies indicate a different entry age for future new entrants than the average for initial participants. In such cases the above characteristic of Class IV funding can be preserved only by assuming for the initial group the same pattern of hiring ages as is indicated for the future. If it is important that future normal costs remain relatively constant, this modification of the usual exact entry age method would seem to be appropriate.

Dr. Seal attempts to put a practical emphasis on what is essentially a theoretical paper. After examining the theoretical possibilities to determine which are acceptable to the Treasury, he produces a list of nine "practical" methods, three of which he limits to certain types of benefit formulas.

Dr. Seal's inclusion of the immediate adjustment form of attained age normal in a list of practical methods is rather surprising, since immediate adjustment for gains or losses is as difficult to make in this method as it is in the aggregate method. Perhaps Dr. Seal is thinking of immediate adjustment in respect to the past service portion (this appears to be feasible), but with a spread of gains or losses arising from the future service portion.

Evidently Dr. Seal finds something not apparent to the author in the *Bulletin on 23(p)*, leading him to the conclusion that spread adjustment for gains is not acceptable under unit credit funding. True, the *Bulletin* does not specifically permit the practice in question; nor does it rule against it. The same situation exists in regard to the technique described by Mr. Immerwahr, which he has found to be acceptable despite its non-inclusion in the *Bulletin*.

Mr. Immerwahr's remarks center around the Treasury regulations with respect to adjustment for actuarial gains. His two conditions under which the spread adjustment technique is acceptable appear to be essentially the same. If I understand him correctly he states that spread adjustment is acceptable provided turnover is adequately recognized—either by a realistic withdrawal assumption, or by sufficient elimination of short service employees from the funding.

It is interesting to note that he has found acceptable a modification of immediate adjustment. This modification appears to amount to the spreading of gains arising within the initial accrued liability over the minimum funding period for such liability, even though greater gains may occur in a particular year.

Mr. Weaver reaches the conclusion that if gains predominate the spread adjustment form of any funding method produces a higher fund than the corresponding immediate adjustment form; but conversely a lower fund is produced by spread adjustment if assumptions are unconservative and losses prevail. The validity of Mr. Weaver's conclusion can be demonstrated rather easily by simple general reasoning. If there is no change in assumptions (and under these conditions a change would seem to be appropriate) spread adjustment tends to exaggerate the overfunding arising from assumptions that prove to be too conservative, and also tends to accentuate any underfunding arising from too liberal assumptions.

The reader of Mr. Weaver's discussion should realize that the comparison there being drawn is between the aggregate method (spread adjustment technique) and the immediate adjustment form of entry age normal. The "frozen initial liability" form of entry age normal produces the same eventual fund and same ultimate contribution as aggregate, even if tabular assumptions are not realized (subject to Mr. Link's exception as to multiple entry ages).

The author does not feel particularly qualified to comment on Mr. Williamson's observations regarding the funding of the Federal Civil Service Retirement System and OASI. Mr. Williamson's comments bring to mind, however, an earlier study of pension funding which might well be brought to the attention of those interested in this subject. I refer to *Actuarial Study No. 10* of the Office of the Actuary, Social Security Board, entitled "Various Methods of Financing Old-Age Pension Plans." Mr. Williamson, Mr. R. J. Myers, and Mr. E. A. Rasor were the authors of this pamphlet, which is an excellent primer on funding method, written in 1938 at the time of the controversy over reserve financing of OASI.

Since the publication of the paper the Treasury position with respect to maximum deductions under individual level premium funding has been changed with the Commissioner's acquiescence in the *Saalfield* decision. It now appears that the contributions called for by individual level premium funding are fully deductible, even if in excess of the "normal cost plus 10%" maximum for entry age normal.

The author wants to thank the several persons who participated in the discussion of the paper, each of whom have in one way or another added to published knowledge regarding methods of pension funding. Even so the author would like to echo Mr. Link's statement to the effect that there is a long way yet to go.

End Notes

1. A peculiarity of the aggregate method is that the assumption of heavier death or withdrawal rates sometimes leads to a higher initial contribution. The higher decrements reduce the average temporary annuity, thereby increasing the percentage *k*. The increase in *k* may be enough to offset the decrease in normal cost and accrued liability.
2. If it seems to the reader that "frozen initial liability" is something of a misnomer for a method under which funding of the accrued liability is contemplated, he may prefer the terminology suggested by Mr. Rae in TSA 1, 274. "Frozen initial liability" might be better applied to the Class I methods described on page 33.
3. Section 29. 23(p)—4 of Regulations 111, as revised by T. D. 5666.