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TERM VERSUS WHOLE LIFE—ACTUARIAL NOTE

D. C. BAILLIE

THE argument is often advanced that a man choosing between equal amounts of Whole Life insurance and, say, Term to Age 65, may be "better off" at age 65 if he takes the cheaper plan. The argument usually assumes that the insured can regularly invest the difference in annual premiums at a strong compound rate of interest—and with even stronger will-power—to accumulate by age 65 a fund which may exceed the cash value then obtainable on the Whole Life plan. Even if the savings fund falls a bit short of the cash value, it is pointed out that the insured has had the advantage of a larger estate on death before 65. It is clear from considerations of actuarial equity that if net level premiums are used in the argument, if the savings fund earns the same net interest rate as the Whole Life reserve, and if the cash value equals the full reserve, then the actuarial value at age 65 of this extra estate available at prior death should just balance the difference between the cash value and the fund.

A layman, however, is not always satisfied by an explanation based on actuarial equity! To eliminate this difficult notion from the comparison, one might envisage a sort of race between two men, both now aged x. The Term man, A, takes out, say, \$10,000 of Term to 65. The Whole Life man, B, takes out \$10,000 of Whole Life. A immediately starts investing the difference between the two gross annual premiums. After a few years, say five or ten, B realizes that A's estate is larger than his, and accordingly takes out some more Whole Life, paying the attained-age gross annual premium. The amount of extra insurance B buys will take into account the fact that A has already had some extra estate coverage, and what is equally important, that A will now start saving a larger annual difference in premiums. After another five or ten years, B must again raise his sights, and A's rate of saving increases correspondingly. At age 65, if we are satisfied that A and B have enjoyed roughly equal estates till that time, it is then valid to compare A's fund with B's total available cash value.

Arithmetical experiments using actual office premiums may be carried out in this way with interesting results. It would, however, be more pleasant if we could predict the size of A's fund with a minimum of arithmetic. In what follows, the two estates are kept in continual balance by the use of continuous premiums. Each estate starts at 1 and grows in t

years to $1 + F_t$, where F_t is A's savings fund. At time t, B is paying a total Whole Life premium at the nominal rate of Π_t per annum. A is still paying his original Term premium, at the rate of, say, T per annum. F_t is earning instantaneous net interest at the nominal rate of δ_1 per annum, which may or may not be the same as the force of interest δ used in computing premiums.

The following two equations apply whether Π_t and T are gross or net premiums.

$$\frac{d}{dt}\Pi_t = {}^c\mathrm{P}_{x+t}\frac{d}{dt}F_t\,, \tag{1}$$

where ${}^c\mathrm{P}_{x+t}$ is the attained age continuous annual premium rate per unit for the increased coverage ΔF_t that B takes out during time Δt. The change in ${}^c\mathrm{P}_{x+t}$ during Δt causes only a 2d order change in Π_t. Also,

$$\frac{d}{dt}F_t = F_t\delta_1 + (\Pi_t - T)\,, \tag{2}$$

since F_t increases in time Δt by interest of approximately $F_t\,\delta_1\,\Delta t$ and new investment of approximately $(\Pi_t - T)\Delta t$.

Writing $(d/dt)F_t$ as $\dot{F}_t$ and $(d^2/dt^2)F_t$ as $\ddot{F}_t$, we have from (2) $\ddot{F}_t = \dot{F}_t\delta_1 + \dot{\Pi}_t$, which is equal to $(\delta_1 + {}^c\mathrm{P}_{x+t})\dot{F}_t$ from (1).

$$\therefore\quad \frac{d}{dt}\log\dot{F}_t = \delta_1 + {}^c\mathrm{P}_{x+t}\,. \tag{3}$$

For simplicity of exposition we now assume that $\delta_1 = \delta$ and that ${}^c\mathrm{P}_{x+t} = \bar{\mathrm{P}}(\bar{\mathrm{A}}_{x+t}) = \bar{\mathrm{M}}_{x+t}/\bar{\mathrm{N}}_{x+t}$. Later we shall make some adjustments removing these restrictions. Since $\bar{\mathrm{M}}_y = \mathrm{D}_y - \delta\bar{\mathrm{N}}_y$ and $(d/dy)\bar{\mathrm{N}}_y = -\mathrm{D}_y$, (3) now becomes

$$\frac{d}{dt}\log\dot{F}_t = \frac{\mathrm{D}_{x+t}}{\bar{\mathrm{N}}_{x+t}} = -\frac{d}{dt}\log\bar{\mathrm{N}}_{x+t}\,,$$

whence

$$\dot{F}_t = (\Pi_0 - T)\frac{\bar{\mathrm{N}}_x}{\bar{\mathrm{N}}_{x+t}}\,,$$

since $\dot{F}_0$ is the initial savings rate of $(\Pi_0 - T)$ per annum.

F_t itself is now

$$(\Pi_0 - T)\,\bar{\mathrm{N}}_x\int_0^t\frac{du}{\bar{\mathrm{N}}_{x+u}}\,,$$

which seems irreducible but can be approximated fairly accurately by the usual methods when $x + t = 65$.

Similarly

$$\dot{\Pi}_t = \frac{\bar{\mathrm{M}}_{x+t}\,(\Pi_0 - T)\,\bar{\mathrm{N}}_x}{(\bar{\mathrm{N}}_{x+t})^2}$$

and

$$\Pi_t = (\Pi_0 - T)\,\bar{N}_x \int_0^t \frac{\bar{M}_{x+u}}{(\bar{N}_{x+u})^2}\,du + \Pi_0\,,$$

again irreducible. We can, however, predict that the sum $F_t\delta + \Pi_t$ must equal

$$(\Pi_0 - T)\frac{\bar{N}_x}{\bar{N}_{x+t}} + T$$

by considering (2). If we further simplify by using net premiums for Π_0 and T, *i.e.*, $\Pi_0\bar{N}_x = \bar{M}_x$ and $T(\bar{N}_x - \bar{N}_{65}) = (\bar{M}_x - \bar{M}_{65})$, we have

$$\begin{aligned} F_t\delta + \Pi_t &= \frac{\bar{M}_{x+t}}{\bar{N}_{x+t}} - \frac{T(\bar{N}_x - \bar{N}_{x+t}) - (\bar{M}_x - \bar{M}_{x+t})}{\bar{N}_{x+t}} \\ &= \bar{P}(\bar{A}_{x+t}) - \frac{{}_t\bar{V}^1_{x:\overline{65-x}|}}{\bar{a}_{x+t}}. \end{aligned} \tag{4}$$

This means that if A and B decide to call a halt to their heroic struggles at time t and continue with level estates of $1 + F_t$, B can do so by merely continuing to pay Π_t, while A can do so by cashing his Term policy for the full reserve (assumed available) and using this value to buy a life annuity of ${}_t\bar{V}/\bar{a}$. With this income and his interest $F_t\delta$, A can then buy 1 unit of Whole Life coverage for a *net* outlay of

$$\bar{P}(\bar{A}_{x+t}) - F_t\delta - \frac{{}_t\bar{V}}{\bar{a}}$$

per annum, giving him a level Whole Life estate of $1 + F_t$ for the same annual outlay as B. Finally, at duration $n = (65 - x)$, the reserve on the Term policy is zero, and (4) becomes

$$F_n\delta + \Pi_n = \bar{P}(\bar{A}_{65})\,. \tag{5}$$

We can check (5) in another way by observing that if A and B are to have the same equity at 65—after having the same coverage to 65 for the same outlay—then A's fund F_n must equal B's reserve, *viz.* $(1 + F_n)\bar{A}_{65} - \Pi_n\bar{a}_{65}$.

$$\therefore\quad F_n(1 - \bar{A}_{65}) + \Pi_n\bar{a}_{65} = \bar{A}_{65}$$

$$\therefore\quad F_n\delta + \Pi_n = \bar{P}(\bar{A}_{65})\,,$$

as required.

As a broad indication of the size of F_n, we can predict that it will much exceed $(\Pi_0 - T)\bar{s}_{\overline{n}|}$ and be much less than $(\Pi_0 - T)(\bar{N}_x/\bar{N}_{65})\cdot n$, which reduces to $n\,\{\bar{P}(\bar{A}_{65}) - \bar{P}(\bar{A}^1_{x:\overline{65-x}|})\}$ if Π_0 and T are net premiums. For example, on the 1941 CSO $2\frac{1}{2}\%$ basis with $x + n = 25 + 40$, we find $F_{40} = 0.88$, using four intervals of 10 years each and either the factors 7, 32, 12, 32, 7, or the Simpson factors 1, 4, 2, 4, 1, as the relative weights for the five ordinates.

GROSS PREMIUMS (NONPARTICIPATING)

To study the effect of premium loadings on F_n, we can now go back to equation (3) and treat ${}^{c}\mathrm{P}_{x+t}$ as the net premium loaded by a percentage k plus a constant c. Also, we can let $\delta_1 = \delta - h$. Equation (3) becomes

$$\frac{d}{dt}\log \dot{F}_t = \frac{(1+k)\,\mathrm{D}_{x+t}}{\bar{\mathrm{N}}_{x+t}} + c - k\,\delta - h\,,$$

whence $\dot{F}_t$ becomes

$$(\Pi_0 - T)\left(\frac{\bar{\mathrm{N}}_x}{\bar{\mathrm{N}}_{x+t}}\right)^{1+k} \cdot e^{(c-h)\,t} \cdot v^{kt}\,,$$

which is the net form previously obtained, multiplied by the factor

$$G(t) = \left(\frac{\bar{\mathrm{N}}_x}{\bar{\mathrm{N}}_{x+t}}\,v^t\right)^k \cdot e^{(c-h)\,t}\,.$$

The new value of F_n may then be thought of as the previous value,

$$(\Pi_0 - T)\,\bar{\mathrm{N}}_x \int_0^n \frac{dt}{\bar{\mathrm{N}}_{x+t}}\,,$$

multiplied by an average value of $G(t)$. In considering the size of $G(t)$ we may perhaps ignore the $(c - h)$ component on the ground that c will be of the order \$3 to \$5 per \$1,000, while h may represent personal income tax at about 20% to 25% of δ, say \$5 to \$8 per \$1,000. In Canada h would not be this large if part of A's fund were invested in shares of taxable Canadian corporations (20% deductible from personal tax), or if part were invested at $3\frac{1}{2}\%$ in Dominion Government Annuities. The other component may be regarded as the kth power of

$$\frac{\bar{a}_x}{{}_tp_x\,\bar{a}_{x+t}}\,, \qquad \text{or of} \qquad \frac{1}{{}_tp_x\,(1 - {}_t\bar{V}_x)}\,,$$

a ratio which can be considerably greater than 1. Its excess over unity is, however, greatly reduced by taking the fractional kth power.

Nothing has yet been said about the relative sizes of the loading scales for Π_0 and T, nor about the fact that T may be computed on a higher mortality basis than Π_0. It is clear that the use of higher factors for the Term premium will reduce the size of $(\Pi_0 - T)$, but the difference is not likely to be reduced much below the net premium difference on a common mortality table. Thus there seems little likelihood that the gross-premium F_n will fall below the net-premium F_n in equation (5).

DISCUSSION OF PRECEDING PAPER

JOHN C. MAYNARD:

It may add something to Professor Baillie's interesting problem to note that his statement on page 384 that $F_t\delta + \Pi_t$ must equal

$$(\Pi_0 - T)\frac{\bar{N}_x}{\bar{N}_{x+t}} + T$$

expresses the fact that A's reserve at time t is equal to B's. This may be seen by rewriting the equation in the form:

$$F_t + T\,\frac{\bar{N}_x - \bar{N}_{x+t}}{D_{x+t}} - \frac{\bar{M}_x - \bar{M}_{x+t}}{D_{x+t}} = (1+F_t)\,\frac{\bar{M}_{x+t}}{D_{x+t}} - \Pi_t\,\frac{\bar{N}_{x+t}}{D_{x+t}}.$$

The left side of this equation is A's reserve, being F_t plus the term reserve viewed retrospectively. The right side is B's reserve viewed prospectively under the assumption that the death benefit remains constant at $1 + F_t$ and the rate of premium constant at Π_t. It is also B's reserve if the death benefit and rate of premium continue to increase, for the value at time t of the excess of future death benefits over $1 + F_t$ is equal to the value of the excess of future premiums over premiums of Π_t. The latter point may be demonstrated by using three identities:

$$\frac{d}{ds}\,\bar{M}_{x+t+s} = -\,D_{x+t+s}\mu_{x+t+s}\,,$$

$$\frac{d}{ds}\,\bar{N}_{x+t+s} = -\,D_{x+t+s}\,,$$

$$\bar{N}_{x+t+s}\,\frac{d\Pi_{t+s}}{ds} = \bar{M}_{x+t+s}\,\frac{dF_{t+s}}{ds}\quad\text{(from equation 1)}\,,$$

and writing an exact expression for B's prospective reserve:

$$\frac{1}{D_{x+t}}\int_{s=0}^{\infty}[\,(1+F_{t+s})\,D_{x+t+s}\mu_{x+t+s}\,ds - D_{x+t+s}\Pi_{t+s}\,ds]$$
$$= \frac{1}{D_{x+t}}\int_{s=0}^{\infty}[\,-\,(1+F_{t+s})\,d\bar{M}_{x+t+s} + \Pi_{t+s}\,d\bar{N}_{x+t+s}]\ .$$

The last expression reduces to the right side of the above equation after two integrations by parts.

CECIL J. NESBITT:

The author discusses an interesting race between a Term man A who purchases a unit of Term to 65 insurance and a Whole Life man B who takes out a unit of Whole Life insurance, there being the further condition that A accumulates a fund from the difference in premiums. It occurs to me that there might be other races of interest that could be set up. An obvious one is between a Term to 65 man and a person who purchases Endowment to age $z (z > 65)$. Relationships analogous to those in the note are readily obtained. For instance, one finds

$$F_t = (\Pi_0 - T)\ (\bar{N}_x - \bar{N}_z) \int_0^t \frac{du}{\bar{N}_{x+u} - \bar{N}_z}.$$

If z were 65, the integral on the right side would become improper for $t = n = 65 - x$. In such case F_t as $t \to n$ must increase indefinitely, as one would be trying to achieve equality between F_n, the amount in the fund at age 65, and $1 + F_n$, the maturity value of the Endowment to 65 insurance.

This difficulty is obviated by choosing z larger than 65. There is another way of avoiding trouble and that is by modifying the conditions of the race. Under the modified conditions B would start off with a unit of Endowment to 65 insurance, but would purchase the increments on a Term to 65 basis. In this case, on the basis of net premiums, $\Pi_0 - T = \bar{P}_{x:\overset{1}{\overline{n}|}} = 1/\bar{s}_{x:\overline{n}|}$, and equations (1) and (3) are replaced by

$$\frac{d}{dt}\Pi_t = \bar{P}\,(\bar{A}^{\,1}_{x+t:\overline{n-t}|})\,\frac{d}{dt}F_t \tag{1'}$$

$$\frac{d \log F_t}{dt} = \delta + \bar{P}\,(\bar{A}^{\,1}_{x+t:\overline{n-t}|}) \tag{3'}$$

$$= \frac{D_{x+t} - D_{x+n}}{\bar{N}_{x+t} - \bar{N}_{x+n}}.$$

It follows that

$$F_t = (\Pi_0 - T) \int_0^t e^{\int_0^k (D_{x+h} - D_{x+n})/(\bar{N}_{x+h} - \bar{N}_{x+n})dh}\, dk.$$

The maturity value of B's insurance is 1 and this must be equaled by F_n. Setting $t = n$, and $(\Pi_0 - T) = 1/\bar{s}_{x:\overline{n}|}$, we obtain the identity

$$\bar{s}_{x:\overline{n}|} = \int_0^n e^{\int_0^k (D_{x+h} - D_{x+n})/(\bar{N}_{x+h} - \bar{N}_{x+n})dh}\, dk. \tag{A}$$

The identity (A) appears to be of no practical value but it is something of a mathematical curiosity. To prove it mathematically, without recourse to actuarial arguments such as were used above, is not trivial. One device for proving it is to consider

$$\bar{s}_{x:\overline{n}|} \quad \text{as} \quad \int_0^n \frac{D_{x+j}}{D_{x+n}}\, dj$$

and to make in the right member of formula (A) a substitution determined by the differential equation

$$\frac{D_{x+j}}{D_{x+n}} \frac{dj}{dk} = e^{\int_0^k (D_{x+h} - D_{x+n})/(\bar{N}_{x+h} - \bar{N}_{x+n})dh} \tag{B}$$

with the initial condition that $j = 0$ for $k = 0$. It turns out that a third member, namely

$$\frac{\bar{N}_{x+j} - \bar{N}_{x+n}}{\bar{N}_{x+k} - \bar{N}_{x+n}}$$

may then be added to equation (B), and it follows that $\lim\limits_{k \to n} j = n$.

Since

$$\frac{D_{x+h} - D_{x+n}}{\bar{N}_{x+h} - \bar{N}_{x+n}} = \frac{\int_h^n D_{x+t}(\mu_{x+t} + \delta)\, dt}{\int_h^n D_{x+t}\, dt} = \frac{\int_h^n D_{x+t}\mu_{x+t}\, dt}{\int_h^n D_{x+t}\, dt} + \delta$$

may be considered to be $\tilde{\mu}_{x+h} + \delta$, where $\tilde{\mu}_{x+h}$ represents a weighted average of the mortality rates μ_{x+t}, $h \leq t \leq n$, we have

$$e^{\int_0^k (D_{x+h} - D_{x+n})/(\bar{N}_{x+h} - \bar{N}_{x+n})dh} = e^{\int_0^k (\tilde{\mu}_{x+h} + \delta)dh} = \frac{\tilde{D}_x}{\tilde{D}_{x+k}}$$

where $\tilde{D}_x$ is based on the forces $\tilde{\mu}_{x+h}$ and δ. It follows that

$$\bar{s}_{x:\overline{n}|} = \int_0^n \frac{\tilde{D}_x}{\tilde{D}_{x+k}}\, dk$$

and, on differentiation in regard to n, we find

$$1 + \bar{s}_{x:\overline{n}|}\,(\mu_{x+n} + \delta) = \frac{\tilde{D}_x}{\tilde{D}_{x+n}}$$

or

$$\bar{s}_{x:\overline{n}|} = \frac{\tilde{D}_x/\tilde{D}_{x+n} - 1}{\mu_{x+n} + \delta}$$

which is reminiscent of

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}.$$

The author is to be congratulated for presenting this interesting note. It provides a good illustration of the versatility and power of continuous functions and methods.

(AUTHOR'S REVIEW OF DISCUSSION)

D. C. BAILLIE:

As Dr. Nesbitt points out, plans of insurance other than Whole Life can be compared with Term to 65 by the method outlined in my note, and I am grateful to him for having taken the trouble to investigate some of these other comparisons in the continuous case.

It appears that my equations (1), (2), (3) are true for a wide variety of combinations of plans. ${}^c\mathrm{P}_{x+t}$ could represent the continuous annual premium rate for any of the conventional plans that include a life insurance element. Π_0 need not necessarily be on the same plan as ${}^c\mathrm{P}_{x+t}$, and Π_t can thus combine two plans (or more, if we think of the ${}^c\mathrm{P}_{x+t}$ plan itself being changed at some duration). Nor does T need to be confined to Term to 65.

I had thought of B as a dyed-in-the-wool Whole Life man, who would buy no other plan under any circumstances. In actual numerical comparisons he may suffer for this loyalty, in so far as he has to buy a substantial amount of Whole Life at age 55, or age 60, the cash value of which may be fairly low at age 65.

Dr. Nesbitt's Mr. B is more flexible. By buying his increments on the Term to 65 basis he avoids the problem of extra cash values at 65, and incidentally makes it possible to predict F_n at once. It is merely the reserve at 65 on whatever plan he used for his original insurance.

$$\text{Thus for } \Pi_0 = \bar{\mathrm{P}}\,(\bar{\mathrm{A}}_{x:\overline{65-x}|})\,, \qquad F_n = 1\,,$$

$$\text{for } \Pi_0 = \bar{\mathrm{P}}\,(\bar{\mathrm{A}}_x)\,, \qquad F_n = {}_n\bar{\mathrm{V}}_x = \bar{\mathrm{A}}_{65} - \Pi_0\bar{a}_{65}$$

$$\text{and for } \Pi_0 = {}_{65-x}\bar{\mathrm{P}}\,(\bar{\mathrm{A}}_x)\,, \qquad F_n = \bar{\mathrm{A}}_{65}.$$

Any one of these results will lead to Dr. Nesbitt's novel identity (A). For the record, here is an alternative proof.

$$\int_0^k \frac{\mathrm{D}_{x+h} - \mathrm{D}_{x+n}}{\overline{\mathrm{N}}_{x+h} - \overline{\mathrm{N}}_{x+n}}\,dh = \left[-\log\left(\overline{\mathrm{N}}_{x+h} - \overline{\mathrm{N}}_{x+n}\right)\right]_0^k$$

$$- \mathrm{D}_{x+n}\int_0^k \frac{dh}{\overline{\mathrm{N}}_{x+h} - \overline{\mathrm{N}}_{x+n}} = \log\left(\frac{\overline{\mathrm{N}}_x - \overline{\mathrm{N}}_{x+n}}{\overline{\mathrm{N}}_{x+k} - \overline{\mathrm{N}}_{x+n}}\right) - \mathrm{D}_{x+n}\cdot I_k\,,$$

where I_k is the second integral.

As k approaches n, each of the two terms grows very large, but their difference remains finite.

The right hand side of (A) is now

$$\int_0^n \frac{\bar{N}_x - \bar{N}_{x+n}}{\bar{N}_{x+k} - \bar{N}_{x+n}} \cdot e^{-D_{x+n}I_k} dk .$$

Since

$$\frac{d}{dk} e^{-D_{x+n}I_k} = \frac{-D_{x+n} e^{-D_{x+n}I_k}}{\bar{N}_{x+k} - \bar{N}_{x+n}} ,$$

this integral equals

$$\frac{\bar{N}_x - \bar{N}_{x+n}}{D_{x+n}} \cdot [e^{-D_{x+n}I_k}]_n^0 = \bar{s}_{x:\overline{n}|} \cdot [1 - \lim_{k\to n} e^{-D_{x+n}I_k}] .$$

Since I_k increases indefinitely as $k \to n$, the limit above is 0, and the identity is established.

Dr. Nesbitt's formula for $\bar{s}_{x:\overline{n}|}$ in terms of a modified life-table is provocative, since it depends on the values $\bar{l}_x$ and $\bar{l}_{x+n}$ but not on any intermediate values $\bar{l}_{x+t}$. Thus any two tables with the same ${}_np_x$ and μ_{x+n} will produce the same $\bar{s}_{x:\overline{n}|}$. A possibly useful approximation occurs when the range x to $x+n$ covers the younger part of a modern mortality table, where μ_{x+t} is small and nearly constant. If we treat it as being constantly equal to an average μ, which in turn equals $(k-1)\delta$, then

$$\bar{s}_{x:\overline{n}|} = \frac{(1+i)^{kn} - 1}{k\delta} = \frac{\bar{s}_{\overline{kn}|}}{k} .$$

This result can be derived directly, as well.

Mr. Maynard has in effect given a simpler explanation of equation (4) than I did. He multiplies it through by $\bar{a}_{x+t}$ and then explains it in terms of reserves. The form I used concentrates attention on Π_t as a function of F_t, rather than as an ingredient of B's reserve.

Before concluding, I should like to record for posterity Mr. Maynard's informal comment: "My sympathies are with B because it is very likely that A will try to slip a little extra into his fund and B will be so busy having continuous medical examinations that he won't notice it." One need hardly point out to Mr. Maynard, or anyone else with his mathematical training, that the continuous calculus is being used here merely as a quick approximation, and guide, to the practical step-by-step comparison outlined in the second paragraph of the note. Also, since A and B are actually one man viewing two possible insurance programs, for A to slip anything past B would be like cheating oneself at solitaire! Mr. Maynard's picture is none the less enjoyable.

I am glad that a note on this touchy subject has produced such good-natured and well-reasoned response, and I thank Messrs. Maynard and Nesbitt very much for their interest.