## NEW POSSIBILITIES IN GRADUATION

KINGSLAND CAMP

SEE PAGE 6 OF THIS VOLUME

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Mr. Camp's paper has been of great value in outlining the problems of performing and testing a Whittaker-Henderson Type B graduation with electronic equipment. In writing a program for the I.B.M. Type 650 Magnetic Drum Data Processing Machine we have made the following modifications of Mr. Camp's approach:

(1) Since a Type A graduation can be considered a special case of a Type B graduation, we are using the same set of instructions to perform both the preliminary and final graduation. This has the advantage of reducing the number of instruction locations needed and of producing the correct preliminary graduation without considering "starter" or "error" values.

(2) To reduce the number of storage locations needed, we are using the Cholesky "square root" method of factoring the matrix represented by the table on page 20. Although this method may not be practical for desk calculators, the extraction of the necessary square roots is no problem for the machine. If $n$ is the number of terms to be graduated, and $z$ is the highest order of differences minimized, then essentially only $n(z+1)$ working values need to be stored under this method where under the Henderson matrix factoring method essentially $n(2z+1)$ working values need to be stored.

Our system is designed to handle graduations of the Whittaker-Henderson general case with $z = 6$ or less and $n = 111$ or less. Without going into the matrix algebra underlying the derivation of the formulas, and trying to adopt Mr. Camp's notation as closely as possible, the square-root method of solving the system of linear equations (20) would in effect replace equations (21), (22), (23) and (24) by:

$$ {}_xS_0 = \sqrt{{}_xR_0 - {}_{x-1}S_1^2 - {}_{x-2}S_2^2 - {}_{x-3}S_3^2 - {}_{x-4}S_4^2} \qquad (21') $$

$$ {}_xS_n = $$

$$ \frac{{}_xR_n - {}_{x-1}S_{n+1}\cdot{}_{x-1}S_1 - {}_{x-2}S_{n+2}\cdot{}_{x-2}S_2 - {}_{x-3}S_{n+3}\cdot{}_{x-3}S_3 - {}_{n-4}S_{n+4}\cdot{}_{x-4}S_4}{{}_xS_0} \qquad (22') $$

$$[W'_x q''_x + {}_{x-1}S_1 \cdot v_{x-1} - {}_{x-2}S_2 \cdot v_{x-2} + {}_{x-3}S_3 \cdot v_{x-3} - {}_{x-4}S_4 \cdot v_{x-4}] \div {}_xS_0 = v_x \quad (23')$$

$$[v_x + {}_xS_1 \cdot q_{x+1} - {}_xS_2 \cdot q_{x+2} + {}_xS_3 \cdot q_{x+3} - {}_xS_4 \cdot q_{x+4}] \div {}_xS_0 = q_x \,. \quad (24')$$

The ${}_xS_n$ functions have the same range (*i.e.*, vanish at the same marginal values) as do the ${}_xR_n$ functions and Mr. Camp's comments on the increasing number of terms in the first five equations for $v_x$ and $q_x$ still hold.

The reason for needing fewer storage locations under this method is that the $S$'s used in (23′) are also used in (24′) although not in the same order. The storage will cause no trouble, for if the $S$'s are stored consecutively as calculated, *i.e.*, ${}_xS_0$, ${}_xS_1$, ${}_xS_2$, ${}_xS_3$, ${}_xS_4$, ${}_{x+1}S_0$, ${}_{x+1}S_1$, ${}_{x+1}S_2$, etc., they will be ready for use in (24′) and a fixed number of locations apart (7 in our program) for use in (23′). For $z$ less than 6 the appropriate number of ${}_xR_n$'s at the end of our equations comparable to (20) are made equal to zero so that one program will fit all types of difference-equation graduations. Using as much of Mr. Camp's material as possible, and the square-root method, one could:

(1) Follow Mr. Camp from the beginning of his paper to equation (2) page 9.

(2) Solve (2) by means of (21′), (22′), (23′) and (24′) where for the $g_2 = 60$ case:

$${}_xR_3 = {}_xR_4 = {}_xR_5 = {}_xR_6 = 0$$

| $x$ | ${}_xR_2$ | ${}_xR_1$ | ${}_xR_0$ |
|---|---|---|---|
| $a$ | 60 | 120 | 61 |
| $a+1$ | 60 | 240 | 301 |
| $a+1<x<\omega-1$ | 60 | 240 | 361 |
| $\omega-1$ | 0 | 120 | 301 |
| $\omega$ | 0 | 0 | 61 |

This could require three passes through the 650, one to calculate the $S$'s, one to calculate the $v_x$'s and one to calculate the $q_x^{a}$'s.

(3) Follow Mr. Camp from paragraph (*c*) page 16 to equation (20) page 20.

(4) Solve (20) by arranging the ${}_xR_n$'s on the card so that the Type 650 program used in step 2 can be used again to calculate the $q_x$'s.

(5) Follow Mr. Camp from III on page 24 to the end of his paper.

It seems possible to break the Whittaker-Henderson graduation problem into 5 steps:

(1) Determining the number of terms to be graduated; the formula (type A, B, etc.); the order or orders of differences minimized; and the graduating coefficient or coefficients. In other words, the determination of the $(\omega - a + 1)$ linear equations of the general form (20) on page 20.
(2) Solving these $(\omega - a + 1)$ linear equations to produce graduated mortality rates.
(3) Determining tests for the graduation—mainly of statistical nature—to measure its acceptability as a graduation, keeping in mind the purpose for which the graduation is being made.
(4) Performing the graduation tests.
(5) Evaluating the results of the graduation tests.

The actuary is primarily concerned with steps 1, 3 and 5 but will probably be called upon for assistance in steps 2 and 4. Step 4 should provide few computational difficulties and step 2 permits a variety of approaches. It may prove easier in step 2 to go back to basic principles and utilize the extensive work that has been done in recent years by mathematicians and others in solving systems of linear equations. At any rate it appears to us that the Henderson *method* of solving the Whittaker difference equation is no longer the most convenient approach to the graduation problem. Many methods are available and the choice should probably depend on the calculating medium available, *i.e.*, desk calculators, small, medium or large electronic calculators, and the preference of the operator.

Mr. Camp is to be congratulated upon his extension of the Whittaker-Henderson graduation theory. The idea of constraining an order of differences toward a geometric progression whose ratio is determined from a preliminary graduation is a significant advance in this field.

## (AUTHOR'S REVIEW OF DISCUSSION)

### KINGSLAND CAMP:

We are much indebted to Mr. Cragoe for presenting a new method for solving the Whittaker-Henderson equations, especially as it enables us easily to handle formulas of still higher order. As he remarks, the new method would not have been useful in the days of desk calculators but becomes so with the versatile electronic equipment now available—another instance of mathematics that was once only theory becoming later also practical.

His suggestion of working the preliminary smoothing process or "A" graduation with the same program scheme (but of course not the same ${}_xR_n$ schedule) is also very good, especially if, as usual, we can be pretty sure how strong a smoothing process will be needed. As readers of my paper may note, my own thought was to incorporate a test of that preliminary job into the same first program with it, and to be sure of accommodating it I used the simpler A-process relation for the work. It does not seem that the starter and error series should make much trouble, considering the comparatively few constants and manipulations (both repeated many times) that are involved.

It may not be amiss to emphasize the peculiar advantage of difference-equation methods for constructing processes of graduation and interpolation. More nearly than any other methods, they operate to let a body of data tell its own story, and may some day even enable us to divine the trend toward which they should constrain (note that they never really fit a curve to) the kind of material operated on. These principles were probably always in the mind of the late Mr. Henderson when selecting and developing processes for practical use, including his interpolation formula. This, I am confident, will be more highly regarded when more of us recognize that the leveling out of third differences (accepted as the index of smoothness) is not a proper test of such a process. (In this connection I much regret carelessly attributing to Mr. H. H. Wolfenden the statement that Henderson's interpolation formula was "only of historical interest." Mr. Wolfenden's words were, "mainly of historical interest.")

Before closing, I should like to point out a practically virgin field of exploration for some of our younger and more flexible mathematical minds: the extension of difference-equation technique to bivariate and multivariate graduation. With the mechanical facilities already available or almost surely in prospect, the great usefulness of such methods for analyzing select experiences and secular trends should be evident.