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THE EFFECT OF VARYING INTEREST RATES

CHARLES H. CONNOLLY

Recent years have witnessed much discussion and activity on the part of various actuaries both in the private and in the public fields in their attempt to resolve the conflict between improvements in mortality on the one hand and statutory requirements related to valuation and deficiency reserves on the other. Considerable progress has been made toward the eventual adoption of a more realistic mortality table for valuation, but the final acceptance by the various states could well be a long way off. In the interim my own Company will use a valuation basis involving the use of two interest rates, which will in our case solve the deficiency reserve problem without requiring the abandonment of either low gross premiums or relatively high withdrawal values. The thought that other members of the Society might be interested in this application of varying interest rates resulted in this note.

The usual formula for the net annual premium for a whole life assurance can be expressed either as $A_x/\ddot{a}_x$ or as $A'_x/\ddot{a}'_x$, where unprimed symbols are derived from an assumption of a level interest rate and primed symbols are derived from an assumption of a varying interest rate. This variation might take on a number of forms, but the type of variation most likely to receive acceptance from regulatory authorities and the insuring public should be quite simple, probably in the form of two specific rates, a rate i' applying for n years followed by a rate i'' for the remaining years of the contract. This type of variation is the principal one treated herein.

Under such an arrangement the formulas for A'_x and $\ddot{a}'_x$ are

$$A'_x = A^{1\,i'}_{x:\overline{n}|} + {}_nE^{i'}_x \cdot A^{i''}_{x+n} \tag{1}$$

and

$$\ddot{a}'_x = \ddot{a}^{i'}_{x:\overline{n}|} + {}_nE^{i'}_x \cdot \ddot{a}^{i''}_{x+n}\,. \tag{2}$$

If i' and i'' are selected so that i' equals $i + k$ and i'' equals $i - k$, as n varies from 0 to $\omega - x$, the value of the resulting premium P^{var}_x will vary from $P^{i''}_x$ to $P^{i'}_x$. Between the extreme values of n a value can be found to produce a premium P_x^{var} equal to a previously selected net between $P_x^{i'}$ and $P_x^{i''}$; one such net might be P_x^i. We would then have

$$P_x^{var} = \frac{A^{1\,i'}_{x:\overline{n}|} + {}_nE_x^{i'} \cdot A^{i''}_{x+n}}{\ddot{a}^{i'}_{x:\overline{n}|} + {}_nE_x^{i'} \cdot \ddot{a}^{i''}_{x+n}} = P_x^i . \tag{3}$$

Comparing the reserves developed by identical premiums, one using varying interest and the other using uniform interest, we see that during the first n years the varying interest reserves will accumulate excess interest at the rate $i' - i$ to offset the lower interest rate i'' applying after the n-year period; no such excess accumulation is required when a level rate i is used throughout. On the varying interest basis this excess interest accumulation amounts to $P_x^i \cdot k/p_x$ in the first contract year, and continues to increase until at the end of n years the excess E is found to be

$$E = A^{i''}_{x+n} - P_x^i \cdot \ddot{a}^{i''}_{x+n} - {}_nV_x^i \tag{4}$$

$$= {}_nV_x^{i''} - {}_nV_x^i + (P_x^{i''} - P_x^i)\, \ddot{a}^{i''}_{x+n} . \tag{5}$$

From the above we see that throughout the duration of the contract the use of a premium P_x^i with rates i' and i'' produces at all durations reserves higher than those developed with a uniform rate i. At some time r previous to the expiration of n years it would be true that

$$P_x^i \cdot {}_ru_x^{i'} - {}_rk_x^{i'} = {}_rV_x^{i''} , \tag{6}$$

and at all durations greater than r reserves would exceed reserves developed on the assumption of an interest rate of i'' throughout. This excess would reach a maximum of $(P_x^{i''} - P_x^i)\ddot{a}^{i''}_{x+n}$.

An example of this approach might be of interest. The net level premium for a whole life assurance of \$1,000 at age 35 assuming 1941 CSO mortality and $2\frac{3}{4}\%$ interest is \$19.80. Using rates of 3% and $2\frac{1}{2}\%$ for i' and i'' and using trial values of n in formula (3), it is found that n equals approximately 23 years. The use of this integral n yields a premium based on varying interest of \$19.82. Terminal reserves developed from this net premium are shown in the table on page 137 as compared with reserves on $2\frac{3}{4}\%$ and $2\frac{1}{2}\%$ interest. The reserves using varying interest exceed $2\frac{1}{2}\%$ reserves after 16 years. This example is based on a value of $i' - i''$ of $\frac{1}{2}\%$ and the selection of P_x^{var} equal to P_x^i. A higher value of $i' - i''$ would yield more striking comparisons, and would also make a wider range within which P_x^{var} could be selected. For example, if the various jurisdictions in which a particular company operated allowed the use of interest rates as high as $3\frac{1}{2}\%$, the values of i' and i'' could be set at 3½% and 2½%, and P_x^{var} could be varied over a range of \$2.60 at 35, or a range of \$2.85 at 55.

CONTROL OF CASH VALUES

Competitive considerations might make desirable the issuance of a contract with cash values approximately those developed from reserves on an interest rate $i - k$, while deficiency reserve statutes or other factors might make the use of a net premium at such a rate undesirable. As noted above, the use of a varying interest approach will develop from a

	Interest Basis		
	Varying	2¾%	2½%
Net Premium:	\$ 19.82	\$ 19.80	\$ 20.50
Duration			
1..........	15.90	15.83	16.49
3..........	48.56	48.22	50.16
5..........	82.37	81.57	84.70
10..........	171.60	168.69	174.39
15..........	266.73	260.01	267.62
16..........	286.35	278.62	286.53
20..........	366.48	353.59	362.44
23..........	427.83	409.84	419.09
25..........	464.58	447.01	456.40
30..........	553.61	537.48	546.80
60..........	907.85	903.15	906.44

net premium at rate i reserves in excess of those at rate $i - k$ after r years. If r is less than the period during which cash values are usually developed from a nonforfeiture factor differing from the net premium, the values can probably be controlled to a sufficient extent to meet the scale of values desired.

If such is not the case, an alternative approach is available. Instead of accepting r as dependent on the premium, the premium can be made dependent on a previously selected value of r. Assume that equality of variable interest reserves and reserves at rate i'' $(=i - k)$ is desired at the end of m years. Setting r equal to m in (6) above, we find

$$P_x^{\text{var}} = \frac{{}_mV_x^{i''} + {}_mk_x^{i'}}{{}_mu_x^{i'}}. \tag{7}$$

Using this premium as a criterion, a value of n can be determined which will produce a premium varying but slightly from P_x^{var} and reserves will then fulfill the condition set out above.

UNDETERMINED INTEREST

Justification for the use of standard values and reserves on substandard insurance contracts can be based on Lidstone's Theorem which results in undetermined mortality rates. A similar approach could yield undetermined interest rates whereby the interest rate in any contract year was the rate required to produce exactly reserves previously selected. Certain legal limits make this approach inapplicable until the reserve reaches a sufficient size to keep the interest rate within the legal maximum. Setting i' at the legal maximum, P_x^{var} could be found from (7) above. After m years the interest rate applicable in any subsequent year t could then be found from the familiar reserve formula

$$({}_tV + P)(1+i) - q_{x+t}(1 - {}_{t+1}V) = {}_{t+1}V\,, \tag{8}$$

where ${}_tV$ and ${}_{t+1}V$ equal the desired reserves at rate i'' and P equals P_x^{var}. Making these substitutions and solving for $1 + i$ gives a value

$$1 + i = \frac{{}_{t+1}V_x^{i''} \cdot p_{x+t} + q_{x+t}}{{}_tV_x^{i''} + P_x^{var}}\,. \tag{9}$$

Substituting for ${}_{t+1}V_x^{i''}$ its equivalent

$${}_{t+1}V_x^{i''} = \frac{({}_tV_x^{i''} + P_x^{i''})(1+i'') - q_{x+t}}{p_{x+t}}\,, \tag{10}$$

formula (9) simplifies to

$$1 + i = \frac{{}_tV_x^{i''} + P_x^{i''}}{{}_tV_x^{i''} + P_x^{var}}(1+i'')\,. \tag{11}$$

This indicates that as long as ${}_tV_x^{i''}$ increases with an increase in t, $1 + i$ decreases and approaches $1 + i''$ as a limit.

The approval by regulatory authorities of such an approach to the interest rate is quite doubtful. This subject of undetermined interest was introduced to point out the lack of interdependence of reserves and premiums when variation in interest is introduced and to suggest the investigation of other methods of varying interest besides the simple two-rate basis used elsewhere in this note.

GAIN AND LOSS EXHIBIT

For many years the comparisons of actual with expected mortality and loading with expenses have been recognized as relatively worthless. Many of us have used interest rates in asset share calculations differing from the reserve interest rate, and have, therefore, made comparisons between actual and "required" interest of doubtful value. The above treatment of interest rates does further violence to the value of the Gain and Loss Exhibit, but certainly not without precedent.

The acceptability of this approach to premiums and valuation hinges on the degree to which company officials are willing to accept the idea of an artificial interest assumption along with already artificial loading and mortality assumptions.

ELECTRONIC DATA PROCESSING MACHINES

The calculation of P_x^{var} and the resulting reserves and nonforfeiture values would become a tedious task because of the varying interest assumption if it were not for the availability of high-speed electronic calculators in the EDPM field. For example, the calculation of extended insurance is the matter of comparing ${}_tCV_x$ with $A^{i'}{}^{1}_{x+t:\overline{k}|}$ until such time as ${}_tCV_x > A^{i'}{}^{1}_{x+t:\overline{n-t}|}$ when ${}_tCV_x$ must be compared with $A^{i'}{}^{1}_{x+t:\overline{n-t}|} + {}_{n-t}E^{i'}_{x+t} \cdot A^{i''}{}^{1}_{x+n:\overline{k}|}$.

With this equipment the calculation becomes a minor problem from a time standpoint. A complete file of reserves and values using varying interest was prepared on two plans of insurance on the standard basis and on twelve substandard tables in about fourteen hours on the I.B.M. type 650 machine.

The more general availability of such equipment will no doubt make the continued study of blocks of business a much easier task, so that comparisons of the actual results forthcoming with the assumptions used in asset shares will make available much more valuable "gain and loss" figures.

CONCLUSION

The use of varying interest in the calculation of premiums and reserves makes possible the lowering of net premiums without the necessity of abandoning the previous scale of cash values. This tool makes possible the elimination or minimization of deficiency reserve problems. The principal disadvantage lies in the fact that the interest basis of reserves will probably differ from the interest assumptions used in arriving at gross premiums. This disadvantage seems minor in comparison with the advantages.