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DISCUSSION OF PRECEDING PAPER

J. B. MABON:

In this note, the authors have gathered together, for convenient reference, the mathematical forms affecting the classical probabilities that exactly r lives or at least r lives out of n lives of different ages will survive t years. They have added very interesting and instructive analyses which will prove very useful under some circumstances.

The note is incomplete since the authors make no explicit reference to or comparison with the direct expansion by multiplication of the continued product in formula (2). Perhaps they regarded this as obvious, but the comparison as to the work involved is important. In the elementary case of 10 lives, five successive pairs of products may be made. Two pairs of these give two products of 4, one of which may be combined with the remaining product of 2 to give a 6th degree product. The advantages of this procedure are: the algebra is elementary and the recording is as easy as possible, several products are formed with one factor fixed on the machine, and the arithmetic can be checked since the sum of the coefficients in each product is unity. In the final product the multiplication may be limited to only those terms which enter into the solution of the problem proposed. The entire 10th degree product was worked in only slightly over two hours time and the bottom figures in Table 1 were verified.

Should anyone be so unfortunate as to require to deal with the complicated case of 8 lives in 28, it is suggested that 14 products of pairs build into 7 products of fours. Two of these give an 8th degree product. Each subsequent multiplication by four of the remaining 4th degree expressions will retain only powers of 8, or less, of x. Summation checks are easily derived. If only one probability is required, the final multiplication will apply only to terms giving the 8th power of x. Assuming that all coefficients remain significant, 289 multiplications, with the advantages mentioned earlier, are required for a single probability. This compares favorably with the authors' figures of 346 and 286. If only the final result is required, the process adds 4 new lives at each step and, in some respects, is an expansion of the authors' formula (1).

If 7 or 8 decimal places are considered sufficient for any probability, a seven figure logarithm table may be used very conveniently to form products of fours. It is easy to set down the combinations for 4 lives. The change from the probability of one life surviving to that of becoming a death requires only the addition of the logarithm of the necessary factor to the logarithm of a previous product. The coefficient of the third power of x, for example, is the sum of all the products of three P's and one Q, and so on.

DONALD R. SCHUETTE AND C. J. NESBITT:

Several years ago at the University of Michigan we had a seminar discussion of a paper, "The $\bar{Z}$ Method for the expression of Multiple-Life Contingencies in terms of Last Survivor Statuses," by S. C. Damle, which appeared in the Institute of Actuaries Students' Society *Journal*, Vol. 9, p. 286. In this paper Mr. Damle introduced the quantity $\bar{Z}^r$ which is defined to be the sum, for all selections of r events out of n, of the probabilities that at least one of the specified r events occurs, irrespective of the outcome of the remaining $n - r$ events. To have something closer to the notation of the present paper, let us denote $\bar{Z}^r$ by $\bar{B}_r$. We (including Mr. Robert Butcher who had a hand in this) investigated by methods of linear algebra some of the interrelationships between the quantities $P_{[r]}$, P_r, B_r and $\bar{B}_r$. One of the relations we have obtained is the formula

$$P_{[0]}c_0 + P_{[1]}c_1 + \ldots + P_{[n]}c_n = c_0 + B_1\Delta c_0 + B_2\Delta^2 c_0 + \ldots + B_n\Delta^n c_0 , \qquad (1)$$

from which the formula of Rasor and Myers, referred to in the present paper, may be obtained as a special case by setting $c_r = 1/(r+1)$. Here the c_r are arbitrary coefficients and are written on the right in order to clarify the application of the operators we shall employ. Our formula (1) is valid even when the n events are not independent; however, if we assume independence of the n events we can obtain the formula rather quickly. To do this, consider the symbolic operator $\phi(E)$ obtained by substituting for x in the generating function $\phi(x)$ of the paper the finite difference operator E,

$$\phi(E) = (p_1E + q_1)(p_2E + q_2)\ldots(p_nE + q_n) .$$

Expanding, we obtain

$$\phi(E) = P_{[0]} + P_{[1]}E + P_{[2]}E^2 + \ldots + P_{[n]}E^n .$$

Hence,

$$\phi(E)\, c_0 = P_{[0]}c_0 + P_{[1]}c_1 + \ldots + P_{[n]}c_n .$$

If we now replace E by $1 + \Delta$, we obtain

$$\phi(E) = (1 + p_1\Delta)(1 + p_2\Delta)\ldots(1 + p_n\Delta)$$
$$= 1 + B_1\Delta + B_2\Delta^2 + \ldots + B_n\Delta^n .$$

Hence, we have that $\phi(E)$ operating on c_0 also produces

$$c_0 + B_1\Delta c_0 + B_2\Delta^2 c_0 + \ldots + B_n\Delta^n c_0 ,$$

which gives us our formula (1).

Another relation we have noticed is

$$c_0 + P_1 c_1 + P_2 c_2 + \ldots + P_n c_n = c_0 + B_1 c_1 + B_2\Delta c_1 + \ldots + B_n\Delta^{n-1} c_1 . \tag{2}$$

This follows from formula (1), since

$$\begin{aligned} c_0 + P_1 c_1 + P_2 c_2 + \ldots + P_n c_n &= P_{[0]} c_0 + P_{[1]} (c_0 + c_1) \\ &\quad + P_{[2]} (c_0 + c_1 + c_2) + \ldots + P_{[n]} (c_0 + c_1 + \ldots + c_n) \\ &= P_{[0]} d_0 + P_{[1]} d_1 + \ldots + P_{[n]} d_n , \end{aligned}$$

where $d_r = c_0 + c_1 + \ldots + c_r$. Since $\Delta d_r = c_{r+1}$, application of formula (1) yields formula (2).

The corresponding formulas in terms of the quantities $\bar{B}_r$ can be obtained in a similar manner, again on the assumption of independence of the n events. For, upon replacement of

$$p_i E \quad \text{by} \quad (1 - q_i) E, \quad i = 1, 2, \ldots, n , \quad \text{in} \quad \phi(E)$$

we have

$$\phi(E) = (E - q_1\Delta)(E - q_2\Delta) \ldots (E - q_n\Delta) .$$

Expanding, we obtain

$$\begin{aligned} \phi(E) &= E^n - \Sigma q_i E^{n-1}\Delta + \Sigma q_i q_j E^{n-2}\Delta^2 + \ldots + (-1)^n q_1 q_2 \ldots q_n \Delta^n \\ &= \Sigma (1 - q_i) E^{n-1}\Delta - \Sigma (1 - q_i q_j) E^{n-2}\Delta^2 + \ldots \\ &+ (-1)^{n-1} (1 - q_1 q_2 \ldots q_n) \Delta^n + \left[E^n - \binom{n}{1} E^{n-1}\Delta + \binom{n}{2} E^{n-2}\Delta^2 + \ldots \right. \\ &\left. + (-1)^n \Delta^n \right] = \bar{B}_1 E^{n-1}\Delta - \bar{B}_2 E^{n-2}\Delta^2 + \ldots + (-1)^{n-1} \bar{B}_n \Delta^n + 1 \end{aligned}$$

since

$$\left[E^n - \binom{n}{1} E^{n-1}\Delta + \binom{n}{2} E^{n-2}\Delta^2 + \ldots + (-1)^n \Delta^n \right] = (E - \Delta)^n = 1^n = 1 .$$

Hence, applying $\phi(E)$ in this form to c_0 and comparing with the original application, we obtain

$$\begin{aligned} P_{[0]} c_0 + P_{[1]} c_1 + P_{[2]} c_2 + \ldots + P_{[n]} c_n \\ = c_0 + \bar{B}_1\Delta c_{n-1} - \bar{B}_2\Delta^2 c_{n-2} + \ldots + (-1)^{n-1} \bar{B}_n \Delta^n c_0 . \end{aligned} \tag{3}$$

We can also obtain the formula

$$c_0+P_1c_1+P_2c_2+\ldots+P_nc_n = c_0+\bar{B}_1c_n-\bar{B}_2\Delta c_{n-1} + \bar{B}_3\Delta^2 c_{n-2}-\ldots+(-1)^{n-1}\bar{B}_n\Delta^{n-1}c_1 \tag{4}$$

by the same technique as was used to obtain formula (2) from formula (1).

All four of our formulas could have application to survivorship annuities. Formulas (2) and (4) would also apply to insurances, but that does not seem to be the case for formulas (1) and (3) since quantities such as

$$A_{\overline{x_1x_2\ldots x_n}}^{[r]}$$

do not appear to have a useful interpretation.

It was most interesting to us to have this paper appear at the same time we were thinking over the foregoing relations, and to find that the generating function of the paper had a bearing on those relations. We were also much interested in the authors' application of the symmetric function identity which appears as formula (4) of the paper. Now, that they have shown the way, the authors' approach seems a most natural one and provides a thought-provoking connection between classical algebra and probability theory.

(AUTHORS' REVIEW OF DISCUSSION)

ROBERT P. WHITE AND T. N. E. GREVILLE:

We want to thank Mr. Mabon and Professor Nesbitt for their interesting and useful discussion. We have tried out in some actual problems a computation method along the general lines suggested by Mr. Mabon and have found it advantageous in many cases.