Pension Funding: A Historical Perspective[1]

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[1] This article is a revised and updated version of Shapiro (1985).

Abstract

An important prerequisite for insights into the future of pension plan funding is a sense of its history. This includes not only the history of traditional topics like actuarial cost methods and assumptions, and the history of past inquiries into the dynamic and stochastic nature of pension costs, but also the perceptions and concerns of pension actuaries of the past. The purpose of this paper is to present this historical perspective.

1. Introduction

It is difficult to ascertain precisely when pension plans first came into being. Military pensions were certainly available at the time of the Roman Empire,[2] and guilds dating back to at least the fourteenth century are recorded as having provided pensions for their members.[3] It was not until the nineteenth century, however, that employers began to establish pension plans for their employees along more or less modern lines,[4] and it was the latter part of that century before the "analysis" of pension costs gained any sophistication.[5]

One of the reasons for the lengthy gestation period for pension cost analysis was that in former times pensions were generally gratuitous in nature and employers could, at their option, choose who was to receive a pension and the scope of the pension that would be made available. Largely for this reason, the cost of pensions was considered to represent only a relatively small expenditure for the employer, and, as a consequence, employers were not particularly concerned with determining the cost associated with the pensions.

A second reason for the slow evolution of pension cost analysis was that early pensions generally were provided on a pay-as-you-go or assessmentism basis. Under this program the incidence of cost is negligible during the early years of the pension plan when there are only relatively few retired employees. It takes a number of years before the ultimate cost of the plan becomes apparent.

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[2] The legionnaires of the Roman army, for example, were provided with pensions upon completion of a specified period of service. The plan was contributory in the sense that the soldiers were required to deposit half of any monies that they accumulated through the fortunes of war in a state-held fund. These sums were returned without interest upon their retirement (Archibald 1981, p. 332).

[3] The St. Catherine's Guild, for example, which was founded sometime during the period 1327–77, provided that if a member was "sick or infirm in his old age, he is to be supported by the Guild" (Walford 1871, pp. 381–82).

[4] Civil service schemes were formalized in Britain, for example, in the Superannuation Act of 1834 (Cobb 1950, p. 10). Railway companies were among the first private companies to provide pensions for their employees. The earliest railway superannuation fund was started about 1852, also in Britain (see Maddex's discussion of Bronson 1949, p. 287). The Grand Trunk Railway of Canada organized the first formal industrial plan on the North American continent in 1874 (Latimer 1932, p. 20).

[5] G. H. Ryan, during the course of a discussion with R. P. Hardy, A. J. Finlison, G. King, and other prominent pension actuaries of the day, in 1895, remarked that "it was a very curious thing that the important question of the valuation of Widows' Funds had on no occasion been treated by contributors to the proceedings of the Institute. In Dr. Sprague's 20-volume Index there was absolutely no reference to the question of Widows' Funds. … He might add that the Index to the Journal contained no entry relating to superannuation schemes, which was again a matter of great public importance" (Hewat and Chatham 1893–95, p. 470).

Even in those instances where there was some obligation, moral or otherwise, to provide pensions, few government officials or employers were concerned with the ultimate financial consequence of a pension plan.[6] Since no one was concerned that the cost of a pension plan could impose a serious financial problem, an analytical solution for this problem was not sought.[7]

Toward the end of the nineteenth century, however, employees were beginning to look to their employers to provide a vehicle for their retirement, and, although it was generally mandatory for employees to contribute to the cost of their own pension, the cost of the pension to the employer was becoming an expense to be reckoned with. Ultimately the employer turned to the actuary for an assessment of this expense. As a consequence, actuaries of the time began to develop the mathematics necessary to cope with this new problem, and by the end of the century, articles on the analysis of pension costs had begun to appear in the actuarial journals and transactions.

This article provides an overview of some of the more notable contributions to the pension funding and analysis literature from that time to the present. However, this version should be considered a work in progress, since the focus is on relevant background topics that probably will not be covered in the other sessions. The final version of this article will include a historical perspective of the other topics as well.

2. The Early Pension Mathematics Literature

One of the first papers published on the mathematics of pensions was an article by Hardy (1892).[8] Hardy, one of the foremost pension actuaries of the day, recognized that old age pensions would ultimately become an important area of actuarial endeavor, and he urged actuaries to expand their horizons so that they would be prepared for that eventuality. The main thrust of his article was to indicate roughly how the cost of old age pensions could be determined and to emphasize the phenomenal amount of money that the old age pensions would ultimately cost. Concerned primarily with the cost of a national old age pension program, Hardy assumed an initially mature group and proceeded to show how to determine the uniform contribution that would be

[6] For an early discussion of the characteristics of a "hopelessly-mismanaged Fund" see Carment (1903, pp. 307–8).

[7] It is not universally accepted that pay-as-you-go schemes are inappropriate. Trowbridge (1977) advocates a system where the cash flow of the contributions and pension benefits are essentially in balance, which is the basis of the French repartition schemes.

[8] Literature on the mathematics of private pensions was contained in the reports and notebooks of Hardy dating as far back as 1875, but efforts made to induce him to publish his methods met without success, except for his short publication in 1892 (King 1905, p. 129 and discussion by Lidstone p. 193).

required from each employed person to fund a national pension program.[9]

A national pension program had many things in common with private pension programs. Under both systems, benefits were sometimes provided for disabled individuals, death benefits were often provided for beneficiaries, and both programs were operated on a collective basis, among other things. However, one important difference between national pension programs and private pension programs was that participants could withdraw from private pension plans. This difference gave impetus to one of the earliest publications on the mathematics of private pension plans.

In 1898 Manly published the first of his many articles that dealt with the mathematics of "staff pension plans" (Manly 1898). The main purpose of the article was to provide solutions to problems arising from the fact that

> Where large bodies of employees have been compelled to join some pension scheme, it has generally been found necessary, in order to prevent disaffection, to introduce the no-loss-under-any-circumstances element, so far as the men's contributions are concerned, by providing that the subscriber shall have his contributions returned to him or his heirs, whether he withdraws, or dies, or enters on his pension. (p. 860)

The presentation, which was somewhat elementary by today's standards but nonetheless a significant contribution at the time, entailed the determination of the premiums and reserves for deferred annuities that allowed refunds under the circumstances mentioned above.

Although the 1898 article touched on some of the actuarial aspects implicit in pension cost determination, Manly's first major contribution came in 1901. At that time he published the first of two articles entitled "On the Valuation of Staff Pension Funds," in which he laid the foundation for what was to become a standard approach to the computation of expected pension costs.[10]

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[9] Although Hardy gave a procedure for valuing national old age pensions through a tax on income, he felt that the working class would not endorse the required tax increase, and that an estate tax would probably provide a more susceptible vehicle for funding the old age pension. He "did not think anybody should object to it: it did not touch the living earner at all" (Hardy 1892, p. 96).

[10] Other actuaries of the day were endeavoring to formulate an appropriate approach for determining the cost of pensions. Prior to the publication of Manly's article, for example, McGowan (1902) discussed the subject. That article was considerably less detailed than Manly's and contained many flaws from a practical point of view, not the least of which was an unsatisfactory approach to salary scales.

Two useful purposes are served by recounting the types of problems for which Manly offered solutions. First, these problems have historical significance, since they indicate the types of pensions that were in vogue at the beginning of the 1900s. Second, and perhaps of greater importance, they form a standard against which current contributions to the mathematics of pensions can be gauged.

The problems were divided into two classes: those that did not include salary scales and those that did. The former set of problems included the determination of the present value of an annuity that was deferred to age 65, was payable on retirement at that age, and was combined with various combinations of a return of premiums with and without interest in the event of death, withdrawal, or early retirement. In the event of early retirement, a linear reduction in the pension benefit was assumed.

The latter set of problems was of both the defined benefit type and the defined contribution type. In the defined benefit type, Manly determined the present value, at the age of entry, of a pension benefit based on both a career average and final average salary and a number of years of service. In the defined contribution type he gave a method of determining the size of the normal retirement benefit that could be obtained with the accumulation of predetermined contributions.[11]

One of the areas that Manly did not address explicitly was the general mechanics for obtaining and coordinating the statistics of the pension plan. This gap in the pension literature was filled by George King (1905). His explanation for transforming raw decrement data into decrement tables was so fundamental that its essential features are still taught to current actuarial students (who study such things) under the appropriate title of “King’s Method.”

In that same paper, King extended the pension literature in four other respects. He differentiated between past and future contributions, he investigated the implications of a provision for the return of contributions at other than the valuation rate of interest, he introduced the assumption that “pensions, taken one with another, are payable continuously throughout the year,”[12] and he devised an

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[11] It should not necessarily be assumed that the data used by Manly (1901) were representative of those used in practice at the time, but it is interesting to note that for the purpose of his paper the accumulation interest rate was assumed to be 4 percent per annum, the average age at entry was 20, the normal retirement age was 65, the defined benefit was 2 percent of total salary, and the defined contribution was 5 percent of salary.

[12] This assumption greatly facilitates the mathematical analysis of pensions since it allows the use of continuous functions.

approximation for the final average salary.

In the years following the articles of Manly and King, most of the advances in the computation of pension cost dealt with the derivation of relevant data and the implementation of those data. Hence, Bacon (1908) developed the concept of salary scales based on time series data, rather than cross-sectional data; Grant (1909) investigated the derivation of mortality data for pensioners; Richmond (1911) commented on the actuarial basis of the state pensions of Austria; and Thomas (1914) introduced the notion of segregating the various classes of participants by category of employment.

These early studies did much to promote an understanding of the mathematics involved in the development of an expected pension cost model. They did, however, have an inherent deficiency. Since the scope of the benefits that were then available was somewhat limited, the models that were developed did not include provisions for some of the subsequent benefits, such as those offered by a vesting provision. Furthermore, consideration was generally limited to single-entry-age models, which further limited the generality of the models. Later contributions often dealt with models that were significantly more general.

3. Actuarial Cost Methods

> "The plan which provides for regular payments to a trust, during the active service of employees, sufficient to accumulate reserves to provide the benefits as they mature ... is the best type of financing with which I am familiar." —George B. Buck (1953 p. 55)

The contributions to pension mathematics prior to 1915 were due almost entirely to the efforts of European actuaries. In fact, only three articles on pension plans appeared in the *Transactions* of the Actuarial Society of America prior to that time, and these were of a qualitative nature (Flynn 1907–8, Grant 1909, and Ferguson 1911).

Walker (1915) set the stage for what was to become a major area of contribution of the American actuary—the development of actuarial cost methods—when he investigated the "initial liability" for benefits based on past service at the commencement of a fund. Walker (1915, p. 141) was concerned with the problem of estimating this liability, and, hence, after showing how this could be done, he simply observed that "the extra contribution [needed to fund this liability] jumps up very rapidly as the age at entry become high . . . [so that if] the company itself did not care to assume any of this charge, it would be practically necessary to reduce the percentage benefit at the older ages and durations so that the employee could himself pay the contribution." It is interesting to note that while Walker, following Manly and Ackland (1912), used the term "initial liability," Hutcheson (1920), in discussing Walker's paper (1915, p. 389), used the term "[initial] accrued liability," a current nomenclature.

Although the need for a detailed analysis of the implications of the accrued liability was firmly established by 1915, a considerable period of time elapsed before actuaries began to quantify systematically the approaches for dealing with this problem. A primary reason, evidently, was the limited practical application for such investigations, since it was only rarely that employers were concerned with this liability. Manly (1901, p. 258) remarked, for example, that although he had performed valuations on many plans, he himself had never started a plan; similarly, Woodward (1925, p. 447) observed that "Systematic provision for meeting the pension liabilities of the future is almost never made at the inception of a new enterprise."

It is interesting to note the early methods that were proposed for dealing with the initial accrued liability. It appears that the first method used was based upon the assumption that the plan will exist indefinitely into the future; hence, only the interest on the accrued liability was paid.[13] Later actuaries who investigated the mathematics for dealing with the accrued liability took various other approaches: Woodward (1925) suggested that the initial accrued liability be funded by the time the original employees retire; Robbins (1929) suggested that it be funded during the life of the original employees; and Corbett (1936) suggested limiting the initial accrued liability to include only those persons aged 40 or over and amortizing it over a 25-year period, under the assumption that the error introduced by such a procedure would be negligible. Apparently, it was Woodward (1925, p. 447) who coined the term "normal cost," defining it to be the amount that if set aside annually would be adequate to fund the pension benefits for new employees entering the plan.

It is obvious that the accrued liability may be funded by any number of methods, the method chosen in any particular case being dependent upon such things as the capital requirements of a firm and income tax considerations.[14] However,

[13] It was mentioned, for example, in connection with the funding of the past service liability of a pension plan governed by the 1866 Metropolitan Superannuation Act of London (Manly and Ackland 1912, and discussion of Tinner, p. 378).

[14] Capital requirements of a firm and income tax considerations impose conflicting constraints on plan

a hallmark in the trend of funding methods in the United States was the publication, in 1945, of the Bulletin on Section 23(p) (1) (A) and (B) of the Internal Revenue Code, which described a number of methods of calculating and amortizing the liabilities of a pension fund. These funding methods have come to be known as actuarial cost methods.[15] Most of the contributions to pension mathematics following that publication dealt with quantifying the opinions expressed in that Bulletin.

Two of the more important articles of this period were the articles of Seal (1952) and Trowbridge (1952). The main purpose of Seal's article was to discuss and quantify the various funding methods described in the Bulletin as "acceptable" for tax-exempt self-insured plans. Trowbridge, using a somewhat more general approach, first discussed the broad characteristics of a funding method and then proceeded to show the quantitative nature of a variety of these methods. In addition, he discussed and exemplified the accrual of funds under each of the methods that he described.

Most of the methods in common use at the time in the United States also were used in the United Kingdom, although, apparently, the aggregate cost method was the most common. A comparison was given by Colbran (1982). Variances existed, however. The qualitative nature of some of these was described by Lyon (1960).[16]

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funding. Ideally, a firm would fund its plan in such a way as to coordinate alternate demands on assets. Thus, for example, a firm may wish to contribute most of the cost of its plan during its most profitable years, while contributing little or nothing to the plan in years when profits are low or other corporate needs seem more compelling.

The Internal Revenue Service severely restricts the maximum contribution a firm can make to a defined benefit pension plan, however, by limiting the portion of a contribution which is deductible in a given year. Similarly, except for firms experiencing substantial business hardships, there are restrictions on minimum contributions to a plan. When one superimposes other considerations such as corporate liability and potential limits on reversion of assets in the event of plan termination, the complexity of the problem grows considerably.

Early researchers who explored pension funding strategies and related issues include Arnott and Gersovitz (1980), Black (1980), Bulow (1979), Friedman (1982), Harrison and Sharpe (1982), Sharp (1976), Tepper (1981), and Treynor (1977).

[15] For a description of actuarial cost methods, see Anderson (1990, Chapters 2 and 7) and McGill et al. (2004, Chapters 22 and 23).

[16] An interesting comparison is afforded by Collinson (2001), which documents actuarial methods and assumptions used in the valuation of retirement benefits in the EU and other European countries.

The foregoing articles tended to stress the uniqueness of the various methods of pension funding. Other articles stressed the formulation of general funding models, of which the previous models were special cases. At least four significant articles have been written on general funding methods.

Trowbridge (1963) wrote an article in which he developed a family of funding methods based on the rate at which the unfunded present value of benefits was being funded. In that article he emphasized the importance of the ratio of the present value of the benefits to the size of the accumulated fund, the "fund ratio," and its progression from year to year, and advocated deemphasizing "confusing concepts such as normal cost, accrued liability, actuarial gains or losses, etc." The funding family that he proposed was characterized by an annual contribution, payable in advance, which was composed of (1) the discounted value of the annual interest on the unfunded liability and (2) a payment to reduce the unfunded liability. By varying the size of the contribution to reduce the unfunded liability, the ultimate contribution and fund of any actuarial cost method could be reproduced.[17] If the population was initially immature, the family was capable of only ultimate values of the fund and contributions for any of the actuarial cost methods.

During the discussion of Trowbridge's paper (1963, p. 171), Nesbitt suggested that the unfunded present value family could be further generalized by distinguishing between the initial accrued liability, if any, and the liability that accrues after that time. Taylor (1967), acting on Nesbitt's suggestion, designed a funding family characterized by contributions, payable in advance, which were composed of (1) a duration-dependent contribution to reduce the unfunded liability and (2) a duration-dependent contribution to reduce the initial accrued liability. By the appropriate choice of both portions of the contribution, the ultimate contribution and fund of any of the actuarial cost methods could be reproduced, including Trowbridge's unfunded present value family.

The third study to interrelate the various actuarial cost methods was that of Cooper and Hickman (1967), in which they developed a general family of actuarial cost methods based on the rate at which retirement income is being purchased, for a member of the pension group, at a given attained age. A major contribution of their study was that the family of actuarial cost methods that they

[17] Pension plan contributions and funds are in their ultimate state when the pension plan population has stabilized and sufficient time has elapsed so that the aggregate contribution and fund are the same each year. The fund is said to be in balance, in the sense that the total contributions to the fund plus the income earned on the fund are equal to the total benefits plus expenses paid out of the fund.

developed permitted the purchase of retirement income at either an accelerating or a decelerating rate.

The Cooper and Hickman article formed the basis of an exposition on contribution theory by Bowers, Hickman, and Nesbitt (1979). Here a general model for a pension plan involving growth with respect to the population, salaries, and retirement benefits was used to study contribution patterns that may arise under different actuarial cost methods.

A by-product of their analysis was the development of a function to denote the average age when normal cost payment can be viewed as centered. This function is closely related to the cumulant generating function of probability and statistics and provides a bridge that permits pension funding ideas to be stated in terms of compound interest functions.

By way of contrast, the 1985 study of the American Academy of Actuaries, "Pension Cost Method Analysis," provides documentation and analysis of the primary actuarial cost methods that were in vogue at the time. They concluded that five actuarial cost methods represented the basis of pension expense for substantially all sponsors of large defined benefit pension plans. These methods were the entry age normal, frozen initial liability, aggregate cost, unit credit, and prorated unit credit/service prorate.

Insofar as small plans were concerned, Shapiro (1983) wrote that two common actuarial cost methods in use at the time were the modified aggregate cost method (MACM) and the individual aggregate cost method (IACM). The distinguishing feature of the MACM was that, instead of weighting by salaries, it was weighted by the present value of future benefits divided by the present value of an active-life annuity, weighted by a salary-scale function, as of some pivotal age. The distinguishing feature of the IACM was that the contribution on behalf of each participant was computed as the present value of future benefits, net of a hypothetical asset allocation divided by an active-life annuity that was weighted by a salary-scale function.

One of the major problems with actuarial cost methods is that they are based on expected values.[18] This deficiency lead Ramsay (1993) to propose a new family of

[18] This has been a perennial concern. In the 1970s, it was pointed out to actuarial students in the Society of Actuaries' Pension Topics Study Note (71-22-76, p. 10), which reads in part, "Ideally, the individual assumptions and the composite results of the valuation should be viewed as the mean of the universe from which the experience of that pension plan will be drawn. The measurement of liabilities would be accompanied by a set of confidence limits based on the combined effect of the entire set of assumptions."

pension cost methods, which he called ◎-percentile cost methods, based on a modification of the traditional cost methods to explicitly include a degree of confidence (assurance) in the payment of future benefits. It is interesting to note that this is the same deficiency this conference is focusing on.

4. The Cost of Vesting

Until the mid-1960s the articles on the mathematics of pensions generally were based on the assumption that employees who voluntarily withdrew from employment are entitled to at most a return of their contributions, if any, with interest. Exceptions, of course, existed. Noback (1950) set down the mathematics for what he termed “deferred annuity vesting,” which provided a deferred annuity, based on a percentage of salary for each year of service, for employees who had completed a given number of years of service. Niessen (1950) also considered the derivation of “deferred annuities coming from withdrawals” when he discussed the projection of pension costs and funds. These exceptions, however, invariably dealt with special cases, rather than a general model.

In the United States a major thrust in this area was the Report of the President’s Committee on Corporate Pension Funds (1965), which stressed the need for vesting in pension plans. Soon thereafter the general mathematics of the cost of vesting began to appear in the pension literature of that country, the most often quoted of these articles being the studies of Marples (1966) and McGinn (1966). Both studies developed vesting cost ratios or indices based on the proportion of vested benefits to nonvested benefits. Marples developed a vesting cost ratio without regard to a particular actuarial cost method, while McGinn developed attained age vesting cost ratios for the normal cost and accrued liability under the individual projected cost method with supplemental cost, the funding method suggested by Woodward. Both studies, including the discussions thereof, provided the basis for a general model of vesting cost ratios, which was later developed independently by Trowbridge (1973) and Winklevoss and Shapiro (1973). The primary thrust of the latter articles was to develop mathematically some of the major results of the former articles.

The foregoing articles were concerned primarily with the cost of vesting to the employer; the impact of vesting on the employee received little attention. An important thrust toward filling this gap in the pension literature was provided by Balcer and Sahin (1979). Their contribution was the development of a probabilistic model for tracking the accumulated benefits that were vested during the working lifetime of an individual. An important characteristic of their model was that it took into account the fraction of jobs offering pension plans and the probability of portability.

The previous studies generally assumed that the vested accrued benefit was the benefit to be valued for vesting purposes. A recent article by Gold (2005) suggests that the term “accrued benefit” is vague and that it would be more appropriate to focus on the “exit benefit,” which he defines as the benefit the employee would be entitled to if he or she left service at year-end. It reflects years of service and pay to date, and excludes nonvested benefits, benefits for which eligibility has not yet been satisfied, and salary scale effects. Gold asserts that exit-cost methods are more accurate, more transparent, and less fraught with opportunities for manipulation.

5. Dynamic Simulation of Pension Costs

> I do not feel that a ‘best’ contribution figure exists. I cannot seriously argue that a change of 10% or even 20% in a best estimate makes it wrong or even less good.
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> — Guy Shannon (1976 p. 103)

Pension actuaries long have realized that pension cost projections provide valuable insight into the cash-flow characteristics of pension plans. Because of this, a pension cost projection typically is appended to the more elaborate pension plan proposals and valuations. Depending on the size of the plan, these range from simple projections that assume a closed group with no terminations other than for retirement to more sophisticated models that introduce the full spectrum of pension plan parameters. This section traces the history of this dynamic simulation of pension costs.

One of the first published accounts of the growth of a pension fund was an article by M’Lauchlan (1908). That article illustrated the necessity of accumulating large invested funds during the early years of a fund’s existence so as to provide for the heavy liability that ultimately will be maturing for payment.

The format advocated by M’Lauchlan for simulating pension fund growth is still current, and it is appropriate, therefore, to mention the procedure that he used. He started with tables of rates of mortality, withdrawals, retirement, and salary increases, an assumed guaranteed annual interest rate, and one thousand entrants at the single entry age 20. Then, under the assumption that the benefits based upon 5 percent of salary had been accurately determined, he tabulated (1) the fund that accumulated over decennial periods, from contributions and interest and the payments incurred by virtue of withdrawals, death in service, pensions, and expenses, and (2) the value of the accumulated fund at the end of each period.

M'Lauchlan was primarily concerned with drawing attention to the natural growth of a normal pension fund and made no attempt to ascertain the sensitivity of the growth of liabilities to changes in assumptions. This inquiry was undertaken by Manly (1911) in a study in which he extended M'Lauchlan's investigation by showing the effect on pension liability growth of variation in the rates of mortality, withdrawal, retirement, and salaries. Manly's primary concern was to show that the ultimate trend of the simulated pension fund depends upon the nature of the assumptions made, and that "what appears to be a simple variation in the incidence of service ... may make vast difference in the contributions required or in the benefits that can be granted" (p. 151). His primary observations, and ones that have been echoed many times since (pp. 169, 182), were that "the experience and conditions of service should be investigated at frequent intervals, in order that the necessary changes, in contributions or pensions, should be made before it becomes practically too late, [and that no] two Funds are exactly alike, and the conditions will vary from time to time in the same Fund."

The use of fund projections has continued to be a major tool in the investigation of pension plan costs. Llewellyn (1938) projected pension payments for a plan through 1979, and compared the present value of that projection with a then-current valuation; Niessen (1950) authored an article titled "Projections—How to Make Them and How to Use Them;" Trowbridge (1952) traced the fund and contributions, under various actuarial cost methods, to their "ultimate" state; Griffin (1966) investigated the impact of both decreasing and increasing populations on the adequacy of various actuarial cost methods; and Winklevoss (1974) investigated this impact on the cost of vesting in pension plans.

Simulation invariably involves sensitivity analysis. However, sensitivity studies have been used in many situations in which simulation has not been involved. The implication of using various interest rates, for example, apparently first investigated by McGowan (1902), when he compared the impact of using 3, 3.5, and 4 percent interest rates on estimated pension costs, was later studied by Carter (1934), Bizley (1951), Adams (1967), and Allison and Winklevoss (1975). The last of these studies also examined the impact of salary scales on expected pension costs. This inquiry also has had a long history and includes the works of Bacon (1908), Bizley (1949), and Marples (1962). The latter two studies were based on mathematical models: Bizley used mathematical analysis, while Marples extended the notion of a family of salary scales that he and Haywood (1949) had developed.

The factors that have most often been used to study the sensitivity of pension costs have been the rates of decrement. Among the researchers in this area were Carment (1903), who was one of the first to suggest using select mortality rates for pensioners; Buck (1916), who tested the impact of select withdrawal rates; and Huffman (1967–68), who studied the impact of early retirements.

The preceding studies, for the most part, concentrated on particular decrement factors and particular situations. Some of the more comprehensive studies were those of Marples (1945), who investigated a family of decrement rates; McGinn (1966), who investigated the impact of a whole array of withdrawal rates and various vesting schedules on the cost of vesting in pension plans; and Grubbs (1973) and Winklevoss (1974), both of whom expanded on the study done by McGinn.

Bowers, Hickman, and Nesbitt, singularly and jointly, produced a number of important papers dealing with dynamic pension cost models. They (Bowers et al. 1976) authored an introductory paper that expounded on the mathematical principles applicable to pension funding under dynamic conditions of population growth, inflation, and automatic adjustment of benefits. The model was deterministic in nature, in the sense that it did not deal with contingency reserves and, for simplicity, was continuous and based only on retirement benefits.

One outgrowth of this study was a paper by Nesbitt (1982), which dealt with vesting. This paper analyzed the situation in which, in addition to the retirement benefit available at the normal retirement age, a termination benefit existed equivalent to the reserve in regard to the terminating participant. A plan with this feature was said to be exactly vested.

A second outgrowth was the paper mentioned previously that dealt with contribution theory.

The authors extended this discussion in yet another paper (Bowers et al. 1982) wherein they discussed a generalized amortization method for the unfunded accrued liability, the impact of gains and losses, and an application of their model to a variable annuity system.

Bowers et al. often illustrated ideas or developed theory that depended on the exponential growth case, since it lends itself to mathematical exploration.

Although they considered that case to be rather special and unrealistic, practical models have been developed using similar techniques. Treuil (1981), for example, used an exponential growth model to study alternatives related to the tail end of long-term projections for social insurance and pension systems, under the assumption that a time is being approached when an assumption of stabilized conditions seems as appropriate as any other.

Another type of analysis that began to appear in the 1970s was the forecast valuation methods. Like the original work of M'Lauchlan, the methods involve the development of expected future population and payroll figures, development of expected future benefit disbursements, and development of future cost figures. The distinguishing feature of these methods, however, was the use of open group techniques, in the sense that future new entrants were anticipated. Early contributions to this area included Howe and Smith (1974), who simply replaced "the deaths and terminations by employees of the same age while replacing retirements and allowing for expansion by introducing employees at selected younger ages," Fleischer (1975), who discussed the use of the forecast valuation method for the first year actuarial valuation of pension plans, and Schnitzer (1977), who discussed its use in the valuation of an ongoing plan. Subsequent forecast valuation studies included Toyoda et al. (1995), who used an open aggregate cost method with past service liability, Clark (2001), who used an analysis of a dynamic pension plan valuation to highlight the value of using a forecast-inclusive valuation method and to discuss the uses for which such a method is best suited, and McCrory (2003), who incorporated the stochastic variability of asset returns and inflation.

Although not exactly on topic, it is worth mentioning that the SOA sponsored research into the most important models used by public policy analysts looking at retirement benefit issues. The results of the study, which were reported in Anderson (2000), resulted in a fairly comprehensive review of these models and their methods, including a review of their capabilities and limitations.[19]

[19] The models reviewed included Dynamic Simulation of Income (DYNASIM), Pension and Retirement Income Simulation Model (PRISM), Cornell Microsimulation Model (CORSIM), a Canadian Microsimulation Model for Public Pension Analysis (DYNACAN), the Social Security Administration Near Term Retirement Income Model (MINT), Social Security Policy Simulation Model (SSASIM), a Macroeconomic-Demographic Model of the U.S. Retirement (MDM), and Pension Insurance Modeling System (PIMS). The last of these was used to forecast and analyze the financial position of the PBGC and its insured plan sponsors.

6. Stochastic Models

> What is needed is an estimate of the contingency reserve necessary to meet adverse fluctuations. —Kenneth B. Piper (1933 p. 240)

Stochastic models are any models having a probabilistic, or random, component. From a pension funding perspective, the endogenous stochastic components can involve the plan demographics, the plan investments, or both. In this section I trace the history of the actuarial analysis of these stochastic components.

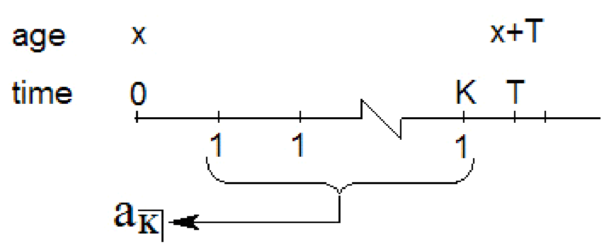
6.1 Stochastic Demographic Models

Life actuaries have been exposed to stochastic models for some time now. Although it has been only during the last 20 years that the Society's actuarial students have formally studied such models in Bowers et al. (1986) and Klugman et al. (1998), and their later editions, a number of earlier articles and texts incorporated these models. Piper (1933), for example, developed contingency reserves for life annuities based on the mean and variance associated with those annuities, using an annuity model based on the equation

$$a_x = \sum_{t=1}^{\infty} {}_{t|}q_x \, a_{\overline{t|}} .$$

This equation was later mentioned in Jordan (1967), although only in passing.[20] The interpretation of this equation, which is now well known, is shown in Figure 1.

Figure 1
Immediate Life Annuity

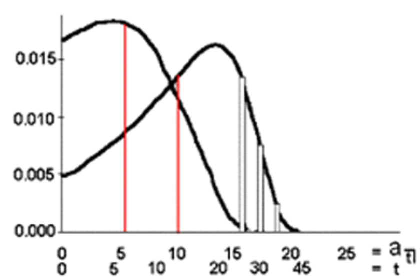


[20] In Jordan (1967), Chapter 2, question 7, the reader was asked to prove this equation algebraically and to explain it by general reasoning. Jordan, however, did not pursue the stochastic model.

As indicated, the life annuity-immediate issued to a life aged x (at time 0) can be represented as the present value of a K-year annuity-certain, where the random variable K is the number of complete years lived by the life aged x (x dies at time T, where K = [T]). Thus, $a_x = E(a_{\overline{K}|})$.

Piper used this model to estimate the mean and variance associated with the contingency reserves for life annuities. The insight he had into this model is summarized in Figure 2, which was taken from his paper.

Figure 2
Immediate Life Annuity Distribution



The curve on the left and the first vertical line represent the distribution for a life annuity for a life aged 80 and its expected value, respectively. The area to the right of the vertical line represents the probability of the annuity exceeding the mean. Similarly, the second curve and the second vertical line represent the distribution for a life annuity for a life aged 65 and its expected value, respectively. The rectangles on the right of the figure represent probabilities of dying during a given year of age ($_{t|}q_x$).

Menge (1937), Hickman (1964), Pollard and Pollard (1969), and Boyle (1974) elaborated on the Piper concept: Menge, Pollard and Pollard, and Boyle using discrete functions and Hickman using continuous functions. Hickman extended the development to include loss functions and a probabilistic consideration of multiple decrement theory, Pollard and Pollard focused on the second and higher-order moments of some common actuarial random variables, and Boyle investigated reversionary annuities.

Early on, although the number of lives that persist to a given age from an initial group of lives is generated by a Bernoulli process, the considerable labor required to generate appropriate distributions under this process resulted in the use of various approximation methods. Hence, Taylor (1967) suggested fitting a

Pearson Type III distribution to the total present value of life annuity costs; Boermeester (1956) applied a Monte Carlo approach to the problem; Fretwell and Hickman (1964) investigated upper bounds for the cost using the inequalities of Tchebychef and Uspensky; and Bowers (1967) investigated the use of the Cornish-Fisher expansion to develop probabilities of sufficient reserves, based on correction factors applied to a standard normal table.

Now, of course, given the proliferation of fast computers, it has become relatively easy to develop simple stochastic models from first principals. One recent example is Czernicki, Harewood, and Taht (2003), which provides a simple overview of stochastic modeling as it relates to mortality.

The foregoing articles focused on annuities. One of the earliest articles dealing with stochastic models of pension costs was by Stone (1948), who investigated the impact of mortality fluctuations on pensions paid to pensioners. The main thrust of that study was the use of probability-generating functions to develop probabilities, at various durations after employees had begun to retire, that the total actual pension payments would differ from the expected total payments. Taylor (1952) investigated the size of the contingency reserve needed to ensure, with a given probability, that the funds on hand would be sufficient to pay all promised pensions. Both these studies dealt exclusively with the retired population, under the assumption that the number of retirees was known.

Papers that considered variability in pension cost estimates for active plan participants include the studies of Seal (1953), Knopf (1957), and Shapiro (1977, 1979). Seal investigated the impact of death benefits in a trusteed plan, using a normal approximation to the binomial distribution to introduce variance minimization into the design of pension plans. Knopf investigated the feasibility of a fully trusteed pension plan using a simplified Monte Carlo approach. Shapiro considered the credibility of projected pension costs using a model based on the direct application of a conditional Bernoulli process and later extended the study to include an unconditional distribution.

6.2 Stochastic Investment Models

The studies in the previous subsection assumed a stochastic pension population coupled with deterministic investment returns, salary increases, and inflation rates. The following studies assumed the accumulation of assets was a random process.

One of the earliest studies, in this regard, was by Pollard (1971), who assumed that the force of interest can be represented by an autoregressive process of order two, AR(2). A number of analysts followed Pollard's lead and incorporated an autocorrelated environment into their models.

Initially the studies focused on annuities. Panjer and Bellhouse (1980) used an autoregressive model to compute moments of annuity functions; Bellhouse and Panjer (1981) published a follow-up paper that focused on conditional autoregressive interest rate models; Giaccotto (1986) developed an algorithm for evaluating present value functions when interest rates are assumed to follow an autoregressive integrated moving average (ARIMA)[21] (*p*, 0, *q*) process or an ARIMA (*p*, 1, *q*) process, as did Dhaene (1989), whose model required less computing time than Giaccotto's method; and Frees (1990) assumed that one-period spot rates are independent, or follow a simple moving average, to introduce volatility and autocorrelation effects into a stochastic life contingencies model.

By the 1990s, studies involving an auto-correlated environment were being directed at the pension area. Haberman (1990) investigated the variability of pension contributions and fund levels using an AR(1) process; Haberman (1993) applied the same process to pension funding with time delays; Haberman (1994) extended the study to different pension funding methods and optimal (variability reducing) amortization periods; Harris (1995) applied an exponential regressive conditional heteroscedasticity (ERCH) model to interest rate series at monthly, quarterly, and annual frequencies; Mandl and Mazurová (1996) used spectral analysis to compute the variances of fund levels and contribution rates, given randomly fluctuating rates of return and numbers of entrants; and Cairns and Parker (1997) introduced the concept of the efficient frontier as a means of choosing an optimal funding strategy and use it and an AR(1) process to plot the unconditional variance of the contribution against its mean.

An alternate approach was to develop stochastic models to simulate the random rates of return. Using this approach, McKenna (1982) simulated probability distributions of pension cost using a model based on the work of Ibbotson and Sinquefield;[22] Dufresne (1988, 1989) investigated the moments of contribution and fund levels for spread gain actuarial cost methods, under the assumption that rates of return are independent and identically distributed (iid) random variables; Beekman and Fuelling (1990, 1991) modeled interest rate randomness as an Ornstein-

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[21] An ARIMA (*p*,*d*,*q*) process is a ARMA (*p*,*q*) process where the series has been differenced *d* times.

[22] Examples of this approach also are provided by Kingsland (1982) and Winklevoss (1982), although their articles provide little insight into the technical aspects or validity of their models.

Uhlenbeck[23] process and a Wiener stochastic process, respectively; Haberman (1990), in addition to the work mentioned previously, investigated the variability of pension contributions and fund levels under the assumption that earned rates of return were iid; Cairns (1995) used a simple stochastic interest model (iid) to provide some intuitively appealing, analytical results related to the variances of the contributions and fund level, and then went on to investigate some Wilkie model–based simulations; Haberman (1997) used an iid model of investment returns to investigate the optimal period for spreading valuation surpluses and deficiencies to minimize contribution rate risk; and Owadally and Haberman (1999) used a stochastic model similar to that described by Dufresne (1988, 1989) to investigate gains or losses and the evolution of the first and second moments of the pension fund and contribution levels.

7. Pension Funding as a Control Problem

Another innovation was the conceptualization of the funding of a pension plan as a control problem. Benjamin (1984) set the stage for this type of inquiry when he applied control theory to aggregate funding, from the perspective of an actuarial laymen.

A common view of the control problem is based on the competing factors of stability and security. In this context, stability focuses on a stable contribution, which generally is a function of the contribution at time t (the beginning of the year), C_t , and the target contribution at time t, CT_t, while security focuses on adequate funds, which is a function of the fund at time t+1 (the end of the year), F_{t+1}, and the target fund at time t+1, FT_{t+1}. This view, which was discussed in some detail in Dufresne (1992), has attracted a number of researchers. The basic proposition of this approach was characterized by Haberman and Sung (1994) as the problem of minimizing

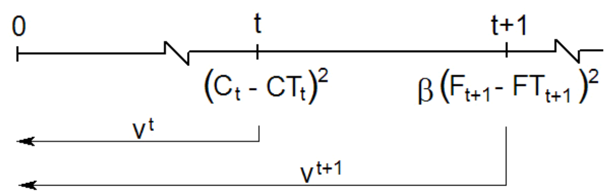
$$\mathrm{E}\left\{\sum_{t=0}^{T-1} v^{s}\left[(C_t - CT_t)^2 + v\beta\,(F_{t+1} - FT_{t+1})^2\right]\right\},$$

where ⊚ is the relative weight associated with the deviation in security. The competing factors during the (t+1)-st year can be represented as shown in

Figure 3.

[23] It is interesting to note that Beekman was an advocate for the Ornstein-Uhlenbeck process before Vasicek used it to describe the stochastic process followed by the interest rate.

Figure 3
Pension Funding Controller



A number of configurations have been investigated using this control model. Representative examples include Benjamin (1989), who investigated minimizing the change in the contribution, $CT_t = C_{t+1}$, with the aim of driving an opening fund and contribution rate to a desired fund and contribution rate in a period of years, using the smoothest path of contribution rates; Haberman (1993), who analyzed the case where the target cost was the expected cost, E[C_t], that is, the first term in

Figure 3 was the variance of the contribution; and Haberman and Sung (1994) , who explored the implications of setting the target contribution and fund to the normal cost, NC_t, and accrued liability, AL_{t+1}, respectively.

The foregoing papers dealt primarily with hypothetical situations. One of the first reported applied applications of control theory to an existing pension plan was documented by Chang (1999), who discussed control theory as it pertains to the funding of the Taiwan public employees retirement system. Using Haberman and Sung (1994) as a guide, Chang used a Bernoulli process to project the population, projected the cash flows, and sought the contribution that minimized the function

$$E\left\{\sum_{s=0}^{T-1} v^s \left[\left(1-\frac{C_s}{NC_s}\right)^2 + v\beta_{s+1}\left(1-\frac{F_{s+1}}{\eta AL_{s+1}}\right)^2\right]\right\}$$

In a follow-up study, Chang, Tzeng, and Miaoc (2003) added two additional components of risks, underfunding risk and overcontribution risk, to accommodate fund managers who had a preference of one over the other.

The previous studies were based on a discrete-time approach. An alternative approach is to use a continuous-time stochastic dynamic pension fund model, where the system is described by a stochastic differential equation, while still incorporating the notion of a target value. Studies that took this approach include O'Brien (1986, 1987), which, following Bowers et al. (1982), focused on a target fund equal to a portion of the present value of future benefits, $FT_t = k * PVFB_t$; Vanderbroek (1990), who extended O'Brien (1987) by relating the contribution target to the total salary, $CT_t = m\ S_t$ and $FT_{t+1} = k\ PVFB_{t+1}$; Taylor (1994), who laid out a framework for stochastic control using a loss function expressed explicitly in terms of fund contribution rate and fund solvency; and Boulier, Trussant, and Florens (1995), Boulier, Michel, and Wisnia (1996), and Cairns (2000), who used similar approaches.

Readers interested in a detailed exposition of the stochastic control of funding systems might read Taylor (2002). That paper presents a model where the employer seeks to minimize contributions, subject to solvency constraints, and includes the investment policy as a control variable.

8. Commentary

What do we glean from this review of the literature? Three things are clear:

First, actuarial science is an evolving field. This comes as no surprise; every actuary knows that.

Second, the actuary's world is stochastic. Piper demonstrated that in 1933.

Third, investment income is every bit as important to actuarial models as life contingencies. Hickman (1985) alluded to this when he posed the question "Why doesn't Actuarial Mathematics start with the premise that the rate of investment income is a random process?"

Of course, these observations need to be qualified.

Although the actuarial societies have implemented continuing education programs to help keep us abreast of our evolving field, much of the energy is focused on "hot topics" and subjects that are ancillary to actuarial science. There are relatively few conferences, like this one, whose expressed purpose is to envision a better system.

Another concern is that our actuarial methodology does not always reflect our stochastic world. Ramsay (1993) put this notion in context when he observed that

> Traditional pension cost methods are based on the actuarial present value of future benefits (which is a mean value). As a consequence, these cost methods are deficient because they cannot ... be used to determine the probability that the accumulation of a particular sequence of contributions will ultimately provide sufficient funds to pay benefits.

You would have thought that after 20 years of Bowers et al., stochastic models (at least to the extent of risk-adjusted variables) would have made more inroads into our current pension valuations. Perhaps it is telling that so many current pension practitioners prefer the deterministic models of Jordan to the stochastic models of Bowers et al., that in 2003 the Society responded by reprinting Jordan.

Finally, the profession generally has been informed of all the technical advances related to finance. However, although the profession has had the advantage of the writings of what Bühlmann (1989) called "actuaries of the third kind," people like David Wilkie, Phelim Boyle, Elias Shiu, and James Tilley, to name a few, pension practitioners, as a group, seem not to be aware of how to implement the ideas of many of these articles.

One can speculate that the reason for these disconnects is a lack of communication. Most of the recent articles in this review were written by academics, and it is not unusual to hear practitioners complain that they cannot relate to many of the academic articles. By the same token, academicians often lack the insights of the practitioner, which, again, would interfere with communication. To the extent that this is a major issue, one way to promote the development of a better system would be to bridge the communications gap between our academic actuaries and our practicing pension actuaries.

9. Acknowledgments

This work was supported in part by the Robert G. Schwartz Faculty Fellowship at the Penn State University. The assistance of Michelle L. Fultz, Joseph D. Heintzelman, Scott W. Nowicki, and Mariasha A. Shapiro is gratefully acknowledged.

References

Adams, W. R. 1967. The Effect of Interest on Pension Contributions. *Transactions of the Society of Actuaries* 19: 170–83.

Allen, E. T., Jr., J. J. Melone, and J. S. Rosenbloom. 1981. *Pension Planning.* 4th ed. Homewood, IL: Richard D. Irwin.

Allison, G. D., and H. E. Winklevoss. 1975. The Interrelationship among Inflation Rates, Salary Rates, and Pension Costs. *Transactions of the Society of Actuaries* 27: 197–210.

American Academy of Actuaries. 1985. Pension Cost Method Analysis.Washington, D.C.: American Academy of Actuaries.

Anderson, A. W. 1990. *Pension Mathematics for Actuaries.* 2nd ed. Providence, RI: ACTEX Publications.

Anderson, J. M. 2000. Computer Models for Retirement Policy. Record 26(1): 1–18.

Archibald, G. C. 1981. Retirement Benefits in Continental Europe. *Transactions of the Faculty of Actuaries* 37: 339–98.

Arnott, R. J., and M. Gersovitz. 1980. Corporate Financial Structure and the Funding of Private Pension Plans. *Journal of Public Economics* 13 (April): 231–47.

Bacon, J. 1908. A Pension Fund Problem: With Some Remarks on the Deduction of Salary-Scales. *Journal of the Institute of Actuaries* 42: 36–60.

Balcer, Y., and I. Sahin. 1979. Probabilistic Models for Pension Benefits. *Journal of Risk and Insurance* 46 (March): 99–123.

Beekman, J. A., and C. P. Fuelling. 1990. Interest and Mortality Randomness in Some Annuities. *Insurance: Mathematics and Economics* 9(2–3): 185–96.

———. 1991. Extra Randomness in Certain Annuity Models. *Insurance: Mathematics and Economics* 10(4): 275–87.

Bellhouse, D. R., and H. H. Panjer. 1981. Stochastic Modelling of Interest Rates with Application to Life Contingencies: Part II. *Journal of Risk and Insurance* 48: 628–37.

Benjamin, S. 1984. An Actuarial Layman Looks at Control Theory. *Proceedings of the Transactions of 22nd International Congress of Actuaries:* 295–310.

———. 1989. Driving the Pension Fund. *Journal of Institute of Actuaries* 116: 717–35.

Bizley, M. T. L. 1949. The Comparison of Salary Scales. *Journal of the Institute of Actuaries' Student Society* 9: 105–20.

———. 1951. The Effect on Pension Fund Contributions of a Change in the Rate of

Interest. *Journal of the Institute of Actuaries' Student Society* 9: 47–51.

Black, F. 1980. The Tax Consequence of Long-Run Pension Policy. *Financial Analysts Journal* 36 (July-August): 21–28.

Boermeester, J. M. 1956. Frequency Distribution of Mortality Costs. *Transactions of the Society of Actuaries* 8: 1–9.

Boulier, J.-F., S. Michel, and V. Wisnia. 1996. Optimizing Investment and Contribution Policies of a Defined Benefit Pension Fund. AFIR Colloquium, Nuremberg, Germany, 593–607.

Boulier, J.-F., E. Trussant, and D. Florens. 1995. A Dynamic Model For Pension Funds Management. AFIR Colloquium, Nuremberg, Germany, 361–84.

Bowers, N. L. 1967. An Approximation to the Distribution of Annuity Costs. *Transactions of the Society of Actuaries* 19: 295–309.

Bowers, N. L., H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt. 1986. *Actuarial Mathematics.* Ithaca, IL: Society of Actuaries.

Bowers, N. L., J. C. Hickman, and C. J. Nesbitt. 1976. Introduction to the Dynamics of Pension Funding. *Transactions of the Society of Actuaries* 28: 177–204.

———. 1979. The Dynamics of Pension Funding: Contribution Theory. *Transactions of the Society of Actuaries* 31: 93–119.

———. 1982. Notes on the Dynamics of Pension Funding. *Insurance: Mathematics and Economics* 1(4): 261–70.

Boyle, P. P. 1974. Note on the Variance of a Widow's Pension. *Journal of the Institute of Actuaries* 101: 103–7.

Bronson, D. C. 1949. Pension—1949. *Transactions of the Society of Actuaries* 1: 219–55.

Buck, G. B. 1916. Valuation of Pension Funds, with Special Reference to the Work of the New York City Pension Commission. *Proceedings of the Casualty Actuarial and Statistical Society of America* 11: 370–86.

———. 1953. Actuarial Soundness in Trusteed and Governmental Retirement Plans. *Journal of the American Association of University Teachers of Insurance* 20(1): 54–58.

Bühlmann, H. 1989. Actuaries of the Third Kind? ASTIN Bulletin, Vol. 19: 5-6.

Bulow, J. I. 1979. Analysis of Pension Funding under ERISA. Working Paper No. 402, November. Cambridge, MA: National Bureau of Economic Research.

Cairns, A. J. G. 1995. Pension Funding in a Stochastic Environment: the Role of Objectives in Selecting an Asset Allocation Strategy. In Proceedings of the 5th AFIR Colloquium, Brussels, September, vol. 1: 429–54.

———. 2000. Some Notes on the Dynamics and Optimal Control of Stochastic Pension Fund Models in Continuous Time. *ASTIN Bulletin* 30: 19–55.

Cairns, A. J., and G. Parker. 1997. Stochastic Pension Fund Modeling. *Insurance: Mathematics and Economics* 21: 43–79.

Carment, D. 1903. Some Practical Notes on the Valuation of Pension Funds. *Transactions of the Faculty of Actuaries* 1: 305–28.

Carter, W. H. 1934. Notes Arising Out of a Pension Scheme. *Journal of the Institute of Actuaries* 65: 94–100.

Chang, S. C. 1999. Optimal Pension Funding through Dynamic Simulations: The Case of Taiwan Public Employees Retirement System. *Insurance: Mathematics and Economics* 24: 187–99.

Chang, S. C., L. Y. Tzeng, and J. C. Y. Miaoc. 2003. Pension Funding Incorporating Downside Risks. *Insurance: Mathematics and Economics* 32(2): 217–28.

Clark, C. 2001. A Look at Dynamic Pension Plan Valuations. *Pension Forum* 13(1): 72–82.

Cobb, A. C. 1950. The Development of Pension Superannuation Schemes. *Journal of the Institute of Actuaries* 76: 3–21.

Colbran, R. B. 1982. Valuation of Final Salary Pension Schemes. *Journal of the Institute of Actuaries* 109: 359-385, and discussions: 386–416.

Collinson, D., ed. 2001. *Actuarial Methods and Assumptions Used in the Valuation of Retirement Benefits in the EU and Other European Countries.* Oxford: European Actuarial Consultative Group.

Cooper, S. L., and J. C. Hickman. 1967. A Family of Accrued Benefit Actuarial Cost Methods. *Transactions of the Society of Actuaries* 19: 53–59.

Corbett, H. R. 1936. Pension Funds: Alternative Funding Methods of Valuation. *Record of the American Institute of Actuaries* 25: 27–32.

Czernicki, D., N. Harewood, and M. Taht. 2003. Stochastic Modeling of Mortality. Presented at the *Stochastic Modeling Symposium* sponsored by the Canadian Institute of Actuaries and the Society of Actuaries.

Dhaene, J. 1989. Stochastic Interest Rates and Autoregressive Integrated Moving Average Processes. *ASTIN Bulletin* 19: 131–38.

Dufresne, D. 1988. Moments of Pension Fund Contributions and Fund Levels When Rates of Return Are Random. *Journal of the Institute of Actuaries* 115: 535–44.

———. 1989. Stability of Pension Systems When Rates of Return Are Random. *Insurance: Mathematics and Economics* 8: 71–76.

———. 1992. On Discounting When Rates of Return Are Random. *Transactions of the 24th International Congress of Actuaries* 1: 27–44.

Ferguson, C. C. 1911. A Pension Fund Method. *Transactions of the Actuarial Society of America* 12: 192–95.

Fleischer, D. R. 1975. The Forecast Valuation Method for Pension Plans. *Transactions of the Society of Actuaries* 27: 93–125.

Flynn, B. D. 1907–8. Staff Pension Funds with Special Reference to a Retirement Plan for United States Civil Service Employees. *Transactions of the Actuarial Society of America* 10: 275–88.

Frees, E. W. 1990. Stochastic Life Contingencies with Solvency Considerations. *Transactions of the Society of Actuaries* 42: 91–129.

Fretwell, R. L., and J. C. Hickman. 1964. Approximate Probability Statements about Life Annuity Costs. *Transactions of the Society of Actuaries* 16: 55–60.

Friedman, B. M. 1982. Pension Funding, Pension Asset Allocation, and Corporate Finance: Evidence from Individual Company Data. Working Paper No. 957, August. Cambridge, MA: National Bureau of Economic Research.

Gershenson, H. 1961. *Measurement of Mortality.* Chicago: Society of Actuaries.

Giacotto, C. 1986. Stochastic Modelling of Interest Rates: Actuarial vs. Equilibrium Approach. *Journal of Risk and Insurance* 53: 435–53.

Gold, J. 2005. Retirement Benefits, Economics and Accounting: Moral Hazard and Frail Benefit Designs. *North American Actuarial Journal* 9(1): 88–111.

Grant, M. D. 1909. Staff Pension Funds: An Investigation of the Mortality of Canadian Civil Service Pensioners. *Transactions of the Actuarial Society of America* 11: 163–71.

Griffin, F. L., Jr. 1966. Concepts of Adequacy in Pension Plans. *Transactions of the Society of Actuaries* 18: 46–63.

Grubbs, D. S., Jr. 1973. *Study of the Cost of Mandatory Vesting Provisions for Private Pension Plans.* Washington, DC: U.S. Government Printing Office.

Haberman, S. 1990. Stochastic Approach to Pension Funding Methods. In *Proceedings of First AFIR Colloquium, Paris,* vol. 4, 93–112.

———. 1993. Pension funding with time delays and autoregressive rates of investment return. Insurance: Mathematics and Economics 13(1): 45-56.

———. 1994. Autoregressive Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Pension Scheme. *Insurance: Mathematics and Economics* 14: 219–40.

———. 1997. Stochastic Investment Returns and Contribution Rate Risk in a Defined Benefit Pension Scheme. *Insurance: Mathematics and Economics* 19: 127–39.

Haberman, S., and J.-H. Sung. 1994. Dynamic Approaches to Pension Funding. *Insurance: Mathematics and Economics* 15: 151–62.

Hardy, R. P. 1892. On the Formula for Determining the Value of Benefits, according to the Principle of Collective Assurance. *Journal of the Institute of Actuaries* 30: 79–87.

Harris, G. 1995. Low Frequency Statistical Interest Rate Models. In *5th AFIR International Colloquium,* 799–831.

Harrison, J. M., and W. F. Sharpe. 1982. Optimal Funding and Asset Allocation Rules for Defined-Benefit Pension Plans. Working Paper No. 935, July. Cambridge, MA: National Bureau of Economic Research.

Haywood, G., and W. F. Marples. 1949. Pension Funds: Some Practical Points. *Transactions of the Faculty of Actuaries* 19: 111–12.

Hewat, A., and J. Chatham. 1893–95. On the Investigation of the Morality and Marriage Experience of Widow's Funds of the Scottish Banks. *Journal of the Institute of Actuaries* 31: 428–67.

Hickman, J. C. 1964. A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory. *Transactions of the Society of Actuaries* 16: 1–16.

———. 1985. Why Not Random Interest? *The Actuary* 19(2): 1, 3.

Howe, D., and A. Smith. 1974. The Practical Applications of Cash Flow Techniques to Pension Plans. *Proceedings: Canadian Institute of Actuaries* 5(2): 222–63.

Huffman, S. H., Jr. 1967–68. Practical Methods of Valuing the Cost of Early Retirement Benefits without Full Actuarial Reduction. *Proceedings of the Conference of Actuaries in Public Practice* 17: 136–45.

Hutcheson, W. A. 1920. Address of the President. *Transactions of the Actuarial Society of America* 21: 327–64.

Jordan, C. W., Jr. 1967. *Life Contingencies.* 2nd ed. Ithaca, IL: Society of Actuaries.

King, G. 1905. On Staff Pension Funds. *Journal of the Institute of Actuaries* 39: 129–92.

Kingsland, L. 1982. Projecting the Financial Condition of a Pension Plan Using Simulation Analysis. *Journal of Finance* 37 (May): 577–84.

Klugman, S.A., H. H. Panjer and G. E. Willmot. 1998. Loss Models: From Data to Decisions. New York: John Wiley and Sons.

Knopf, M. 1957. A Practical Demonstration of the Risk Run by a Very Small Company with a Trusteed Pension Plan. *Proceedings of the Conference of Actuaries in Public Practice* 6: 230–43.

Latimer, M. W. 1932. *Industrial Pension Systems.* New York: Industrial Relations Counselors.

Lidstone, G. J. 1905. Discussion of G. King "On Staff Pension Funds." Journal of the Institute of Actuaries 39: 193-197.

Llewellyn, G. F. 1938. On a Method of Forecasting the Future Pension Outgo of a Staff Superannuation Fund. *Journal of the Institute of Actuaries* 69: 172–78.

Lyon, C. S. S. 1960. Controlled Funding Methods for Group Pension Schemes. *Journal of the Institute of Actuaries' Student Society* 16: 130–46.

Mandl, P., and L. Mazurová. 1996. Harmonic Analysis of Pension Funding Methods. *Insurance: Mathematics and Economics* 17: 203–14.

Manly, H. W. 1898. On the Solution of Some Problems Which Frequently Arise Out of the Rules of Pension Funds and Friendly Societies. *Transactions of the Second International Actuarial Congress:* 860–67.

———. 1901. On the Valuation of Staff Pension Funds. *Journal of the Institute of Actuaries* 36: 209–76.

———. 1902. On the Valuation of Staff Pension Funds. *Journal of the Institute of Actuaries* 37: 193–244.

———. 1911. On Staff Pension Funds: The Process of the Accumulation of the Funds; The Identity of a Valuation with the Future Progress of a Fund; The Manner of Dealing with Funds Which Are Insolvent; and Sundry Observations. *Journal of the Institute of Actuaries* 45: 149–211.

Manly, H. W., and T. G. Ackland. 1912. On the Superannuation an Pension Funds of Certain Metropolitan Borough Councils, Their Establishment, Administration, and Actuarial Investigation. *Journal of the Institute of Actuaries* 46: 327–70.

Marples, W. F. 1945. An Analysis of a Pension Fund. *Journal of the Institute of Actuaries* 73: 66–86.

———. 1962. Salary Scales. *Transactions of the Society of Actuaries* 14: 1–30.

———. 1966. Cost of Vesting in Pensions. *Transactions of the Society of Actuaries* 18: 277–95.

McCrory, R. T. 2003. Pension Funding without Liabilities. In *The Great Controversy: Current Pension Actuarial Practice in Light of Financial Economics Symposium.* Schaumburg, IL: Society of Actuaries.

McGill, D. M., K. N. Brown, J. J. Haley, and S. J. Schieber. 2004. *Fundamentals of Private Pensions.* 8th ed. Oxford: Oxford University Press.

McGinn, D. F. 1966. Indices to the Cost of Vested Pension Benefits. *Transactions of the Society of Actuaries* 18: 187–242.

McGowan, J. 1902. An Enquiry into the Methods and General Principles to Be Adopted in Calculating the Rates of Contribution, Percentage Deduction from Salary, in the Case of Superannuation Funds and Pension Funds Generally. *Journal of the Institute of Actuaries* 37: 15–37.

McKenna, F. W. 1982. Pension Plan Cost Risk. *Journal of Risk and Insurance* 49 (June): 193–217.

Menge, W. O. 1932. Forces of Decrement in a Multiple-Decrement Table. *Record of the American Institute of Actuaries* 21: 41–46.

———. 1937. A Statistical Treatment of Actuarial Functions. *Record of the American Institute of Actuaries* 26: 65–88.

M'Lauchlan, J. J. 1908. The Fundamental Principles of Pension Funds. *Transactions of the Faculty of Actuaries* 4: 195–227.

Nesbitt, C. J. 1982. Exploration of Pension Funding in Case of Exact Vesting. *Insurance: Mathematics and Economics* 1(4): 255–60.

Niessen, A. M. 1950. Projections—How to Make Them and How to Use Them. *Transactions of the Society of Actuaries* 2: 235–53.

Noback, J. C. 1950. The Valuation of Self-insured Retirement Plans. *Transactions of the Society of Actuaries* 2: 49–69.

O'Brien, T. 1986. A Stochastic-Dynamic Approach to Pension Funding. *Insurance: Mathematics and Economics* 5: 141–46.

———. 1987. A Two-Parameter Family of Pension Contribution Functions and Stochastic Optimization. *Insurance: Mathematics and Economics* 6: 129–34.

Owadally, M. I., and S. Haberman. 1999. Pension Fund Dynamics and Gains/Losses Due to Random Rates of Investment Return. *North American Actuarial Journal* 3: 105–17.

Panjer, H. H., and D. R. Bellhouse. 1981. Stochastic Modelling of Interest Rates with Applications to Life Contingencies. *Journal of Risk and Insurance* 47(1): 91–110.

Pilch, M., and V. Wood. 1979. *Pension Schemes: A Guide to Principles and Practice.* Farnborough: Gower Press.

Piper, K. B. 1933. Contingency Reserves for Life Annuities. *Transactions of the Actuarial Society of America* 34: 240–49.

Pollard, A. H., and J. H. Pollard. 1969. A Stochastic Approach to Actuarial Functions. *Journal of the Institute of Actuaries* 79–113.

Pollard, J. H. 1971. On Fluctuating Interest Rates. Bulletin de l'Association Royal des Actuaries Belges 66: 68-97.

President's Committee on Corporate Pension Funds and Other Private Retirement and Welfare Programs. 1965. *Public Policy and Private Pension Programs.* Washington, DC: U.S. Government Printing Office.

Ramsay, C. M. 1993. Percentile Pension Cost Methods: A New Approach To Pension Valuations. *Transactions of Society of Actuaries* 45: 351–424.

Richmond, G. W. 1911. State Insurance against Invalidity and Old Age—The Actuarial Basis of the Austrian Method. *Journal of the Institute of Actuaries* 45: 336–76.

Robbins, R. B. 1929. The Accrued Liability of Retirement Plans. *Record of the American Institute of Actuaries* 18: 41–44.

Schnitzer, R. J. 1977. Characteristics and Operation of Projection Valuation Methods for Pension Plan Funding. *Transactions of the Society of Actuaries* 29: 269–98.

Seal, H. L. 1952. Acceptable Funding Methods for Self-insured Pension Plans. *Proceedings of the Conference of Actuaries in Public Practice* 2: 17–44.

———. 1953. The Mathematical Risk of Lump-Sum Death Benefits in a Trusted Pension Plan. *Transactions of the Society of Actuaries* 5: 135–42.

Seres, J. S. D., and J. W. Selley. 1980. *Pensions: A Practical Guide.* London: Oyez Publishing.

Shannon, G. 1976. Discussion of P. C. Bassett "Accrued Benefit Level Cost Method," *Proceedings of the Conference of Actuaries in Public Practice* 26: 103-4.

Shapiro, A. F. 1977. The Relevance of Expected Persistency Rates When Projecting Pension Costs. *Journal of Risk and Insurance* 44: 623–38.

———. 1979. A Bayesian Approach to Persistency Rates in the Projection of Retirement Costs. *Transactions of the Society of Actuaries* 31: 337–65.

———. 1983. Modified Cost Methods for Small Pension Plans. *Transactions of Society of Actuaries* 35: 11–36.

———. 1985. Contributions to the Evolution of Pension Cost Analysis. *Journal of Risk and Insurance* 52: 81–99.

Sharpe, W. F. 1976. Corporate Pension Funding Policy. *Journal of Financial Economics* 3 (June): 183–93.

Stone, D. G. 1948. Actuarial Note: Mortality Fluctuations in Small Self-insured Pension Plans. *Transactions of the Actuarial Society of America* 49: 82–91.

Taylor, G. 1994. Optimal control of funding, solvency and asset allocation for defined retirement benefits. Presented at the Seminar on Risk Theory, Mathematisches Forschungsinstitut, Oberwolfach, September 18-24.

———. 2002. Stochastic Control of Funding Systems. *Insurance: Mathematics and Economics* 30: 323–50.

Taylor, J. R. 1967. The Generalized Family of Aggregate Actuarial Cost Methods for Pension Funding. *Transactions of the Society of Actuaries* 19: 1–12.

Taylor, R. H. 1952. The Probability Distribution of Life Annuity Reserves and Its Application to a Pension System. *Proceedings of the Conference of Actuaries in Public Practice* 2: 100–150.

Tepper, I. 1981. Taxation and Corporate Pension Policy. *Journal of Finance* 36 (March): 1–13.

Thomas, E. C. 1914. On the Valuation of Benefits Dependent upon Promotion to a Higher Status. *Journal of the Institute of Actuaries* 48: 217–49.

Tinner, T. 1912. Discussion of H. W. Manly and T. G. Ackland "On the Superannuation an Pension Funds of Certain Metropolitan Borough Councils, Their Establishment, Administration, and Actuarial Investigation." Journal of the Institute of Actuaries 46: 377–78.

Toyoda H., S. Nishie and I. Katayose. 1995. The Influence of the Movement of Population on Corporate Pension Plan. Transactions, International Congress of Actuaries, Brussels, 1: 373-384.

Treuil, P. 1981. Fund Development of an Earnings-Related Social Insurance Plan under Stabilized Conditions. *Transactions of the Society of Actuaries* 33: 231–46.

Treynor, J. L. 1977. The Principles of Corporate Pension Finance. *Journal of Finance* 32 (May): 627–38.

Trowbridge, C. L. 1952. Fundamentals of Pension Funding. *Transactions of the Society of Actuaries* 4: 17–43.

———. 1963. The Unfunded Present Value Family of Pension Funding Methods. *Transactions of the Society of Actuaries* 15: 151–69.

———. 1973. Cost of Vesting in Private Pension Plans. *Transactions of the Society of Actuaries* 24: 394–410.

Trowbridge, J. R. 1977. Assessmentism—An Alternative to Pensions Funding *Journal of the Institute of Actuaries* 104: 173–204.

Vanderbroek, M. 1990. Pension Funding and Optimal Control. *Mitteilungen der schweizerische Vereinigung der Versicherungsmathematiker* 2: 313–25.

Walford, C. 1871. *The Insurance Cyclopaedia,* vol. 1. London: C & E. Layton.

Walker, D. A. 1915. A Staff Pension Fund. *Transactions of the Actuarial Society of America* 16: 109–44.

Winklevoss, H. E. 1974. Cost Sensitivity Analysis of Mandatory Funding and Vesting Standards in Pension Plans. *Journal of Risk and Insurance* 41 (March): 57–73.

———. 1982. Plasm: Pension Liability and Asset Simulation Model. *Journal of Finance* 37 (May): 585–94.

Winklevoss, H. E., and A. F. Shapiro. 1973. Estimating the Cost of Vesting in Pension Plans. *Transactions of the Society of Actuaries* 24: 373–93.

Woodward, J. H. 1925. The Valuation of Liabilities under Industrial Pension Plans. *Transactions of the Actuarial Society of America* 26: 443–79.