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CASH VALUES AND REDUCED PAID-UP INSURANCE—ACTUARIAL NOTE

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This investigation arose because of the question from a policyholder: "Why is the increase in the cash value greater than the increase in the amount of paid-up insurance purchased by the cash value?" It was thought that an examination of the mathematics involved might furnish the basis for an answer to the policyholder.

We chose to consider a coterminous endowment insurance where the cash value is equal to the net level reserve. We wish to determine the relationship between the rate of increase in the cash value and the rate of increase in the amount of paid-up insurance purchased by the cash value. Continuous functions are used throughout.

Rate of Increase in Cash Value

$$\frac{d}{dt}[{}_t\bar{V}(\bar{A}_{x:\overline{n}|})] = \frac{d}{dt}\left[1-\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}}\right] = \frac{1}{\bar{a}_{x:\overline{n}|}}(\bar{A}_{x+t:\overline{n-t}|}-\mu_{x+t}\bar{a}_{x+t:\overline{n-t}|}) .$$

Rate of Increase in Reduced Paid-up Insurance

$$\frac{d}{dt}[{}_t\bar{W}(\bar{A}_{x:\overline{n}|})] = \frac{d}{dt}\left[\frac{{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{\bar{A}_{x+t:\overline{n-t}|}}\right]$$

$$= \frac{\frac{d}{dt}[{}_t\bar{V}(\bar{A}_{x:\overline{n}|})]}{\bar{A}_{x+t:\overline{n-t}|}} + {}_t\bar{V}(\bar{A}_{x:\overline{n}|})\frac{d}{dt}(1-\delta\bar{a}_{x+t:\overline{n-t}|})^{-1}$$

$$= \frac{\frac{1}{\bar{a}_{x:\overline{n}|}}(\bar{A}_{x+t:\overline{n-t}|}-\mu_{x+t}\bar{a}_{x+t:\overline{n-t}|})}{\bar{A}_{x+t:\overline{n-t}|}}$$

$$- \frac{\delta\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})\,(\bar{A}_{x+t:\overline{n-t}|}-\mu_{x+t}\bar{a}_{x+t:\overline{n-t}|})}{(1-\delta\bar{a}_{x+t:\overline{n-t}|})^2}$$

$$= [\bar{A}_{x+t:\overline{n-t}|}-\mu_{x+t}\bar{a}_{x+t:\overline{n-t}|}]\left[\frac{1}{\bar{a}_{x:\overline{n}|}\bar{A}_{x+t:\overline{n-t}|}}-\frac{\delta\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{(\bar{A}_{x+t:\overline{n-t}|})^2}\right] .$$

Relationship between the Rates of Increase

$$\frac{d}{dt}[{}_t\bar{V}(\bar{A}_{x:\overline{n}|})] - \frac{d}{dt}[{}_t\overline{W}(\bar{A}_{x:\overline{n}|})]$$

$$= [\bar{A}_{x+t:\overline{n-t}|} - \mu_{x+t}\bar{a}_{x+t:\overline{n-t}|}]\left[\frac{1}{\bar{a}_{x:\overline{n}|}} - \frac{1}{\bar{a}_{x:\overline{n}|}\bar{A}_{x+t:\overline{n-t}|}} + \frac{\delta\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{(\bar{A}_{x+t:\overline{n-t}|})^2}\right].$$

The second term of the above product may be reduced as follows:

$$\frac{1}{\bar{a}_{x:\overline{n}|}} - \frac{1}{\bar{a}_{x:\overline{n}|}\bar{A}_{x+t:\overline{n-t}|}} + \frac{\delta\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{(\bar{A}_{x+t:\overline{n-t}|})^2}$$

$$= \frac{(\bar{A}_{x+t:\overline{n-t}|})^2 - \bar{A}_{x+t:\overline{n-t}|} + (\delta\bar{a}_{x:\overline{n}|} - \delta\bar{a}_{x+t:\overline{n-t}|})}{\bar{a}_{x:\overline{n}|}(\bar{A}_{x+t:\overline{n-t}|})^2}$$

$$= \frac{(\bar{A}_{x+t:\overline{n-t}|})^2 - \bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}(\bar{A}_{x+t:\overline{n-t}|})^2}.$$

Therefore,

$$\frac{d}{dt}[{}_t\bar{V}(\bar{A}_{x:\overline{n}|})] - \frac{d}{dt}[{}_t\overline{W}(\bar{A}_{x:\overline{n}|})]$$

$$= \frac{(\bar{A}_{x+t:\overline{n-t}|} - \mu_{x+t}\bar{a}_{x+t:\overline{n-t}|})}{\bar{a}_{x:\overline{n}|}(\bar{A}_{x+t:\overline{n-t}|})^2}[(\bar{A}_{x+t:\overline{n-t}|})^2 - \bar{A}_{x:\overline{n}|}].$$

If μ_x is an increasing function of x, then the expression $(\bar{A}_{x+t:\overline{n-t}|} - \mu_{x+t}\,\bar{a}_{x+t:\overline{n-t}|})$ is greater than zero, as may be seen by examining the function when expressed as an integral:

$$\int_0^{n-t} v^s\,{}_sp_{x+t}(\mu_{x+t+s} - \mu_{x+t})\,ds + A_{x+t:\overset{1}{\overline{n-t}|}}.$$

Therefore, when μ_x is an increasing function of x, we conclude that

$$\frac{d}{dt}[{}_t\bar{V}(\bar{A}_{x:\overline{n}|})] \gtrless \frac{d}{dt}[{}_t\overline{W}(\bar{A}_{x:\overline{n}|})]$$

according as

$$(\bar{A}_{x+t:\overline{n-t}|})^2 \gtrless \bar{A}_{x:\overline{n}|}.$$

While we felt that the above result was interesting, it is obvious that it was of little help in answering our policyholder's question.

There are, of course, more general cases that could be considered. For example, if in the above development the cash value is taken to be equal to the present value of future benefits less the present value of future adjusted premiums, where the adjusted premium P^a is defined by

$$P^a \bar{a}_{x:\overline{n}|} = \bar{A}_{x:\overline{n}|} + E \ ,$$

the criterion is that

$$\frac{d}{dt}\,{}_t\mathrm{CV} \gtreqless \frac{d}{dt}\,{}_t\mathrm{W}$$

according as

$$(\bar{A}_{x+t:\overline{n-t}|})^2 \gtreqless \frac{\bar{A}_{x:\overline{n}|} + E}{1 + E} \ .$$

DISCUSSION OF PRECEDING PAPER

PAUL W. NOWLIN:

The policyholder's question as to why the increase in cash value exceeds the increase in paid-up insurance might possibly be answered as follows. One might think that the increase in paid-up insurance should exceed the increase in cash value, because the amount of the paid-up insurance which the increase in cash value would purchase, at the attained age when the increased cash value is available, naturally exceeds the increase in cash value. But this ignores the fact that the rest of the cash value, namely the cash value before the increase, will purchase, at the same attained age, a smaller amount of paid-up insurance than it would have purchased at the younger age before the increase.

Although it would not make it any clearer for the policyholder, it is interesting to give the mathematical relationship between the increases in reserve, paid-up insurance, and premium rate per dollar of insurance. Ignoring infinitesimals of higher order, this is the differential equation:

$$d\,{}_t\bar{V} = (d\,{}_t\bar{W})\,\bar{A}_{x+t:\overline{n-t}|} + {}_t\bar{W}\,(d\bar{A}_{x+t:\overline{n-t}|})\,.$$

The first term on the right is the increase in reserve (or single premium for the paid-up insurance) resulting from the increase in paid-up insurance. The second term is the increase in reserve resulting from the increase in the premium rate per dollar of insurance. This equation follows immediately upon taking differentials of both sides in ${}_t\bar{V} = {}_t\bar{W}\cdot\bar{A}_{x+t:\overline{n-t}|}$.

Since ${}_t\bar{W}$ is greater than ${}_t\bar{V}$ except for $t = 0$ and $t = n$ where the two are equal, from geometric considerations it appears that if ${}_t\bar{W}$ and ${}_t\bar{V}$ are fairly smooth functions the increase in ${}_t\bar{W}$ will be greater than the increase in ${}_t\bar{V}$ up to a certain time after which the increase in ${}_t\bar{W}$ will be less. The same conclusion follows from the authors' criterion for the relation between the derivatives of the reserve and the paid-up insurance. When $t = 0$, $(\bar{A}_{x+t:\overline{n-t}|})^2 < \bar{A}_{x:\overline{n}|}$; and when $t = n$, $(\bar{A}_{x+t:\overline{n-t}|})^2 > \bar{A}_{x:\overline{n}|}$. Thus if $\bar{A}_{x+t:\overline{n-t}|}$ is monotonic increasing, as would almost always be the case, $(\bar{A}_{x+t:\overline{n-t}|})^2$ is less than $\bar{A}_{x:\overline{n}|}$ up to a certain time and greater thereafter.

BARNET N. BERIN:

While the policyholder's question may have remained unanswered, the algebra contained in this note is interesting and complete. There is, however, another approach to this problem which directly produces the result demonstrated and can be interpreted by general reasoning. At time t, the generalized life-endowment relations are:

$$\log {}_t\bar{W} = \log {}_t\bar{V} - \log \bar{A}_t$$

$$\frac{d}{dt}\,{}_t\bar{V} = \frac{{}_t\bar{V}}{{}_t\bar{W}}\frac{d}{dt}\,{}_t\bar{W} + \frac{{}_t\bar{V}}{\bar{A}_t}\frac{d}{dt}\,\bar{A}_t$$

$$= \left(\frac{d}{dt}\,{}_t\bar{W}\right)\left[\bar{A}_t + \frac{{}_t\bar{V}}{\bar{A}_t}\,\frac{(d/dt)\,\bar{A}_t}{(d/dt)\,{}_t\bar{W}}\right]$$

$$= \left(\frac{d}{dt}\,{}_t\bar{W}\right)\left[\bar{A}_t\left(1 + \frac{\delta\,{}_t\bar{V}}{\bar{P}}\right)\right]. \qquad (1)$$

This result follows because

$$\frac{d}{dt}\,{}_t\bar{W} = -\frac{\bar{P}}{\delta}\frac{d}{dt}\left(\frac{1}{\bar{A}_t}\right).$$

Therefore, we conclude that

$$\frac{d}{dt}\,{}_t\bar{V} \gtreqless \frac{d}{dt}\,{}_t\bar{W}$$

according as

$$\bar{A}_t\left(1 + \frac{\delta\,{}_t\bar{V}}{\bar{P}}\right) \gtreqless 1\,.$$

An equivalent inequality is

$${}_t\bar{V}\,\bar{A}_t \gtreqless \bar{A}\,\frac{\bar{a}_t}{\bar{a}}$$

$$(\bar{A}_t - \bar{A})\,\bar{A}_t \gtreqless \bar{A}\,(1 - \bar{A}_t)$$

$$(\bar{A}_t)^2 \gtreqless \bar{A}\,.$$

As Mr. Hummel and Mr. Stedman have shown, the increase in the net level reserve may be greater than, equal to, or less than the increase in paid-up insurance purchased by the net level reserve. The rate of change in net level reserve at time t is equal to the rate of change in paid-up insurance multiplied by a function of t, as shown in (1) above. The first factor in this function is the net single premium, at the attained age, for the same benefit maturing at the same time that the original policy matures. The second factor adds an interest increment. Therefore, the ratio of increase in reserve to increase in paid-up insurance has a physical interpretation.

C. W. HARTOG:

The criterion that was derived in this paper may be obtained more simply by determining the derivative of the amount of paid-up insurance, using the cash value as variable, as follows:

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}} $$

$$ \bar{A}_{x+t:\overline{n-t}|} = {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|}) + \bar{A}_{x:\overline{n}|} $$

$$ \frac{d\,[{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) / \bar{A}_{x+t:\overline{n-t}|}]}{d\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})} = \frac{d\,\{{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) / [{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|}) + \bar{A}_{x:\overline{n}|}]\}}{d\,{}_t\bar{V}(\bar{A}_{x:\overline{n}|})} $$

$$ = \frac{1}{{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|}) + \bar{A}_{x:\overline{n}|}} - \frac{{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|})}{[{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|}) + \bar{A}_{x:\overline{n}|}]^2} $$

$$ = \frac{\bar{A}_{x:\overline{n}|}}{[{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) \cdot (1 - \bar{A}_{x:\overline{n}|}) + \bar{A}_{x:\overline{n}|}]^2} $$

$$ = \frac{\bar{A}_{x:\overline{n}|}}{(\bar{A}_{x+t:\overline{n-t}|})^2} . $$

By an algebraic transformation, a criterion is developed which states that the increase in the amount of paid-up will exceed the increase in the cash value as long as the sum of the paid-up and the cash value is less than the face amount of insurance. This transformation is shown below:

$$ (\bar{A}_{x+t:\overline{n-t}|})^2 \lesseqgtr \bar{A}_{x:\overline{n}|} $$

$$ 1 - \bar{A}_{x+t:\overline{n-t}|} \gtreqless 1 - \frac{\bar{A}_{x:\overline{n}|}}{\bar{A}_{x+t:\overline{n-t}|}} $$

$$ 1 - \bar{A}_{x:\overline{n}|} - \bar{A}_{x+t:\overline{n-t}|} + \bar{A}_{x:\overline{n}|} \gtreqless \frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{\bar{A}_{x+t:\overline{n-t}|}} $$

$$ 1 - \frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}} \gtreqless \frac{1}{\bar{A}_{x+t:\overline{n-t}|}} \cdot \frac{\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}} $$

$$ 1 - {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) - \frac{{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{\bar{A}_{x+t:\overline{n-t}|}} \gtreqless 0 . $$

(AUTHORS' REVIEW OF DISCUSSION)

THOMAS J. HUMMEL AND JOHN A. STEDMAN:

We are gratified by the interest that was shown in our paper. Each of the authors has produced an equivalent criterion by a simple and direct method.

We did answer the policyholder's letter as suggested in Mr. Nowlin's discussion. We also used actual values to demonstrate the point. The policyholder did not write again.

The criterion developed by Mr. Hartog is especially interesting. It would seem that such a simple relationship between the face amount, the reserve, and the amount of paid-up insurance would submit to general reasoning, but we are unable to produce a general reasoning interpretation.