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LIDSTONE'S FORMULA FOR THE PRESENT VALUE OF THE PROFITS OF A POLICY—ACTUARIAL NOTE

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RECENTLY, in course of an investigation into the value of business in connection with the mutualization of Canadian life insurance companies, it was necessary to review the actuarial literature on the subject. Although, owing to the special features of Canadian law and practice, the question of the dollar value of existing business was not made the criterion of the investigation, but rather the profitability of the company's operations, yet one point arose which is worthy of special note.

The point is one in the theory of life contingencies, but no assistance could be obtained from the Society of Actuaries' Textbook by C. W. Jordan or the British Institute of Actuaries texts by King, Spurgeon, and P. F. Hooker and L. H. Longley-Cook. What were required were the formulas for the present value of the future profits under a policy, on given assumptions of interest, mortality and expenses to be experienced, both individually and in total.

The formula for the present value of the total *future* profits under a policy is

> Valuation Reserve *minus* Reserve on experience bases of interest and mortality and with a premium valued of the gross premium deloaded for expenses

The formula applies not only to nonparticipating business but also to the accruing profit on participating business before any subdivision of such profit is made as between policyholders, shareholders and surplus.

This may be considered by some to be "obvious from general considerations," yet its undoubted importance and particularly its breakdown into its component parts of the present value of the profit due to interest, mortality and loading merit actuarial demonstration, and, as outlined below, Lidstone's results are readily misquoted.

Historically, according to *RAIA*, *TASA* and *TSA* the point was first raised in "The Worth of Business" by Harris E. Vineberg in *RAIA* IV, pages 35 on. There Mr. Vineberg assumed that the value of business was "the excesses of the tabular reserves carried by the company over the reserves thus brought out," the latter being the reserves on a gross premi-

um valuation with standards of interest, mortality and expenses approximating to actual experience. Nonparticipating business only was the subject of the enquiry. The date of the paper is June 1915.

The discussion of the paper made no reference to the principles or formula used, although there was much criticism of the results brought out, namely, for age 35 at entry, a value at the end of ten years on the ordinary life plan of \$4.53 to \$27.90 per mille sum assured, varying according to the expense formula used, and for the 20 payment life plan \$15.38 to \$38.75.

The next reference appears in A. A. Rydgren's paper "Value of Business Reinsured in Bulk" in *TASA* XXII. The date of the paper is May 1921. The most important point brought out in this paper was the major changes in values resulting from relatively minor changes in mortality, interest and expenses. No general formulas were developed, but the arithmetical results of the profits from various sources, year by year, were obtained and valued.

In the discussion of Mr. Rydgren's paper, Mr. J. B. Maclean drew attention to Lidstone's paper in *JIA* XXXII as giving the general formula for the calculation of "the 'price' paid for the expected profits from interest, mortality and net loading, or, in other words . . . the present value of such future profits." Unfortunately Lidstone's formula is incorrectly quoted and anyone who compares Mr. Maclean's formula with that given in Lidstone's paper and the demonstration given below will appreciate how readily this can be done.

It is mainly with a view to clarifying this point and preventing future misunderstandings, as well as to draw attention to the brilliant piece of manipulation of actuarial functions of which Lidstone was a master as given in the Appendix to his paper, that this note has been prepared. The final result is of considerable importance.

To avoid misunderstanding let us vary somewhat the symbols by which Lidstone expresses his final results.

On the valuation basis let R be the reserve on any policy anniversary, B the value of the future benefits and $\ddot{a}$ the annuity due at that time, and P the net premium, so that $R = B - \mathrm{P}\ddot{a}$.

Let R'' be the reserve on the *experience* bases of interest and mortality, with B'' and $\ddot{a}''$ the corresponding functions, and with a premium valued, P'', of the gross premium deloaded for expenses, so that

$$R'' = B'' - \mathrm{P}''\ddot{a}'' .$$

Lidstone demonstrates that the present value of the profits from interest and mortality together is

$$R - (B'' - \mathrm{P}\ddot{a}'') ,$$

which can be written

$$R - (B'' - \mathrm{P}''\ddot{a}'') + (\mathrm{P} - \mathrm{P}'')\ddot{a}''$$

or

$$R - R'' + (\mathrm{P} - \mathrm{P}'')\ddot{a}'' . \tag{i}$$

This is Lidstone's formula (12).

As the value of the profit from *loading* is

$$(\mathrm{P}'' - \mathrm{P})\ddot{a}'' , \tag{ii}$$

it follows that the present value of the profit from interest, mortality *and* loading is

$$R - R'' , \tag{iii}$$

namely, the Valuation Reserve less the reserve based on experience rates of interest and mortality and with a premium valued of the gross premium deloaded for expenses.

It may appear surprising that Lidstone, in giving formula (i), did not go on to indicate formula (iii), but in 1895 profit from mortality was a minor item and Lidstone gave his attention to the two items, (1) loading and (2) interest and mortality combined.

The following statements giving corresponding formulas for these separate items should be of interest.

Assume that R', B', P', $\ddot{a}'$ and d' express the unaccented functions but where the experience rate of interest replaces the valuation rate of interest. Lidstone demonstrates in the same Appendix that the present value of the interest profit alone calculated at the experience rate of interest is

$$(\mathrm{P} + d')\ddot{a}' - (\mathrm{P} + d)\ddot{a} ,$$

which may be expressed as

$$(\mathrm{P}' + d')\ddot{a}' - (\mathrm{P} + d)\ddot{a} + (\mathrm{P} - \mathrm{P}')\ddot{a}' .$$

Noting generally that

$$R = 1 - (\mathrm{P} + d)\ddot{a} ,$$

this formula becomes

Value of Interest Profit

$$R - R' + (\mathrm{P} - \mathrm{P}')\ddot{a}' , \tag{iv}$$

which for ordinary whole life insurances is Lidstone's formula (5) and is expressed in general form in paragraph 36 of his paper.

As the expression for the present value of interest and mortality combined is formula (i) above, namely

$$R - R'' + (\mathrm{P} - \mathrm{P}'')\ddot{a}'' ,$$

we get by subtraction the corresponding formula for the value of the mortality profit only:

Value of Mortality Profit

$$R' - R'' + (\mathrm{P} - \mathrm{P}'')\ddot{a}'' - (\mathrm{P} - \mathrm{P}')\ddot{a}' . \qquad \text{(v)}$$

As the value of the Loading Profit formula (ii) has already been given above, the sum of the three gives again the main formula $R - R''$.

An interesting form to which (v) reduces is

$$(B' - \mathrm{P}\ddot{a}') - (B'' - \mathrm{P}\ddot{a}'') .$$

It is to be hoped that the demonstration of these formulas, at least for an ordinary whole life policy, will be included in text books on life contingencies. Lidstone's paper is one of the classic papers in *JIA* analyzing the equity of the British uniform reversionary bonus system.

The expressions for the present value of the future profits under an ordinary whole life policy are:

Interest Profit	$(\mathrm{P} + d')\ddot{a}' - (\mathrm{P} + d)\ddot{a}$
Mortality Profit	$(\mathrm{P} + d')\ddot{a}'' - (\mathrm{P} + d')\ddot{a}'$
Loading Profit	$(\mathrm{P}'' - \mathrm{P})\ddot{a}''$

DISCUSSION OF PRECEDING PAPER

J. B. MACLEAN:

I generally agree with anything that Mr. Pedoe says. However, I am not sure that I agree with his suggestion that formulas such as are described in this note should have a place in the standard textbooks of actuarial science.

These formulas depend, for any practical value or significance they may have, on the ability to make "realistic" assumptions as to the rates of interest, mortality and expense (as well as the rate of voluntary termination, not mentioned in the note) over long periods in the future. It is one thing to make *conservative* assumptions in regard to these factors, as we must do in calculating premiums and reserves. It is quite another thing to attempt to actually predict the future and make so-called "realistic" assumptions. In fact, it is impossible. Any calculations professing to show the value of future profits or of existing business can, in my opinion, determine no more than a figure which may be considered as not entirely unreasonable.

As Mr. Pedoe points out, Vineberg's methods were criticized because small variations in the "realistic" assumptions produced very large differences in the resulting values, while Rydgren's paper and its discussion emphasized the same point.

A few years ago I made a gross premium valuation of the business of a company which had been purchased by another company. The purpose, in that case, was not, directly, to estimate the value of future profits but to determine the minimum amount required to mature all existing policy obligations. A great many "realistic" assumptions were involved. In fact, we had a perfect example of Lidstone's "delicate mosaic." In the lawsuit which followed, all the assumptions were hotly contested, not only by the lawyers but by the actuaries of our opponents. If I were doing the job today I am sure that practically all the assumptions would be different.

The point is that the value of business or the value of future profits is actually incapable of calculation. The only significant value is what a purchaser is willing to pay for it, and this may be, and usually is, determined, or affected, by considerations quite independent of the "probable" future rates of mortality, interest, expense and so on. Thus, to include the formulas discussed in this note in textbooks might give students the wrong idea that the calculation of future profits is actually possible.

May I say, in conclusion, that when one makes a mistake (as I appear to have done), I think it should be considered as decently buried after thirty-five years!

(AUTHOR'S REVIEW OF DISCUSSION)

ARTHUR PEDOE:

The main purpose of this actuarial note was to ensure that somewhere in our actuarial literature there should be a statement and a demonstration of the formula that the present value of the total future profits of a policy is $R - R''$, namely the valuation reserve less the reserve based on experience rates of interest and mortality and with a premium valued of the gross premium deloaded for expenses.

It was a privilege to have Mr. J. B. Maclean discuss the note and confirm from his wide experience what has already been noted: that the value of future profits of existing business is not something which can be determined by a simple actuarial formula but rather by existing special circumstances and business judgment. However, I cannot agree with Mr. Maclean that just because an actuarial formula may be misused by a student it should not appear in actuarial texts. Further, in my opinion, the pages in the actuarial texts—old and new—dealing with analysis of surplus are unsatisfactory. The formulas given in the note should give a student food for thought, which is the purpose of actuarial training.

With apologies to Mr. Maclean for digging up this ancient matter which has become of considerable current importance, may I point out that the formula in question was *not* stated by Lidstone, and we owe it to Mr. Maclean for drawing the attention of actuaries on this side to Lidstone's analysis which has led to this actuarial note.

It was my wish to save reference to Lidstone's paper in *JIA* XXXII by including in the note a demonstration of each of the parts of the formula, namely the values of the profits due to interest, mortality and loading separately, only the first of which was given by Lidstone. This is now included in the discussion below, for which I am indebted to Mr. Peter Madders, an Associate of the Society. The following proofs appear to me to be far neater than Lidstone's procedure and, what is equally important, are closely related, as were the formulas for the ordinary whole life plan given in the note. I take full responsibility for the form in which the following demonstration appears, but the credit for the ideas belongs entirely to Mr. Madders.

Consider the profits arising from a paid-up *pure endowment* payable in n years: the reserve is ${}_nV^x$. Instead of assessing the position annually,

suppose the company accumulates the interest earned and at the end of the n years, if the insured is still alive, pays the sum insured and allows interest and mortality profit to emerge at that date.

The *interest profit* at the end of n years will be

$$v^n {}_np_x [(1+i')^n - (1+i)^n] ,$$

where the primed symbol represents the experience rate.

The present value of the interest profit will be

$$v'^n v^n {}_np_x [(1+i')^n - (1+i)^n] = v^n {}_np_x - v'^n {}_np_x . \qquad (1)$$

The probability of paying the sum insured in n years is ${}_np'_x$ and the present value of the *mortality profit* is therefore

$$v'^n ({}_np_x - {}_np'_x) . \qquad (2)$$

Following the notation in the actuarial note, the reserve on the valuation basis can be expressed symbolically as $R = B - P\ddot{a}$, that is, the value of the sum insured S payable on death less the value of the future net premiums payable according to the valuation basis. Expressing this in its year-to-year form of value of benefits less value of net premiums, the reserve equals, on a policy anniversary at attained age x:

$$S (vq_x + v^2 {}_2p_x q_{x+1} + v^3 {}_2p_x q_{x+2} + \ldots + v^n {}_{n-1}p_x q_{x+n-1} + \ldots) - P (1 + vp_x + v^2 {}_2p_x + \ldots + v^{n-1} {}_{n-1}p_x + \ldots) . \qquad (3)$$

As

$${}_np_x q_{x+n} = {}_np_x - {}_{n+1}p_x , \qquad (4)$$

the nth term of the expression (3) becomes

$$S v^n ({}_{n-1}p_x - {}_np_x) - P (v^{n-1} {}_{n-1}p_x) ; \qquad (5)$$

and, applying formula (1) to this, the present value of the future *interest profit* is represented by an expression the nth term of which is

$$S [(v^n {}_{n-1}p_x - v'^n {}_{n-1}p_x) - (v^n {}_np_x - v'^n {}_np_x) - P (v^{n-1} {}_{n-1}p_x - v'^{n-1} {}_{n-1}p_x) . \qquad (6)$$

The first part of this expression can be written

$$S [v^n ({}_{n-1}p_x - {}_np_x) - v'^n ({}_{n-1}p_x - {}_np_x)] ,$$

which from (4) can be written

$$S [{}_{n-1}p_x q_{x+n-1} (v^n - v'^n)] ;$$

so that (6) represents the nth term of the expression

$$(B - P\ddot{a}) - (B' - P\ddot{a}') , \qquad (7)$$

where the primed functions are computed at the experience rate of interest but with the valuation mortality; and this is equal to

$$R - R' + (\mathrm{P} - \mathrm{P}')\,\ddot{a}' ,$$

which is formula (iv) of the actuarial note.

Similarly for the present value of the future *mortality profit*, applying formula (2) to formula (5) the nth term of the expression becomes

$$S\,v'^{n}\,[\,({}_{n-1}p_x - {}_{n-1}p'_x) - ({}_{n}p_x - {}_{n}p'_x)\,] - \mathrm{P}\,v'^{n-1}\,({}_{n-1}p_x - {}_{n-1}p'_x)\,. \quad (8)$$

The first part of this expression is

$$S\,v'^{n}\,[{}_{n-1}p_x\,q_{x+n-1} - {}_{n-1}p'_x\,q'_{x+n-1}]\,,$$

so that (8) is the nth term of the expression

$$(B' - \mathrm{P}\ddot{a}') - (B'' - \mathrm{P}\ddot{a}'')\,, \quad (9)$$

which is the second form of formula (v) in the actuarial note.

Thus the present value of future profits under a policy when divided into its component parts can be expressed as follows:

Interest Profit	$(B - \mathrm{P}\ddot{a}) - (B' - \mathrm{P}\ddot{a}')$
Mortality Profit	$(B' - \mathrm{P}\ddot{a}') - (B'' - \mathrm{P}\ddot{a}'')$
Loading Profit	$(\mathrm{P}'' - \mathrm{P})\ddot{a}''$
which add up to	$(B - \mathrm{P}\ddot{a}) - (B'' - \mathrm{P}''\ddot{a}'')$
or	$R - R''$.

Some of our members, including Mr. Madders, have expressed the opinion to me that, while the formula for the total value has some significance, the formulas for the separate parts of the total profit are quite limited in their practical use. However, the analysis is of interest to students of life contingencies.