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GAIN AND LOSS ANALYSIS FOR PENSION FUND VALUATIONS

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THE actuary concerned with valuation of liabilities under self-administered retirement plans and retirement plans funded through group annuity contracts of the Deposit Administration or Immediate Participation Guarantee (IPG) type has three paramount problems, namely:

(1) testing the mathematical correctness of his periodic actuarial valuations,
(2) examining changes in liabilities and contribution levels from one valuation date to another, and
(3) reviewing the valuation assumptions to assess their continuing appropriateness for estimation of the retirement plan liabilities.

Analysis of each element contributing to an actuarial gain or loss between two valuation dates is useful for solution of these problems. Gain and loss analysis aids in solving the first problem by providing an independent check of the actuarial technique and mathematics. The gain or loss related to each valuation assumption can be computed independently of the current valuation; an aggregate gain or loss can be computed by comparing a projection of the unfunded accrued liability from the previous valuation with the unfunded accrued liability calculated on the current valuation date. By measuring this aggregate gain or loss against the sum of the individual gains and losses, a significant error will be revealed if the two are not equal or within an acceptable tolerance. If they are substantially the same, either the valuation is mathematically correct or any errors are offsetting and presumably negligible. The second problem originates with the valuation review, which is more effective if changes in the liabilities and contribution levels can be traced and identified with precision. Also, subsequent explanation and interpretation of results to the client or policyholder are simplified and made more meaningful by the gain and loss analysis. Finally, by interpreting the aggregate financial effect of each valuation assumption, gain and loss analysis provides a quantitative measure of its validity for estimation of future liabilities, as well as a means of gauging the adjustment required if the assumption is to more reliably predict future experience.

The growing importance of retirement plans, as a major segment of national savings, a vital subject of labor-management negotiations and the object of intensive commercial interest by banks and insurance companies, demands that gain and loss analysis and other tools of actuarial science, such as experience studies of termination rates, mortality, and salary scales, be employed in the estimation of retirement plan liabilities and the calculation of pension fund contributions.

The British Institute of Actuaries textbook[1] makes brief reference to gain and loss analysis and most actuarial offices probably are familiar with the technique, but there is only brief treatment of the subject in American actuarial literature.[2] It is hoped that this paper will stimulate discussion and interest and lead to a broader knowledge and appreciation of gain and loss analysis among students and practicing actuaries.

Purpose

This paper will attempt:

(1) to outline the theory of gain and loss analysis for pension fund valuations by defining each element which contributes to the gain or loss and by demonstrating numerically and by general reasoning that the sum of these parts equals the total gain or loss emerging during the valuation interval;

(2) to indicate the practical problems met in applying the theory to two model plans, one self-administered and the other insured, and to comment on the presentation of results to the policyholder or client;

(3) to show, by analogy with the classical Fackler accumulation formulas, the similarity between gain and loss analysis for pension funds and analysis of reserve techniques familiar to insurance actuaries;

(4) to provide a reference for students and other actuaries concerned with this problem and to propose a notation suitable for general usage.

Gain and Loss, in General

If each valuation assumption exactly anticipated the experience under a retirement plan from year to year there would never be an actuarial gain or loss. Since this will not happen, the actuary's valuation is in error to the extent that his chosen valuation basis in aggregate overstates or understates the retirement plan liabilities. The consequence of an

[1] *Pension and Widows' and Orphans' Funds*, by R. J. W. Crabbe, F.I.A. and C. A. Poyser, F.I.A., Cambridge University Press.

[2] "Gain-and-Loss Analysis in a Self-Insured Retirement Plan," Gabriel M. Hellman, *The Proceedings*, Conference of Actuaries in Public Practice, 1956–1957. "A Method of Allocating Gains and Losses in a Pension Fund," Jack M. Elkin, *The Proceedings*, Conference of Actuaries in Public Practice, 1957–1958.

aggregate overstatement is an actuarial gain; conversely an aggregate understatement produces a loss.

The actuarial gain equals (*a*) excess of net actual release of liability over the release predicted by the valuation basis *plus* (*b*) the excess of expected disbursements over actual disbursements. (This general statement considers that salary increases produce a negative release of liability; that interest earnings on invested funds are a negative disbursement; that an actuarial loss is a negative gain; etc.)

Elements of Gain and Loss

The principal elements of an actuarial gain or loss can be grouped under three headings:

1. Events during the valuation period which influence the pension fund, such as:
   a) Payment of benefits upon death before or after retirement; annuity benefits to retired and disabled members and their beneficiaries; refund and severance benefits upon termination of employment
   b) Payment of administrative expenses—actuarial, legal, investment and accounting fees; taxes; commissions; other amounts not paid to a participant or beneficiary under the plan
   c) Receipt of investment income—dividends, interest, amortization of premium or discount, etc., in self-administered funds; guaranteed interest or investment income credits under group annuity contracts
   d) Capital gains or losses, both realized and unrealized
   e) Group annuity dividends
2. Events during the valuation period which affect the future liabilities under the retirement plan, such as:
   a) Deaths, withdrawals, disabilities and retirements among active employees
   b) Deaths among retired employees
   c) Salary changes affecting benefit liabilities
   d) New entrants into the eligible employee group
3. Items which affect the estimated actuarial liabilities, but are not directly related to experience during the valuation period, such as:
   a) Errors in reporting or processing the valuation data
   b) Changes in amounts and types of benefits provided under the retirement plan
   c) Changes in the actuarial valuation method
   d) Changes in the pension fund asset valuation method
   e) Changes in the actuarial assumptions underlying the valuation.

$+ B_t^{Er}$ (payments to retired employees and their beneficiaries).

Under Deposit Administration plans, B_t^{Er} and possibly B_t^{Ei} and B_t^{Ew} would include the full purchase price of annuity benefits vesting or commencing during the year, rather than dollar amounts of expected benefit payment during the year. This procedure focuses the analysis of gain and loss on the Deposit Administration fund rather than on the policyholder dividend account. Recognitions of gains or losses attributable to participants for whom annuity benefits have been purchased will be retarded until dividends are declared under the contract and credited to the Deposit Administration fund. Means of reflecting and allocating these factors, as well as excess interest credits and expense savings related to active employees, are discussed hereafter.

B_t^A = Actual amount of benefits due and accrued during valuation year t

$= B_t^{Ad} + B_t^{Aw} + B_t^{Ai} + B_t^{Ar}$.

These symbols are directly analogous to the symbols in B_t^E above.

R_t^E = Expected release of liability due to change of participant status during the year, computed as of the end of valuation year t

$= R_t^{Ed} + R_t^{Ew} + R_t^{Ei} + R_t^{Er}$.

Again, these symbols and those of R_t^A below are analogous to B_t^E.

R_t^A = Actual release of liability due to change of participant status during the year, computed at end of valuation year t

$= R_t^{Ad} + R_t^{Aw} + R_t^{Ai} + R_t^{Ar}$.

S_t^E = Expected increase in accrued liability at end of valuation year t because of anticipated salary increases during the year.

S_t^A = Actual increase in accrued liability at end of valuation year t because of salary changes during the year.

N_t^E = Expected addition to accrued liability at end of valuation year t because of participants who enter during the year and survive as active employees to the year end.

N_t^A = Actual addition to accrued liability at end of valuation year t because of new entrants during the year who continued as active employees to the year end.

Factors affecting benefits to widows, children and other dependents, of the type included under the Federal Old-Age, Survivors, and Disability Insurance Act (OASDI) and many state and municipal retirement systems, have not been mentioned specifically. The analysis of gain and loss attributable to these benefits can be constructed by analogy with the basic principles governing gain and loss analysis for participants' benefits.

Definitions

The definitions set out below presume annual actuarial valuations. The extensions for biennial or quinquennial valuations are obvious.

$(AL)_t^A$ = Actual accrued liability as of the beginning of valuation year t, in accordance with funding method and valuation assumptions then in use.

For example, under the aggregate cost method, this would be the present value of all future expenses, refunds and benefit payments; under the entry age and attained age normal cost methods, it would be the present value of all future expenses, refunds and benefit payments less the present value of future normal costs; under the unit credit normal cost method, it would be the present value of benefit credits earned prior to the valuation date. When one of the frozen initial liability methods is in use, it is desirable to define $(AL)_t^A$ as the present value of all future expenses, refunds and benefit payments.

$(NC)_t$ = Normal cost for valuation year t as defined by the funding method, calculated on the assumption that it is due at the beginning of year t.

Under the aggregate cost and frozen initial liability methods, $(NC)_t$ will be zero for the purpose of analysis, since these methods are analogous to single premium or paid-up policies.

$(NC)_t$ will reflect all death, retirement and other benefits, as well as expenses paid from the pension fund.

B_t^E = Expected amount of benefits for which liability will be incurred during valuation year t

= B_t^{Ed} (death benefits prior to retirement for active employees and for terminated employees with vested rights)

+ B_t^{Ew} (withdrawal benefits)

+ B_t^{Ei} (payments to disabled employees and their beneficiaries)

E_t^E = Expected noninvestment expenses during valuation year t. This item is included within $(NC)_t$ for the unit credit, attained age and entry age normal cost methods, but is part of $(AL)_t^A$ for the aggregate cost and frozen initial liability methods.

E_t^A = Actual noninvestment expenses during valuation year t.

I_t^E = Expected interest earnings on the pension fund during year t, accumulated to the year end and net of investment expenses.

I_t^A = Actual interest earnings on the fund during valuation year t, accumulated to the year end and net of investment expenses.

C_t^E = Expected net realized and unrealized capital gains and losses during valuation year t if assets are valued at market; expected net realized capital gains and losses during the year, if assets are valued at cost.

C_t^A = Actual net realized and unrealized capital gains and losses during valuation year t if assets are valued at market; actual net realized capital gains and losses during the year, if assets are valued at cost.

$(AL)_{t+1}^E$ = Expected accrued liability at end of valuation year t, if all valuation assumptions are exactly realized.

$(AL)_{t+1}^A$ = Actual accrued liability at end of valuation year t, based upon active and retired members at that date, with same benefit design, funding method and valuation assumptions underlying the valuation at beginning of year t.

$(AL)_{t+1}^{A'}$ = Actual accrued liability at end of valuation year t, including any modification of benefit design, funding method or valuation assumptions.

$(AL)_{t+1}^S$ = Accrued liability at end of valuation year t if the population at the beginning of year t is static (*i.e.*, if all active and retired participants survive the year without change of status, if salary scale assumptions are exactly realized, and if the expected new entrants come into the eligible group).

$(AL)_{t+1}^V$ = Accrued liability at end of valuation year t if contingency V occurs during the year to entire population at the beginning of year t.

F_t = Fund at the beginning of any year t, valued on an initial cost, amortized cost, or some other book basis.

F_t' = Fund at the beginning of any year t, valued at market.

i = Valuation interest rate.

There may be more than one interest rate assumed in the valuation, *e.g.*, when funds attributable to active employees are invested by a corporate trustee, annuities are purchased from an insurance company when employees retire, and differences between the yield rate of the pension fund segment held by the trustee and that held by the insurance company justify a distinction in the actuarial valuation.

j = Assumed rate of capital appreciation during valuation year.

k = Fraction of the year between any date of payment and the valuation year end.

P_t^A = Actual employer and employee contributions during valuation year t.

P_t^E = Expected employer and employee contributions during valuation year t.

Elements of Gain

Each of the elements composing the total gain or loss during the valuation year is analyzed below,[3] with comments about practical problems of calculation and presentation to the client or policyholder. Numerical examples are included as Appendix A. It should be observed that all of the "expected" items can be computed at the beginning of the valuation year, as a by-product of the valuation then being made. The "actual" items are unknown until the end of the valuation year, but their calculation can be made part of the basic valuation procedure for the succeeding year.

(1) Gain from expenses

$$= E_t^E - E_t^A,$$

where E_t^E might be zero (if administrative expenses are not paid out of the trust fund) or a percentage of contribution or a percentage plus a dollar amount. Under a Deposit Administration group annuity contract this item typically would be a percentage of contributions plus the contract administration charge.

and E_t^A is a cash accounting item furnished by the trustee or the retirement plan committee, if the retirement plan is self-administered, or the total expenses charged against the fund under an IPG contract. For

[3] The effect of each dependent variable has been isolated arbitrarily. Other divisions are equally defensible. For example, if we have a_0, b_0 at the beginning of the year and a_1, b_1 at the year end, the effect of a and b can be computed as $f(a_1, b_0) - f(a_0, b_0)$ and $f(a_1, b_1) - f(a_1, b_0)$, respectively. The allocation between a and b might equally well be $f(a_1, b_1) - f(a_0, b_1)$ and $f(a_0, b_1) - f(a_0, b_0)$, respectively. See Hellman for an expanded treatment of this point.

Deposit Administration plans, this item should theoretically be adjusted by a fraction of any dividend paid during the valuation year.

(2) Gain from interest earnings

$$= I_t^A - I_t^E,$$

where I_t^A is an accounting item equal, for self-administered plans, to the gross income reported by the trustee of the fund, adjusted for due and accrued income and reduced by the investment expenses. Under an IPG contract, it is the investment income credited by the insurer; under a Deposit Administration group annuity contract, it is the sum of contractually guaranteed interest credits to the Deposit Administration fund and that portion of any dividend which is derived from excess interest earnings.

If dividends have been anticipated by introduction of a valuation interest rate higher than that incorporated in the annuity purchase rates or guaranteed on the Deposit Administration fund, there may be merit to the inclusion of the entirety of any dividends in I_t^A. However, the interpretation of these results to the policyholder should point out that dividends represent the cumulative effect of experience and are not simply the result of one year's operation of the pension fund.

and $$I_t^E = F_t'i + P_t^E[(1+i)^k - 1] - B_t^E[(1+i)^k - 1] - E_t^E[(1+i)^k - 1]$$

P_t^E suggests a scheduled amortization of unfunded accrued liabilities or fixed contributions and presumes prior knowledge of the amount and date of employer contributions for the year. It has three elements: employer normal cost contribution, employer contribution to amortize unfunded accrued liability, and employee contributions. It is simpler to wait until the year end and define $P_t^E = P_t^A$, rather than speculate about P_t^E; this has practical appeal and, furthermore, avoids some complicated interest adjustments when the total gain or loss is equated to the sum of its elements. (See section entitled "Equation of Aggregate Gain with Net Gain from All Sources.")

The fractional interest calculations may be based on simple interest or, if size and distribution justify, k may be taken as $\frac{1}{2}$.

Excepting small funds, the gain from interest earnings will generally be more meaningful to the IPG policyholder or the client with a self-administered pension fund if presented as:

$$100\left[\frac{2I_t^A}{F_t+F_{t+1}-I_t^A}-i\right]\% \quad \text{or} \quad 100\left[\frac{2I_t^A}{F'_t+F'_{t+1}-I_t^A}-i\right]\%.$$

For a Deposit Administration plan the analogous device for presenting the gain from interest earnings is:

$$100\left[\frac{2I_t^A}{V'_t+V'_{t+1}-I_t^A}-i\right]\%,$$

where V'_t is the total of actuarial and contingency reserves on active and retired employees at the beginning of year t, or

$$100\,(i'-i)\%,$$

where i' is the credited interest rate provided by the current year's dividend formula.

(3) Gain from appreciation of fund

$= C_t^A - C_t^E$,

where $C_t^A = F'_{t+1} - F'_t - P_t^A + B_t^A + E_t^A - I_t^A$, if assets are valued at market, or $F_{t+1} - F_t - P_t^A + B_t^A + E_t^A - I_t^A$, if assets are valued at cost.

This element of gain will probably be more properly related to the mean assets, in the same manner as the gain from interest earnings described above.

Even though the pension fund is valued on a book basis and only realized capital gains are reflected directly in the gain and loss analysis, it may be informative to the client if a subsidiary calculation identifying the unrealized gains is included in the valuation report. The net unrealized gains and losses are equal to

$$[F'_{t+1} - F_{t+1}] - [F'_t - F_t]$$

and C_t^E depends on the method, if any, used to discount capital gains. Capital gains on equity investments can be discounted by modifying the valuation interest rate. For example, if a trustee intends to invest 25% of the assets in common stocks and an average annual appreciation of 2% is deemed reasonable, the valuation interest rate could be increased ½ of 1% in anticipation of these capital gains. Since this assumption implies increases in productivity and/or inflation and, consequently, rising salary levels, it is imperative that it be coupled with an adequate salary scale if liabilities are to be properly estimated. In consideration of this relationship it is more common that no allowance for capital gains is included in the valuation and a somewhat flatter salary scale is adopted. However, if the technique is employed,

$$C_t^E = F_t'(j) + P_t^E[(1+j)^k - 1] - B_t^E[(1+j)^k - 1] - E_t^E[(1+j)^k - 1],$$

on a market basis, or

$$F_t(j) + P_t^E[(1+j)^k - 1] - B_t^E[(1+j)^k - 1] - E_t^E[(1+j)^k - 1],$$

on a book basis.

This formula presumes no capital appreciation on investment earnings during the year. There is a simpler method of computing C_t^E since it is closely approximated by

$$\frac{j}{i} I_t^E .$$

(4) Gain from deaths prior to retirement, gain from withdrawals, gain from disabilities and gain from retirements
$= R_t^{Av} - B_t^{Av} - R_t^{Ev} + B_t^{Ev}$, where v is the contingency under analysis.

R_t^{Av} is calculated with respect to each participant under the plan whose status has changed because of contingency v during the year and equals the excess of the accrued liability at the end of year t, if the status had not changed, over that on the new status.

B_t^{Av} is an accrual accounting item supplied by the retirement plan administrator, the trustee or the insurance company.

R_t^{Ev} is calculated either as $[(AL)_{t+1}^{S} - (AL)_{t+1}^{V}]q_y^v$ or as

$$(AL)_{t+1}^{S} - [(AL)_t^A + (NC)_t](1+i) + B_t^E(1+i)^k + E_t^E(1+i)^k$$

depending upon the number of decrements and the complexity of benefits.

R_t^{Ev} could be computed as of the assumed date of decrement and increased by interest at the valuation rate for the remainder of the year; a parallel revision of R_t^{Av} would be required. The suggested definition is easier to calculate and either definition should give approximately the same value to the aggregate quantity $[R_t^{Av} - R_t^{Ev}]$.

B_t^{Ev} is $B_t^v \times q_y^v$ for refund and severance benefits and $B_t^v \times p_y^v$ for survivor benefits, where B_t^v is the benefit payable if the contingency v materializes during the valuation year.

The gain from nonvested withdrawals and disabilities which occur before eligibility for annuity benefits may be combined with the gain from deaths prior to retirement.

If postponements of retirement have a significant effect on the actuarial liabilities of the retirement plan, it may be pertinent to separate the gain or loss due to election of retirement and the gain or loss due to deaths after retirement.

The gain from each of these elements will often be more pointedly communicated to the client or policyholder if related to the effect on the contribution level. For instance, if the normal cost is presented as a percentage of eligible compensation and a frozen initial liability or aggregate cost funding method is used, the actuarial gain or loss from any element can be divided by the present value of 1% of future eligible compensation. It is not theoretically correct that $(NC)_t$, as a percentage of eligible compensation at time t, plus each element of gain or loss (related to the eligible compensation at time $t+1$) equals $(NC)_{t+1}$ as a percentage of the latter base, but the relation holds approximately and it will indicate the general influence of each valuation assumption.

(5) Gain from salary changes

$$= S_t^E - S_t^A.$$

The formulation of S_t^E and S_t^A is complicated when benefits are integrated with or offset by OASDI benefits, when an n year average salary determines the benefit and the participant is within the n year range; or when past service and future service benefits are not determined by the same formula. In the simple case where the benefit credit is a constant percentage of salary during the last n years of service and the participant is more than n years from retirement,

$$S_t^E = (AL)_{t+1}^E \cdot \frac{s_{y+1} - s_y}{s_{y+1}}$$

$$S_t^A = (AL)_{t+1}^S \cdot \frac{s_y}{s_{y+1}} \cdot \frac{(AS)_{y+1} - (AS)_y}{(AS)_y},$$

where

y = Valuation age at end of valuation year t

$(AS)_y$ = Annual salary rate for (y) at end of valuation year t

s_y = Salary scale factor for (y) at end of valuation year t.

During the n years before retirement, the benefit can be divided between a fixed benefit, based on earnings since the commencement of the n year period, and a prospective benefit, based on the estimated future earnings. Only this prospective benefit is affected by salary changes after age y. A plan with past service benefits determined from some known earnings base would present similar problems of analysis.

For plans with different benefit credits above and below some salary level, such as the maximum wage level upon which OASDI taxes are paid, S_t^E and S_t^A also must be modified for those employees whose salaries have crossed or are expected to cross this breaking point during the year. It would probably be simplest to consider the benefits of these plans as a "total salary" benefit less an "OASDI tax base" benefit; in which case the analysis reduces, as above, to consideration of a "prospective benefit" and, for employees over the breaking point, a "fixed" benefit.

This gain or loss from salary change can be related to the normal cost by the same device outlined above for gains from death, withdrawal, disability and retirement.

(6) Gain from new entrants

$= N_t^E - N_t^A,$

where N_t^E is generally ignored or taken as zero. However, if the employer wishes the total contribution, including the normal cost and a contribution toward the unfunded accrued liability, to remain a level percentage of eligible compensation or if the age distribution of new entrants is likely to change significantly in the future or, in a plan with fixed or legislated contributions, if the covered group is expected to enlarge, the valuation may include an estimate of the liabilities for new entrants, their future salaries and future contributions on their behalf.

In practice there is often an implicit assumption about new entrants. For example, if the normal cost is computed according to the aggregate cost funding method as a percentage of eligible compensation at the beginning of any year t and this percentage is applied monthly to all eligible compensation (including compensation of new entrants into the eligible group), the contributions related to salaries of new entrants are probably greater than the value of benefit credits earned during the year *unless* the age distribution of new entrants resembles the attained age distribution of active participants at the beginning of the year. In practice this is unlikely and the resulting excess contributions result in a gain.

N_t^A is $(AL)_{t+1}^A$ for participants at the end of year t who were not included in the previous valuation.

(7) Gain from miscellaneous errors

$= \Delta_t$

This item arises when there are errors in the reporting of census data, such as misstatements of ages or incorrect benefits, or errors introduced because of valuation techniques, such as grouping of attained ages, use of a single average entry age, adjustments of the valuation entry age to account for breaks in credited service, etc. Also included in this element would be the combined effect of approximations used to calculate the expected or actual figures composing the various elements of gain.

(8) Gain from change in benefits, funding methods or valuation assumptions

$= (AL)_{t+1}^A - (AL)_{t+1}^{A'}.$

This element of gain is derived by duplicate valuation of all survivors to the end of valuation year t, incorporating the revised benefits, computed according to two funding methods or reflecting two sets of valuation assumptions, as the case may be. It may be desirable, if several elements of the valuation basis are shifted, to isolate the effect of one or more of the revised assumptions.

(9) Gain from change in asset valuation method

$= F^2_{t+1} - F^1_{t+1}$.

If valuation is changed from book to market, Basis 2 is market and Basis 1 is book; if the change is from market to book, vice versa.

Analogy with the Fackler Formulas

The traditional Fackler accumulation formulas offer a useful test of the logic supporting any element of the actuarial gain or loss. This comparison gives one a refreshing reminder that the various branches of actuarial practice stem from the same discipline.

The fundamental Fackler accumulation formula is

$$V_x l_x + P l_x + (V_x l_x + P l_x) i - d_x = V_{x+1} l_{x+1} \tag{1}$$

or, in words,

Reserves at age x *plus* Premiums due at age x *plus* One year's interest on premium and reserves at the valuation rate *less* Expected death claims (paid at year end) *equals* Expected reserves at year end.

But, since the experience rates vary from valuation assumptions, the annual accumulation is

$$V_x l_x + P l_x + (V_x l_x + P l_x) i' - d'_x = V'_{x+1} l'_{x+1} + \text{Gain}\,, \tag{2}$$

where i', d'_x and l'_{x+1} are the experience results and V'_{x+1} is the unit reserve on the valuation basis adopted for the year end. By subtraction,(2) − (1),

$$\text{Gain} = (V_x l_x + P l_x)(i' - i) - (d'_x - d_x) + V_{x+1} l_{x+1} - V'_{x+1} l'_{x+1}\,. \tag{3}$$

But $l_{x+1} = l_x - d_x = l'_{x+1} + d'_x - d_x$ and therefore,

$$V_{x+1} l_{x+1} - V'_{x+1} l'_{x+1} = V_{x+1}(d'_x - d_x) + (V_{x+1} - V'_{x+1})\, l'_{x+1}\,. \tag{4}$$

By substitution,

$$\text{Gain} = (V_x l_x + P l_x)(i' - i) - (d'_x - d_x) + V_{x+1}(d'_x - d_x) + (V_{x+1} - V'_{x+1}) l'_{x+1}\,; \tag{5}$$

which is to say,

Gain = Excess interest earnings

less

Benefit payments in excess of those expected

plus

The difference between actual release of reserves and expected release of reserves, using the former valuation basis,

plus

Reduction in reserves on survivors due to change of valuation basis.

Now, from (1),

$$V_x l_x(1+i) + P l_x(1+i) - d_x = V_{x+1}(l_x - d_x)\ ,$$

and by transposition

$$V_{x+1} d_x = V_{x+1} l_x - [V_x l_x + P l_x + (V_x l_x + P l_x) i - d_x] \qquad (6)$$

or, in words,

Expected release of reserve = The reserve if all survive

less

The sum of initial reserves and premiums increased with interest and reduced by expected benefit payments (or, expected reserves at the year end).

These fundamental equations have analogies in all retirement plan funding methods and the gain or loss attributable to each actuarial assumption can be computed and examined by direct reference to them. Some of these analogies are set out in the following paragraphs.

If there were no actuarial gains or losses, the progressions of retirement plan accrued liability would be (cf. Equation [1])

$$(AL)_t^A + (NC)_t + [(AL)_t^A + (NC)_t]i - B_t^E(1+i)^k - E_t^E(1+i)^k = (AL)_{t+1}^E \qquad (7)$$

r, in words,

Accrued liability at beginning of valuation year t

plus

Normal cost assumed due during valuation year t

plus

Interest at the valuation interest rate on $(AL)_t^A$, $(NC)_t$ and all transactions, accumulated to end of valuation year t,

less

Expected benefit payments during valuation year t

less

Expected expenses during valuation year t

equals

Expected accrued liability at end of valuation year t.

However, experience departs from the assumptions and, from Equation (2),

$$(AL)_t^A + (NC)_t + [(AL)_t^A + (NC)_t]i' - B_t^A(1+i')^k - E_t^A(1+i')^k = (AL)_{t+1}^{A'} + \text{Gain}\,. \tag{8}$$

This Gain, by extension of Equation (5), is

$$\begin{aligned}\text{Gain} = (I_t^A - I_t^E) + (C_t^A - C_t^E) + (R_t^A - R_t^E) + (S_t^E - S_{t+1}^A) \\ + (N_{t+1}^E - N_t^A) + (E_t^E - E_t^A) + (B_t^E - B_t^A) \\ + \Delta_t + [(AL)_{t+1}^A - (AL)_{t+1}^{A'}] + (F_{t+1}^2 - F_{t+1}^1)\end{aligned} \tag{9}$$

or, in words,

Gain is the sum of

a) the differences between actual and expected results for those elements which reduce the accrued liability, such as interest earnings, capital gains and changes of status within the covered group,

b) the differences between expected and actual results for those elements which increase the accrued liability, such as salary increases, new entrants, expenses and benefit payments, and

c) changes in accrued liability due to errors, shifts in valuation basis, funding method or benefit formula and shifts in the asset valuation method.

Another useful relationship, derived from Equation (6), is

$$R_t^E = (AL)_{t+1}^S - \{(AL)_t^A + (NC)_t + [(AL)_t^A + (NC)_t]i - B_t^E\,(1+i)^k - E_t^E\,(1+i)^k\} \tag{10}$$

or, in words,

Expected release of accrued liability due to changes of status during valuation year t

equals

Accrued liability if all participants at beginning of valuation year t survive without change of status, expected salary changes occur and new entrants as predicted enter the covered group

less

The algebraic equivalent of expected accrued liability at the end of valuation year t, *i.e.*, $(AL)_{t+1}^E$. (See Equation [7].)

Another pertinent relationship is

$$V_{x+1}l'_{x+1} = V_{x+1}l_x - V_{x+1}d'_x$$

or, in words,
Actual reserve at year end
equals
Reserve if all survive
less
Actual release of reserve.

From this identity it follows that

$$(AL)^A_{t+1} = (AL)^S_{t+1} - R^A_t + [S^A_t - S^E_t] + [N^A_t - N^E_t] - \Delta_t \,. \quad (12)$$

Equation of Aggregat Gain with Net Gain from All Sources

Appendix A demonstrates the numerical equivalence between the total actuarial gain or loss and the sum of its elements, as defined in the body of the paper, for two hypothetical employee groups. By analogy with the Fackler formulas, the total gain defined in the Internal Revenue Service Bulletin on Section 23(p) of the 1939 Internal Revenue Code can be equated generally with the sum of the individual gain and loss elements. The necessary sequence of equations is outlined below.

1. Gain, according to the principles underlying the IRS Bulletin on Section 23(p) of the 1939 Code and assuming a shift from market to book value of assets,

$$= [(AL)^A_t - F'_t + (NC)_t](1+i) - P^A_t\,(1+i)^k - C^E_t$$
$$- [(AL)^{A'}_{t+1} - F_{t+1}].$$

2. Gain, according to Equation (9) in the preceding section,

$$= (I^A_t - I^E_t) + (C^A_t - C^E_t) + (R^A_t - R^E_t) + (S^E_t - S^A_t)$$
$$+ (N^E_t - N^A_t) + (E^E_t - E^A_t) + (B^E_t - B^A_t) + \Delta_t$$
$$+ [(AL)^A_{t+1} - (AL)^{A'}_{t+1}] + (F^2_{t+1} - F^1_{t+1}) \,.$$

3. $F'_t = F'_{t+1} - P^A_t - [B^A_t - E^A_t + I^A_t + C^A_t]$, by general reasoning.

4. $F'_{t+1} = F^1_{t+1}$ and $F_{t+1} = F^2_{t+1}$, by definition.

5. $0 \equiv (AL)^A_{t+1} - (AL)^A_{t+1}$.

6. $(AL)^A_{t+1} \equiv (AL)^S_{t+1} - R^A_t - [S^E_t - S^A_t] - N^E_t - N^A_t] - \Delta_t$ from Equation (12).

7. $(AL)^S_{t+1} \equiv R^E_t + (AL)^A_t + (NC)_t + [(AL)^A_t + (NC)_t]i$
$- B^E_t\,(1+i)^k - E^E_t(1+i)^k$, from Equation (10).

8. $F'_t i \equiv I^E_t - P^A_t[(1+i)^k - 1] + B^E_t[(1+i)^k - 1] + E^E_t[(1+i)^k - 1]$, from (2) under "Elements of Gain."

9. Gain, by successive substitution from Equations 3, 4, 5, 6, 7, and 8 into Equation 1

$$= (I^A_t - I^E_t) + (C^A_t - C^E_t) + (R^A_t - R^E_t) + (S^E_t - S^A_t)$$

$$+ (N^E_t - N^A_t) + (E^E_t - E^A_t) + (B^E_t - B^A_t) + \Delta_t$$

$$+ (AL)^A_{t+1} - (AL)^{A'}_{t+1} + (F^2_{t+1} - F^1_{t+1})$$

= Gain, according to Equation 2 above.

APPENDIX A

NUMERICAL ILLUSTRATIONS

1. *Introduction*

The principles of gain and loss analysis were applied to two small hypothetical groups, one with benefits funded through a self-administered pension trust and another with benefits funded through a Deposit Administration group annuity contract. The results of those analyses are set forth in the following sections.

It will be observed that the self-administered example combines the entry age normal cost funding method and a salary scale, while the Deposit Administration example provides for the unit credit normal cost funding method and no salary scale. This was done in an effort to illustrate a variety of problems in compact form and does not imply that one set of techniques is used by self-administered plans and the other by insured plans.

Notation is consistent with the Society Study notes for Part 8.

2. *Example 1:*

A. *Summary of Characteristics*

1) Funding medium—a self-administered pension trust
2) Effective date—July 1, 1950
3) Valuation date—July 1
4) Benefits and eligibility requirements
   a) No employee contributions

b) Normal retirement—at age 65. Postponement of retirement at the option of the employee. Normal and postponed retirement benefit—1% of average salary in 10 years prior to age 65 for each year of service, payable 1/12 monthly for life.

c) Early retirement—at the employee's option after attainment of age 55 and completion of 15 years of service. Early retirement benefit—the actuarial equivalent of a normal retirement benefit based on service to the date of retirement and average earnings in the 10 immediately prior years.

d) No death benefits and no disability and withdrawal benefits other than early retirement benefits.

B. *Statement of Actuarial Assumptions and Valuation Methods*

1) Funding method—entry age normal cost
   a) With immediate recognition of actuarial gains and losses, and
   b) With gains spread by frozen initial liability method
2) Asset valuation method—amortized cost for bonds and preferred stocks; market for common stocks.
3) Interest rate—3%; capital gains rate—1/2%; total investment yield rate, net of investment expenses—3 1/2%.
4) Administrative expenses—$500 a year
5) New entrants—one per year at age 25 and salary $3,600 $(1.02)^{t-1950}$, where t = calendar year of employment.
6) Service table. (Copies of commutation functions are available upon request to the author.)
   a) Sources
      (1) Mortality rates—1951 Group Annuity Experience Table, projected by Scale C for year of birth 1917. For males, "experience" rates in 1951 are 10/9 of basic rates in the 1951 Group Annuity Table.[1]
      (2) Withdrawal rates—based on historical study of reserve released upon employment terminations among clerical and management employees.
      (3) Salary scale—based upon examination of secular trend among bank employees. Combines allowance for merit and seniority increases with provision for increases due to inflation.
      (4) Retirement rate—arbitrarily selected. Approximately equivalent to retirement at average age 67.
   b) Rates and values

[1] *TSA* IV, 279.

TABLE 1

X	$1,000q_x^{ad}$	$1,000q_x^{aw}$	$\frac{s_{x+1}}{s_x}$
20.......	.684	180.0	1.0636
25.......	.842	88.0	1.0688
30.......	1.101	50.0	1.0613
35.......	1.508	23.0	1.0538
40.......	2.061	9.0	1.0468
45.......	3.463	1.0	1.0409
50.......	5.883	0.0	1.0341
55.......	8.903	0.0	1.0274
60.......	12.462	0.0	1.0208
64.......	16.812	0.0	1.0193

TABLE 2

X	t	$1,000q_x^{ad}$	$1,000q_{[65]+t}^{ar}$	$\frac{s_{x+1}}{s_x}$	n	$v^{1/2}\ddot{a}_{x+n}^{(12)}$	
						$i=3.25\%$	$i=3.5\%$
65.........	0	18.370	400	1.000	1/2	11.5335	11.2808
66.........	1	20.202	250	1.000	1/2	11.1301	10.8928
67.........	2	22.091	300	1.000	1/2	10.7265	10.5037
68.........	3	23.898	350	1.000	1/2	10.3212	10.1125
69.........	4	25.714	450	1.000	1/2	9.9121	9.7128
70.........	5	27.767	1000	.000	0	9.8628	9.6839

t = duration since first eligible for normal retirement.
n = average time of retirement within duration t.

TABLE 3

X	$1,000q_x$	$\ddot{a}_x^{(12)}$	
		$i=3.25\%$	$i=3.5\%$
70...........	27.767	9.8628	9.6829
75...........	45.938	7.7784	7.6524
80...........	81.408	5.8602	5.7649
85...........	137.644	4.3295	4.2272
90...........	222.882	3.3127	3.0930

C. *Hypothetical Employee Data*

1) Active employees (all males)

Employee Number	Entry Age X	Attained 7/1/59 y	$(AS)_y$	Pension Base Earnings Prior to 7/1/59
A1.....	25	25	\$ 3,500	....
A2.....	25	30	4,000	....
A3.....	25	35	4,200	....
A4.....	35	40	5,000	....
A5.....	30	50	7,000	....
A6.....	45	60	10,000	\$47,000
A7.....	30	64	4,000	32,000
A8.....	40	68	8,000	76,000

2) Retired employees

Retired Member	Attained Age 7/1/59 y	Monthly Benefit 7/1/59
R1.........	70	\$175.00
R2.........	75	200.00

D. *Valuation on July 1, 1959*

1) Active employees

Employee Number	y	$(AS)_y$	Estimated Annual Retirement Benefit at 65 or y, if Greater (1)	P. V. of Ret. Benefit (2)	$(NC)_{x,y}$ (3)	$\frac{^sN_y}{^sD_y}$ (4)	P. V. of $(NC)_{x,y}$ (3)×(4) (5)	$(AL)_y^A$ (2)−(5) (6)
A1...	25	\$ 3,500	\$ 6,091	\$ 6,471	\$ 224	\$28.94	\$ 6,471	0
A2...	30	4,000	5,174	9,486	256	31.36	8,013	\$ 1,473
A3...	35	4,200	4,158	11,096	268	29.63	7,949	3,147
A4...	40	5,000	2,923	10,315	340	25.24	8,593	1,722
A5...	50	7,000	3,211	16,827	469	15.38	7,180	9,647
A6...	60	10,000	1,982	16,720	708	67.86	4,806	11,914
A7...	64	4,000	1,260	12,331	264	32.99	871	11,460
A8...	68	8,000	2,128	20,597	553	16.05	887	19,710
		\$45,700	\$26,927	\$103,843	\$3,080	....	\$44,770	\$59,073

$$.01\,(AS)_y \cdot \frac{^SN_y}{^SD_y} = \text{Present Value of 1\% of Future Compensation}$$

$$= 6{,}789$$

2) Retired employees

Retirement Number	y (1)	Monthly Benefit (2)	$\ddot{a}_y^{(12)}$ (3)	$(AL)_y^A$ $12\times(2)\times(3)$ (4)
R1.......	70	$175.00	9.683	$20,334
R2.......	75	200.00	7.652	18,365
		$375.00	....	$38,699

3) Future entrants

a) Present Value of Benefits

$$= \sum_{t=1960}^{\infty} v_{.035}^{t-1959} (AS)_t \cdot \frac{.01\,{}_{10}s_{65}}{{}^{s}D_{25}} (40M_{65}^{ra} + \bar{R}_{65}^{ra} - \bar{R}_{70}^{ra})$$

$$= .01 \times 3{,}600\, v_{.035} (1.02)^{10} \cdot \frac{{}_{10}s_{65}}{{}^{s}D_{25}} (40M_{65}^{ra} + \bar{R}_{65}^{ra} - \bar{R}_{70}^{ra}) \times \sum_{t=0}^{\infty} v_{.035}^{t} (1.02)^{t}$$

$$= 36 \times 1.1778 \times 184.87 \times 69$$

$$= 540{,}870$$

b) Present Value of Normal Cost
= Present Value of Benefits
= 540,870

c) Present Value of Liability for Future Entrants
= (Present Value of Benefits) − (Present Value of Normal Cost)
= 0

d) Present Value of 1% of Future Entrants' Compensation

$$= \sum_{t=1960}^{\infty} .01\, v_{.035}^{t-1959} (AS)_t \cdot \frac{{}^{s}N_{25}}{{}^{s}D_{25}}$$

$$= .01 \times 3{,}600\, v_{.035} (1.02)^{10} \cdot \frac{{}^{s}N_{25}}{{}^{s}D_{25}} \sum_{t=0}^{\infty} v_{.035}^{t} (1.02)^{t}$$

$$= 36 \times 1.1778 \times 28.938 \times 69$$

$$= 84{,}663$$

4) Present Value of Future Expenses

$$= 500 \sum_{t=0}^{\infty} v^{t}_{.035}$$

$$= 14{,}786$$

5) Normal Cost, $(NC)_{59}$

a) Immediate Recognition of Gains

$$(NC)_{59} = 500 + \Sigma(NC)_{x,\,y}$$

$$= 3{,}580$$

$$= 7.83\% \text{ of } \Sigma(AS)_{y}$$

b) Frozen Initial Liability Method

Assume:

1. Assets = $78,400
2. Frozen Initial Liability = $85,000
3. Unfunded Frozen Initial Liability, 7/1/59 = $25,000

Then

$$(NC)_{59} = \frac{\text{P.V. of Benefits} - \text{Assets} - \text{Unfunded F.I.L., } 7/1/59}{\text{P.V. of } 1\% \text{ of Future Compensation}} \times \Sigma(AS)_{y} + 500$$

$$= 3{,}398$$

$$= 7.44\% \text{ of } \Sigma(AS)_{y}$$

6) If no allowance for new entrants is included in the set of actuarial assumptions, $(NC)_{59}$ would be $3,135, 6.86% of payroll.

E. *Expected Release of Reserve,* R_{59}^E, *and Expected Benefits,* B_{59}^E

1) Active employees

EMPLOYEE NUMBER	y	$\frac{D_y}{D_{y+1}}$ (7)	$(AL)_{y+1}^S$ Frozen Initial Liab. (8)	$(AL)_{y+1}^S$ Immediate Recognition of Gains (8)	B^{Er} (9)	R_y^E (8)+(9)×$1.035^{1/2}$ −[(3)+(6)]1.035 Frozen Initial Liab. (10)	R_y^E Immediate Recognition of Gains (10)
A1......	25	1.1359	$ 7,350	$ 254	....	$ 653	$ 22
A2......	30	1.0907	10,346	1,886	....	528	96
A3......	35	1.0610	11,773	3,623	....	289	88
A4......	40	1.0466	10,796	2,158	....	120	23
A5......	50	1.0411	17,517	10,531	....	103	62
A6......	60	1.0481	13,524	13,229	....	219	165
A7......	64	1.0527	12,981	12,341	....	218	207
A8......	68	1.6531	21,153	20,601	386	228	22
		....	$109,440	$64,846	....	$2,358	$685

Notes:

1. $$(AL)_{y+1}^S = (\text{Col. } 2)\cdot\frac{D_y}{D_{y+1}} - [(\text{Col. } 5) - (\text{Col. } 3)]\cdot\frac{D_y}{D_{y+1}}, \qquad y<65$$

$$= \left[(\text{Col. } 2) - (\text{Col. } 1)\cdot\frac{y-x+1}{y-x}\cdot\frac{\bar{C}_y^{ra}}{D_y}\right]\cdot\frac{D_y}{D_{y+1}} - [(\text{Col. } 5) - (\text{Col. } 3)]\cdot\frac{D_y}{D_{y+1}}, \qquad y\le 65$$

2. $B_y^{Er} = 0, \qquad y<65;$

$$B_y^{Er} = \tfrac{1}{2}(\text{Col. } 1)\cdot\frac{y-x+1}{y-x}\cdot q_y^{ar}, \qquad 65\le y<70$$

3. B_y^{Ed} and B_y^{Ei} are zero; B_y^{Ew} is zero since $q_y^{aw}=0$ for $y\ge 55$

4. With the Frozen Initial Liability method of spreading gains and losses, set (Col. 3) = (Col. 5) = 0 and substitute (Col. 2) for (Col. 6).

Employee Number	y	q_y^{ad} (11)	q_y^{aw} (12)	q_y^{ar} (13)	$(AL)^r_{t+1}$ (14)	Immediate Recognition of Gains: R_y^{Ed} (8)×(11) (15)	R_y^{Ew} (8)×(12) (16)	R_y^{Er} [(8)−(14)]×(13) (17)
A1....	25	.000842	.088	......	....	0	22	....
A2....	30	.001101	.050	......	....	2	94	....
A3....	35	.001508	.023	......	....	5	83	....
A4....	40	.002061	.009	......	....	4	19	....
A5....	50	.005883	.000	......	....	62	....	....
A6....	60	.012462	.000	......	....	165	....	....
A7....	64	.016812	.000	......	....	207	....	....
A8....	68	.023898	.000	.35	21,944	492	....	−470
		....	....	......	....	937	218	−470

$$\Sigma R_y^E = \Sigma R_y^{Ed} + \Sigma R_y^{Ew} + \Sigma R_y^{Er}$$

$$\Sigma(9) = \Sigma(15) + \Sigma(16) + \Sigma(17)$$

$$685 = 937 + 218 - 470$$

$$= 685$$

Frozen Initial Liability

$$\Sigma R_y^{Ed} = 1{,}103;\ \Sigma R_y^{Ew} = 1{,}532;\ \Sigma R_y^{Er} = -277$$

2) Retired employees

Retirement Number	y	q_y (5)	B^{Er} 12×(2) ×[1−1/2 ×(5)] (6)	$\ddot{a}_{y+1}^{(12)}$ (7)	$(AL)^E_{t+1}$ 12×(2)×(7) (8)	R_y^E (5)×(8) (9)
R1.......	70	.027767	2,071	9.273	19,473	541
R2.......	75	.045938	2,345	7.256	17,414	800
		....	4,416	....	36,887	1,341

3) Total

$$B^E_{59} = \Sigma B^{Er}_y \text{ (active)} + \Sigma B^{Er}_y \text{ (retired)}$$
$$= 386 + 4{,}416$$
$$= 4{,}802$$

F. *Expected Expenses, E^E_{59}, Expected Interest, I^E_{59}, and Expected Capital Gains, C^E_{59}*

1) $E^E_{59} = 500$

2) I^E_{59} and C^E_{59} are not known unless P^E_{59} is known, and will be calculated at the year end with $P^E_{59} = P^A_{59}$

G. *Expected Liability Increase Due to New Entrants, N^E_{59}, and Salary Changes, S^E_{59}*

1) N^E_{59}

a) Immediate recognition of gains

$N^E_{59} = 0$

b) Frozen Initial Liability

$$N^E_{59} = .01 \times 3{,}600\,(1.02)^{10} \cdot \frac{10\,s_{65}}{{}^{s}D_{25}} \left(40 M^{ra}_{65} + \bar{R}^{ra}_{65} - \bar{R}^{ra}_{70}\right)$$
$$= 36 \times 1.219 \times 184.87 = 8{,}113$$

2) $$S^E_y \doteq \left[\frac{65-y}{10}\right]_{\not< 0,\ \not> 1} (AL)^E_y \cdot \frac{s_{y+1} - s_y}{s_{y+1}}$$

A precisely correct calculation would separate $(AL)^E_y$ between liability for benefits based on actual earnings from age 55 to age y and liability for benefits based on estimated earnings between ages y and 65 and compute S^E_y with reference to the latter expression.

Employee Number	$(AL)^E_{y+1}$ (8) − (10) Frozen Initial Liab.	Immediate Recognition of Gains	$\frac{s_y}{s_{y+1}}$	S^E_y $\left[\frac{65-y}{10}\right]_{\not>1} \times (18) \times \{1-(19)\}$	
	(18)		(19)	(20)	
1	$ 6,697	$ 232	.9356	431	15
2	9,818	1,790	.9423	566	103
3	11,484	3,535	.9489	587	181
4	10,676	2,135	.9553	477	95
5	17,416	10,469	.9670	575	345
6	17,305	13,064	.9797	175	133
7	12,763	12,134	.9811	23	23
8	20,918	20,579	1.0000	0	0
	$107,077	$63,938	....	2,834	895

H. *Valuation on July 1, 1960*

1) Changes during valuation year

a) Funding method—no change

b) Benefits and eligibility requirements—no change

c) Asset valuation method—change to cost basis of valuing common stocks

d) Actuarial assumptions—change to 3 1/4% net investment yield rate

e) Hypothetical employee data

(1) Active employees (all males)

Employee Number	Status 7/1/60	Attained Age 7/1/60 $y+1$	$(AS)_{y+1}$	Pension Base Earnings Prior to 7/1/60
A1........	Active	26	\$ 3,800	....
A2........	Quit	....	....	....
A3........	Active	36	4,300	....
A4........	Dead	....	....	....
A5........	Active	51	7,500	....
A6........	Active	61	11,000	\$58,500
A7........	Retired	....	....	36,000
A8........	Active	69	8,000	76,000
A9........	New	30	3,800	....
A10.......	New	45	4,680	....
A11.......	New	25	3,600	....

(2) Retired employees (all males)

Retirement Number	Status 7/1/60	Attained Age 7/1/60 $y+1$	$(AS)_{y+1}$
R1........	Retired	71	\$175.00
R2........	Dead	....	....
R3 (A7)...	Retired	65	102.00

2) Valuation

a) Active employees

Employee Number	$(AS)_{y+1}$	Estimated Annual Retirement Benefit at 65 or *y*, if Greater (1)	3 1/2% P. V. of Ret. Benefit (2)	$(NC)_{x,y+1}$ (3)	$\frac{^sN_{y+1}}{^sD_{y+1}}$ (4)	P. V. of $(NC)_{x,y+1}$ (3)×(4) (5)	$(AL)_{y+1}$ (2)−(5) (6)
A1	$ 3,800	$ 6,187	$ 7,466	$ 243	$29.69	$ 7,208	$ 258
A3	4,300	4,039	11,436	275	28.82	7,917	3,519
A5	7,500	3,327	18,152	500	14.47	7,241	10,911
A6	11,000	2,059	18,206	751	5.941	4,462	20,601
A8	8,000	2,204	21,155	554	1.000	554	0
A9	3,800	4,301	7,946	253	31.36	7,946	0
A10	4,600	1,453	6,557	324	20.45	6,557	0
A11	3,600	6,265	6,655	230	28.94	6,655	0
	$46,680	$29,835	$97,573	$3,130	....	$48,540	$49,033

Present Value of 1% of Future Compensation = 7,352.

Employee Number	3 1/4% P. V.′ of Ret. Benefit (7)	$(NC)'_{x,y+1}$ (8)	$\frac{^sN'_{y+1}}{^sD'_{y+1}}$ (9)	P. V.′ of $(NC)_{x,y+1}$ (8)×(9) (10)	$(AL)'_{y+1}$ (7)−(10) (11)
A1	$ 8,411	$262	$31.08	$ 8,133	$ 278
A3	12,575	296	29.85	8,839	3,736
A5	19,248	535	14.74	7,886	11,362
A6	18,846	789	5.981	4,716	14,130
A8	21,589	585	1.000	585	21,004
A9	8,865	271	32.70	8,865	0
A10	7,055	340	20.76	7,055	0
A11	7,515	248	30.31	7,515	0
	$104,104	$3,326		$53,594	$50,510

Present Value of 1% of Future Compensation = $7,596 *b*) Retired employees

Retirement Number	$y+1$ (1)	Monthly Retirement Benefit (2)	$\ddot{a}_{y+1}^{(12)}$ at .035 (3)	$\ddot{a}_{y+1}^{(12)}$ at .0325 (4)	$(AL)_{y+1}^{A}$ 12×(2)×(3) (5)	$(AL)_{y+1}^{A'}$ 12×(2)×(4) (6)
R1.....	71	$175.00	9.273	9.441	$19,473	$19,826
R2.....	65	102.00	11.674	11.925	14,289	14,596
		$277.00	....	....	$33,762	$34,422

c) Future entrants

		3 1/2%	3 1/4%
Present Value of Benefits	=	$551,687	$747,538
Present Value of 1% of Future Compensation	=	86,356	108,549

d) Future expenses

		3 1/2%	3 1/4%
Present Value of Future Expenses	=	$ 14,786	$ 15,885

I. *Actual Release of Reserve*, R_{59}^{A}

Employee Number	Cause v	Reserve on Original Status: Frozen Initial Liab. (1)	Reserve on Original Status: Immediate Recognition of Gains (1)	$(AL)_{y+1}^{v}$ (2)	R_y^A (1)−(2) (3) (a)	R_y^A (1)−(2) (3) (b)
A2........	*w*	$ 9,749	$ 1,777	0	$ 9,749	$ 1,777
A4........	*ad*	10,313	2,062	0	10,313	2,062
A7........	*r*	12,981	12,341	$14,289	− 1,308	− 1,948
R1........	*rd*	17,414	17,414	0	17,414	17,414

J. *Actual Change in Liability Due to Salary Changes, S_{59}^{A}*

$$S_y^A \doteq \overset{\not< 0,\ \not> 1}{\left[\frac{65-y}{10}\right]} \cdot \frac{s_y}{s_{y+1}} (AL)^S_{y+1} \cdot \frac{(AS)_{y+1} - (AS)_y}{(AS)_y}$$

Employee Number	$(AL)^S$ (Col. 8) 1959 (12) Frozen Initial Liab.	$(AL)^S$ (Col. 8) 1959 (12) Immediate Recognition of Gains	(Col. 19) 1959 (13)	$\frac{(AS)_{y+1}-(AS)_y}{(AS)_y}$ (14)	$\doteq S_y^A$ (12)×(13)×(14) $\times \overset{\not> 1}{\left[\frac{65-y}{10}\right]}$ (15) Frozen Initial Liab.	$\doteq S_y^A$ (15) Immediate Recognition of Gains
A1....	$ 7,350	$ 254	.9356	.0857	$ 589	$ 20
A3....	11,773	3,623	.9489	.0238	266	82
A5....	17,519	10,531	.9670	.0714	1,210	727
A6....	17,524	13,229	.9797	.1000	858	648
A8....	21,153	20,601	1.0000	.0000	0	0
	$75,319	$48,238	....	....	$2,923	$1,477

K. *Trustee's Reports*

1) Statement of Assets

	Old Valuation Basis	New Valuation Basis
July 1, 1959		
Bonds....................	$50,300	$50,300
Stocks....................	24,700	20,000
Cash....................	3,000	3,000
Accrued Income...........	400	400
Total..................	$78,400	$73,700
July 1, 1960		
Bonds....................	$52,800	$52,800
Stocks....................	28,000	23,000
Cash....................	3,100	3,100
Accrued Income...........	500	500
Total................	$84,400	$79,400

2) Operating Statement: 7/1/59–6/30/60

Cash, 7/1/59			\$ 3,000	
Contributions: 9/1/59	\$ 3,000			
1/1/60	3,500	\$6,500		
Proceeds of Sales		4,500		
Income		2,950	+13,950	\$ 16,950
Administrative Expenses		515		
Benefits		3,300		
Investment Expenses		410		
Purchase of Securities		9,590		
Accrued Interest on Purchases		35		−\$13,850
Cash, 6/30/60				\$ 3,100

L. *Calculation of* B^A_{59}, E^A_{59}, P^A_{59}, I^A_{59}, C^A_{59}, I^E_{59}, *and* C^E_{59}

1) B^A_{59} = \$3,300

2) E^A_{59} = 515

3) P^A_{59} = 6,500

4) I^A_{59}

Gross Income	\$ 2,950	
Accrued Interest, 7/1/60	+ 500	
	\$ 3,450	
Accrued Interest, 7/1/59	− 400	
	\$ 3,050	
Investment Expenses	− 410	
I^A_{59} =		2,640

5) C^A_{59}

a) F_{60}, common stocks at market	\$84,400	
F_{59}, common stocks at market	−78,400	\$6,000
B^A_{59}	3,300	
E^A_{59}	+ 515	+3,815
		9,815
P^A_{59}	6,500	
I^A_{59}	+ 2,640	−9,140
C^A_{59}		\$ 675

b) F_{60}, common stocks at market . . .	\$84,400	
F_{59}, common stocks at market . . .	−78,400	\$6,000
F_{60}, common stocks at cost.	79,400	
F_{59}, common stocks at cost.	−73,700	5,700
Net Unrealized Capital Gains.		\$ 300
c) C_{59}^{A}		675
Net Unrealized Capital Gains.		300
Net Realized Capital Gains.		\$ 375

6) I_{59}^{E}

a) $F_{59} \times .030$.		\$2,352
b) P_{59}^{A}		
(1) \$3,000 × .03 × 10/12.	\$ 75	
(2) \$3,500 × .03 × 6/15.	+ 53	+ 128
		\$2,480
c) $B_{59}^{E} \times .015$.	50	
d) $E_{59}^{E} \times .015$.	+ 8	− 58
e) I_{59}^{E}.		\$2,432

7) $E_{59}^{E} = \frac{.005}{.030} I_{59}^{E} = \$405$

M. *Calculation of Gain*

1) Gain from expenses

		Immediate Recognition of Gains	Frozen Initial Liab.
a) E_{59}^{E}.	\$ 500		
b) E_{59}^{A}.	− 515	\$−15	\$−15

2) Gain from interest earnings

a) I_{59}^{A}.	\$ 2,640		
b) I_{59}^{E}.	−2,432	\$208	\$208

3) Gain from appreciation of fund

a) C_{59}^{A}.	\$ 675		
b) C_{59}^{E}.	− 405	\$270	\$270

4) Gain from deaths and withdrawals before retirement

a) Immediate Recognition of gains

				Immediate Recognition of Gains	Frozen Initial Liab.
(1) R_{59}^{Ad}	$ 2,062				
(2) R_{59}^{Aw}	1,777	$ 3,839			
(3) $B_{59}^{Ad} = B_{59}^{Aw}$		0	$ 3,839		
(4) R_{59}^{Ed}	$ 937				
(5) R_{59}^{Ew}	218	$ 1,155			
(6) $B_{59}^{Ed} = B_{59}^{Ew}$		0	$ 1,155		
(7) Gains from deaths and withdrawals before retirement				$ 2,684	

b) Frozen Initial Liability method

				Immediate Recognition of Gains	Frozen Initial Liab.
(1) R_{59}^{Ad}	$10,313				
(2) R_{59}^{Aw}	9,749	$20,062			
(3) $B_{59}^{Ad} = B_{59}^{Aw}$		0	$20,062		
(4) R_{59}^{Ed}	$ 1,103				
(5) R_{59}^{Ew}	1,532	$ 2,635			
(6) $B_{59}^{Ed} = B_{59}^{Ew}$		0	$ 2,635		
(7) Gain from deaths and withdrawals before retirement					$17,427

5) Gain from retirements and deaths after retirement

a) Immediate recognition of gains

				Immediate Recognition of Gains	Frozen Initial Liab.
(1) R_{59}^{Ar}	$−1,948				
(2) R_{59}^{Ard}	17,414	$15,466			
(3) B_{59}^{Ar}	− 3,300		$12,166		
(4) R_{59}^{Er}	− 470				
(5) R_{59}^{Erd}	1,341	$ 871			
(6) B_{59}^{Er}		4,802	− 3,931		
(7) Gain from retirements and deaths after retirement				$16,097	

b) Frozen Initial Liability method

				Immediate Recognition of Gain	Frozen Initial Liab.
(1) R_{59}^{Ar}	$−1,308				
(2) R_{59}^{Ard}	17,414	$16,106			
(3) B_{59}^{Ar}		3,300	$12,806		
(4) R_{59}^{Er}	− 277				
(5) R_{59}^{Erd}	1,341	$ 1,064			
(6) B_{59}^{Er}		4,802	− 3,738		
(7) Gain from retirements and deaths after retirement					$ 16,544

6) Gain from Salary changes

a) Immediate Recognition of Gains

		Immediate Recognition of Gain	Frozen Initial Liab.
(1) S_{59}^{E}	$ 895		
(2) S_{59}^{A}	−1,477	$ − 582	

b) Frozen Initial Liability

		Immediate Recognition of Gain	Frozen Initial Liab.
(1) S_{59}^{E}	$ 2,834		
(2) S_{59}^{A}	2,923		$ −89

7) Gain from new entrants

		Immediate Recognition of Gain	Frozen Initial Liab.
a) N_{59}^{E}	$ 8,113		
b) N_{59}^{A}	21,158		$ −13,045

8) Gain from change of valuation interest rate

a) Immediate Recognition of Gains

		Immediate Recognition of Gain	Frozen Initial Liab.
(1) $(AL)_{60}^{A}$	$82,795		
(2) $(AL)_{60}^{A'}$	−84,932	$−2,137	

b) Frozen Initial Liability

		Immediate Recognition of Gain	Frozen Initial Liab.
(1) $(AL)_{60}^{A}$	$697,808		
(2) $(AL)_{60}^{A'}$	−901,949		$−204,141

9) Gain from change of asset valuation method

		Immediate Recognition of Gain	Frozen Initial Liab.
a) F_{60}	$ 79,400		
b) F'_{60}	−84,400	$−5,000	$ −5,000

		Immediate Recognition of Gain	Frozen Initial Liab.
10) Net gain		$ 11,525	$−187,841

N. *Check of Net Gain from Section M against Aggregate Gain from Application of Principles Underlying IRS Bulletin on Section 23(p) of the 1939 Internal Revenue Code*

1) 23(p) Gain

	Immediate Recognition of Gains	Frozen Initial Liab.
a) $(AL)^A_{59}$	$97,772	$ 698,198
b) F'_{59}	78,400	78,400
c) (*a*)–(*b*)	$19,372	$ 619,798
d) $(NC)_{59}$	$ 3,580	$ 64,142
e) P^A_{59}		
(1) $3,000 on 9/1/59		
(2) $3,500 on 1/1/60		
(3) Total	$ 6,500	$ 6,500
f) Interest at 3 1/2% on		
(1) (*c*)	$ 678	$ 21,693
(2) (*d*)	125	0
(3) (*e*)(1)	88	88
(4) (*e*)(2)	61	61
(5) (*c*) + (*d*) − [(*e*)(1) + (*e*)(2)]	654	21,544
g) C^E_{59}	405	405
h) $(AL)^{A'}_{60}$	84,932	901,949
i) F_{60}	79,400	79,400
j) (*h*) − (*i*)	$ 5,532	$ 822,549
k) (*c*) + (*d*) − (*e*)(3) + (*f*)(5) − (*g*) − (*j*) = Gain	$11,169	$−188,112

2) Reconciliation

a) (1)(*k*)	$11,169	−$188,112
b) Net gain from Section M	11,525	− 187,841
c) Δ_{59}	356	271

The absolute value of the gain elements, exclusive of the losses from changes in the asset valuation method and actuarial assumptions, is $19,856 for the immediate recognition of gains method and $47,598 for the frozen initial liability method. Δ_{59} is approximately 2% and 1/2%, respectively, of these amounts. It therefore is reasonable to accept the correctness of the valuation and allocate Δ_{59} arbitrarily among the individual gain elements.

An acceptable solution would be to split Δ_{59} proportionately between gain from deaths and withdrawals before retirement and gain from retirements and deaths after retirement. The revised items would be:

	Immediate Recognition of Gains	Frozen Initial Liab.
Gain from deaths and withdrawals before retirement	$ 2,684	$ 17,427
	− 56	− 137
	$ 2,628	$ 17,290
Gain from retirements and deaths after retirement	$16,097	$ 16,544
	− 300	− 134
	$15,797	$ 16,410

O. *Reporting the Analysis to the Client*

1) The gain, per the Bulletin on Section 23(p), would be computed.
2) The dollar amount of gain or loss from each valuation assumption would be tabulated.

	Immediate Recognition of Gains	Frozen Initial Liab.
(1) Loss from expenses	$ (15)	$ (15)
(2) Gain from interest earnings	208	208
(3) Gain from appreciation of pension fund assets	270	270
(4) Gain from deaths and withdrawals before retirement	2,628	17,290
(5) Gain from retirements and deaths after retirement	15,797	16,410
(6) Loss from salary changes	(582)	(89)
(7) Loss from new entrants	0	(13,045)
(8) Loss from change of valuation interest rate	$(2,137)	(204,141)
(9) Loss from change of asset valuation method	(5,000)	(5,000)
(10) Net gain (loss) for year	$11,169	$(188,112)

3) The source of each gain or loss would be discussed. (See Example 2 for elaboration of this point.)
4) It is meaningful to trace the change in the normal cost contribution rate for the frozen initial liability method. However, if the valuation basis has been changed the present value of future compensation may change sharply and distort a comparison

with the previous years' contribution rate. That is the result with the example being considered here. To illustrate the technique the change of valuation interest rate was ignored; the resulting normal cost contribution rate is analyzed below.

	As Percentage of Payroll
a) Normal Cost Contribution Rate for 1959	7.44%
b) Normal Cost Contribution Rate for 1960 (Assuming no change of valuation basis)	7.28
c) Decrease of Contribution Rate	.16%
d) Effect of Actuarial Gains and Losses (Dollar amount of gain and loss divided by Present Value of 1% of Future Compensation)	
(1) Loss from expenses	.00%
(2) Gain from interest	− .00
(3) Gain from fund appreciation	− .00
(4) Gain from deaths and withdrawals before retirement	− .18
(5) Gain from retirements and deaths after retirement	− .17
(6) Loss from salary changes	.00
(7) Loss from new entrants	.14
(8) Loss from change of asset valuation	.05
(9) Total Effect	− .16%

5) The effect of the change in valuation interest rate would be illustrated.

Valuation Interest Rate	Normal Cost Contribution Rate
3 1/2%	7.28% of Payroll
3 1/4%	7.83% of Payroll
Increase	.55% of Payroll

6) The experience interest and capital gains rates would be presented.

a) Net Investment Income		$ 2,640
Assets, 7/1/59 (market)		78,400
Assets, 7/1/60 (market)		84,400
Mean Assets		81,400
Yield Rate		3.30%
b) Capital Gains		$ 675
Realized	$375	
Unrealized	300	
Capital Gains Rate		.83%

7) Expenses would be related to an index understandable to the client, such as current contributions.

3. *Example 2:*

A. *Summary of Characteristics*

1) Funding medium—a Deposit Administration Group Annuity Contract
2) Effective Date—July 1, 1950
3) Valuation Date—July 1
4) Benefits and Eligibility Requirements
   a) Employee contributions—3% of salary.
   b) Normal retirement date—Age 65. No postponement of retirement. Normal retirement benefit—1 1/4% of career average salary for each year of service, payable 1/12 monthly for life.
   c) Early retirement benefits—available at employee's option after attainment of age 55 and completion of 15 years of service. Benefit—the actuarial equivalent of a normal retirement benefit based on service and average earnings to the date of retirement.
   d) Death benefits
      (1) Before normal retirement date—return of contributions with 3% interest.
      (2) After normal retirement date—pension guaranteed for 120 months.
   e) Disability retirement benefit—upon completion of 15 years of service and total and permanent disability persisting for six months. Benefit—an early retirement benefit without actuarial reduction. Disability benefits and 5% expense charge by insurance company paid out of Deposit Administration fund until normal retirement date.
   f) Withdrawal benefits—same as death benefit before qualification for early retirement benefits.

B. *Statement of Actuarial Assumptions and Valuation Methods*

1) Funding method—Unit Credit Normal Cost.
2) Asset valuation method—book value determined by insurance company.
3) Investment yield rate, net of investment expenses—3 1/4%.
4) Administrative expenses—$500 a year.
5) New entrants—none.
6) Service table (copies of commutation functions are available upon request to the author)

a) Sources

(1) Mortality rates—1951 Group Annuity Experience Table, projected by Scale C for year of birth 1917.

(2) Withdrawal rates—based on historical study of reserve released upon employment terminations among hourly paid employees.

(3) Salary scale—none.

(4) Disability rates—selected from Period 2 experience under Benefits 1 and 2, as reported in the *TSA* 1952 Reports, Table 2.

(5) Disability annuity values—interpolated from Period 2 experience under Benefit 5 with 3% interest, as reported in *TSA* 1952 Reports, Table 14.

(6) Retirement rate—100% at age 65.

b) Rates and values

X	$1,000q_x^{ad}$	$1,000q_x^{aw}$	$1,000q_x^{ai}$	$\ddot{a}_{x+1/2}^{ai(12)}$
20	.684	289.0	.3	2.96
25	.842	156.0	.3	2.91
30	1.101	96.3	.4	2.96
35	1.508	52.2	.5	3.17
40	2.061	28.6	.7	3.54
45	3.463	11.5	1.0	4.07
50	5.883	0.0	1.8	4.70
55	8.903	0.0	3.6	5.46
60	12.462	0.0	9.0	6.29
64	16.812	0.0	22.2	7.05

C. *Hypothetical Active Employee Data* (*all males*)

Active	X	Y	Accrued Benefit	$(AS)_y$	C_y
1	25	25	0	$ 3,500	0
2	25	30	$ 225	4,000	$ 510
3	25	35	220	4,200	495
4	35	40	240	5,000	510
5	30	50	1,600	7,000	4,800
6	45	60	1,440	10,000	3,500
7	30	64	1,210	4,000	3,600

D. *Valuation on July 1, 1959*

1) Valuation

Employee Number	y	$(AS)_y$	Accrued Benefit	P.V. of Accrued Benefit		Normal Cost		Employee Contributions		
				Ret.	Dis.	Ret.	Dis.		P.S.L.*	N.C.
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
A1...	25	$ 3,500	$ 0	$ 0	$ 0	$ 25	$ 2	$ 0	$ 0	$ 81
A2...	30	4,000	225	307	26	68	6	510	309	71
A3...	35	4,200	220	524	44	125	11	495	208	51
A4...	40	5,000	240	842	63	219	16	510	146	42
A5...	50	7,000	1,600	9,363	704	512	39	4,800	672	29
A6...	60	10,000	1,440	13,144	629	1,141	55	3,500	240	20
A7...	64	4,000	1,210	14,019	186	579	8	3,600	61	2
		$37,700	$4,935	$38,199	$1,652	$2,669	$137	$13,415	$1,637	$296

* Past Service Liability.

2) *Normal Cost*, $(NC)_{59}$

$(NC)_{59}$ $= \$500 + \Sigma(5) + \Sigma(6) + \Sigma(9)$

$= \$3,602$

Employee Contributions $= \$1,131$

Employer Normal Cost $= \$2,471$

$= 6.55\%$ of $\Sigma(AS)_y$

E. *Calculation of Expected Release of Reserve*, R^E_{59}, *and Expected Benefits*, B^E_{59}

Employee Number	y	$\frac{D_y}{D_{y+1}}$ (10)	$(AL)^s_{y+1}$ (11)	B^{Ed}_y (12)	B^{Ew}_y (13)	B^{Ei}_y (14)
A1....	25	1.2250	120	0	8	0
A2....	30	1.1444	838	1	55	0
A3....	35	1.0917	1,019	1	29	0
A4....	40	1.0659	1,397	1	17	0
A5....	50	1.0405	11,733	29	0	0
A6....	60	1.0551	15,933	45	0	0
A7....	64	1.0744	15,691	62	0	0
		....	46,731	139	109	0

EMPLOYEE NUMBER	y	q_y^{ad} (14)	q_y^{aw} (15)	q_y^{ai} (16)	$(AL)_{t+1}^{i}$ (17)	R_y^{Ed} (11)×(14) (18)	R_y^{Ew} (11)×(15) (19)	R_y^{Ei} [(11)−(17)] ×(16) (20)
A1......	25	See Table on page 626			0	0	19	0
A2......	30				0	1	81	0
A3......	35				0	2	53	1
A4......	40				0	3	40	1
A5......	50				7,923	69	0	7
A6......	60				9,466	199	0	58
A7......	64				8,657	264	0	155
					26,046	538	193	222

Notes:

1. $(AL)_{y+1}^{S} = [(\text{Col. } 3) + (\text{Col. } 5)]\,(\text{Col. } 10)$

$$+\left\{(\text{Col. } 4) + (\text{Col. } 6) - \left[\frac{.0125}{2}(\text{Col. } 1) + (\text{Col. } 2)\right]\frac{\bar{C}_y^{ai*}}{D_y}\right\}(\text{Col. } 10)$$

$$+\left\{(\text{Col. } 8) + (\text{Col. } 9) - \left[\frac{.03}{2}(\text{Col. } 1) + (\text{Col. } 7)\right]\cdot\frac{\bar{C}_y^{ad} + \bar{C}_y^{aw} + \bar{C}_y^{ai\dagger}}{D_y}\right\}(\text{Col. } 10)$$

2. $B^{Ed} = \left[\frac{.03}{2}(\text{Col. } 1) + (\text{Col. } 7)\right] q_y^{ad}$

3. $B^{Ew} = \left[\frac{.03}{2}(\text{Col. } 1) + (\text{Col. } 7)\right] q_y^{aw}$

4. $B^{Ei} = [.03\,(\text{Col. } 1) + (\text{Col. } 7)]\, q_y^{ai}$

for $x + 15 < y$ or $y < 55$

$= 0$, since waiting period is six months,

for $x + 15 \geq y$ and $y \geq 55$

5. $B^{Er} = 0$, since all employees are under age 65.

6. $(AL)_{t+1}^{i} = \left[\frac{.0125}{2}(\text{Col. } 1) + (\text{Col. } 2)\right]\ddot{a}^{i(12)}_{[y+1/2]+1/2}$

* If eligible for disability retirement benefit.

† If *not* eligible for disability retirement benefit.

F. *Expected Expenses, E^E_{59}, and Expected Interest, I^E_{59}*

1) E^E_{59} = $500

2) I^E_{59} is not known unless P^E_{59} is known and will be calculated at the year end with $P^E_{59} = P^A_{59}$.

G. *Valuation on July 1, 1960*

1) *Changes during Valuation Year*

a) Funding method—no change

b) Benefits and eligibility—no change

c) Asset valuation method—no change

d) Actuarial assumptions—no change

e) Hypothetical employee data

Employee Number	X	Y	Status	y+1	Accrued Benefit	$(AS)_{y+1}$	C_{y+1}
A1.....	25	25	Active	26	$ 45	$3,800	$105
A2.....	25	30	Quit; Refund Benefit—$550				
A3.....	25	35	Active	36	273	$4,300	$635
A4.....	35	40	Dead; Refund Benefit—$610				
A5.....	30	50	Disabled 9/1/59; Benefit—$125 per month				
A6.....	45	60	Dead; Refund Benefit—$3,625				
A7.....	30	64	Retired 6/1/60; Benefit—$105 per month				
A8.....	29	......	New	30	$ 45	$3,800	$ 90
A9.....	45	......	New	45	0	4,600	0
A10....	25	......	New	25	0	3,600	0

Employee Number	y+1	$(AS)_y$	P. S. Bft.	P. V. of P. S. Bft.		N. C.		Employee Contribs.	P.S.L.	N. C.
				Ret.	Dis.	Ret.	Dis.			
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
A1......	26	$ 3,800	$ 45	$ 32	$ 3	$ 34	$ 3	$105	$ 80	$ 84
A3......	36	4,300	273	710	60	140	12	635	246	49
A8......	30	3,800	45	61	5	65	5	90	55	67
A9......	45	4,600	0	0	0	269	13	0	0	30
A10.....	25	3,600	0	0	0	26	2	0	0	84
		$20,100	$363	$803	$68	$534	$ 35	$830	$381	$314

$$(AL)^A_{60} \text{ (actives)} = \Sigma(3) + \Sigma(4) + \Sigma(8)$$
$$= \$1{,}252$$

$$N^A_{59} = \sum_8^{10} (3) + \sum_8^{10} (4) + \sum_8^{10} (8)$$
$$= \$121$$

Disabled Employees (being paid from D.A. Fund)

Retirement Number	$y+1$ (1)	Monthly Disability Benefit (2)	$\ddot{a}^{ai(12)}_{[y+1/2]+1/2}$ (3)	$(AL)^d_{y+1}$ $12\times(2)\times(3)$ (4)
D1.......	51	\$125	4.82	\$7,230

$$(AL)^A_{60} = \$1{,}252 + \$7{,}230 = \$8{,}482$$

H. *Calculation of Actual Release,* R^A_{59}

Employee Number	Cause v	$(AL)^S_{y+1}$ (Col. 8) 1959 (1)	$(AL)^v_{y+1}$ (2)	R^v_y (1)−(2) (3)
A2........	w	\$ 838	\$ 0	\$ 838
A4........	d	1,397	0	1,397
A5........	i	11,733	7,230	4,503
A6........	d	15,933	0	15,933
A7........	r	15,691	0	15,691
		\$45,592	\$7,230	\$38,362

I. *Calculation of Actual Change in Liability Due to Salary Changes,* S^A_{59}

Employee Number	.0125 (Col. 1)$_{59}$ +(Col. 2)$_{59}$ (10)	$[(2)-(10)]\left[\frac{(3)+(4)}{(2)}\right]$ (11)	.03 (Col. 1)$_{59}$ +1.03 (Col. 7)$_{59}$ (12)	$[(7)-(12)]\left[\frac{(8)}{(7)}\right]$ (13)
A1.........	\$ 43.75	\$1	\$105	\$0
A3.........	272.50	1	636	0
	....	\$2	\$741	\$0

$$S^A_{59} = \Sigma(11) + \Sigma(13) = \$2$$

J. *Insurance Company Report*

a) Deposit Administration Fund Report

Fund Balance, July 1, 1959		$24,000
Contributions	$ 6,500	
Guaranteed Interest Credits	825	
Dividend	955	8,280
		$32,280
Contract Administration Charge, Deducted June 30, 1960	500	
Cost of Retirement Benefits, Deducted June 1, 1960	17,110	
Refunds of Employee Contributions	4,785	
Disability Benefits, Paid Out of Fund	525	22,920
Fund Balance, July 1, 1960		$ 9,360

K. *Calculation of* B_{59}^A, E_{59}^A, P_{59}^A, I_{59}^A, *and* I_{59}^E

1) $B_{59}^A = \$17{,}110 + \$4{,}785 + \$525 = \$22{,}420$

2) $E_{59}^A = \$500$

3) $P_{59}^A = \$6{,}500$

4) I_{59}^A = Guaranteed Interest Credit .... $ 825

The apportionment of the dividend is discussed in Section N.

5) I_{59}^E

a) F_{59}		$ 780
b) P_{59}^A		
(1) $3{,}000 \times .0325 \times \frac{10}{12}$	$ 81	
(2) $3{,}500 \times .0325 \times \frac{6}{12}$	+ 57	$+138
		$ 918
c) B_{59}^E		
(1) $\$17{,}110 \times .0325 \times 1/12$		
(2) $\$4{,}785 \times .0325 \times 6/12$		
(3) $\$525 \times .0325 \times 2/12$	$ 127	
d) $E_{59}^E \times .0325 \times 0/12$	− 0	$ 127
e) I_{59}^E		$ 791

L. *Calculation of Gain*

1) Gain from expenses

a) E_{59}^E	$ 500	
b) E_{59}^A	− 500	$ 0

2) Gain from interest earnings

a) I_{59}^{A}	$ 825	
b) I_{59}^{E}	791	$ 34

3) Gain from deaths and nonvested withdrawals

a) R_{59}^{Ad}	$17,330			
R_{59}^{Aw}	838			
R_{59}^{Ai*}	10	$18,168		
b) B_{59}^{Ad}	$ 4,235			
B_{59}^{Aw}	550			
B_{59}^{Ai*}	0	$−4,785	$13,383	
c) R_{59}^{Ed}	$ 538			
R_{59}^{Ew}	193			
R_{59}^{Ei*}	2	$ 733		
d) B_{59}^{Ed}	$ 139			
B_{59}^{Ew}	109			
B_{59}^{Ei*}	0	$ 248	− 485	
e) Gain from deaths and withdrawals before retirement				$12,898

4) Gain from disability retirements†

a) R_{59}^{Ai}	$ 4,503		
b) B_{59}^{Ai}	525	$ 3,978	
c) R_{59}^{Ei}	$ 220		
d) B_{59}^{Ei}	0	220	$ 3,758

5) Gain from retirements and deaths after retirement

a) R_{59}^{Ar}	$ 15,691		
B_{59}^{Ar}	17,110	$−1,419	
b) R_{59}^{Er}	$ 0		
B_{59}^{Er}	− 0	− 0	
c) Gain from retirements			$−1,419

* Disability before eligibility for disability retirement benefits.

† Disability after eligibility for disability retirement benefits.

6) Gain from salary changes

a) S^E_{59}	$ 0	
b) S^A_{59}	− 13	$− 13

7) Gain from new entrants

a) N^E_{59}	$ 0	
b) N^A_{59}	− 121	$− 121

8) Net gain .... $ 15,137

M. *Check of Net Gain from Section L against Aggregate Gain from Application of Principles Underlying IRS Bulletin on Section 23(p) of 1939 Internal Revenue Code*

1) 23(p) Gain

a) $(AL)^A_{59}$		$41,488
b) F_{59}		24,000
c) (*a*) − (*b*)		17,488
d) $(NC)_{59}$		3,602
e) P^A_{59}		
(1) $3,000 on 9/1/59		
(2) $3,500 on 1/1/60		
(3) Total		6,500
f) Interest at 3 1/4% on		
(1) (*c*)	$568	
(2) (*d*)	117	
(3) (*e*)(1)	81	
(4) (*e*)(2)	57	
(5) (*c*) + (*d*) − [(*e*)(1) + (*e*)(2)]		$ 547
g) $(AL)^A_{60}$		8,482
h) F_{60}		9,360
i) (*g*) − (*h*)		− 878
j) (*c*) + (*d*) − (*e*)(3) + (*f*)(5) − (*i*) = Gain		$16,015

2) Reconciliation

a) (1)(*j*)	$16,015
b) Net gain from Section L	15,137
c) (*a*) − (*b*)	$ 878
d) Dividend	955
e) Unreconciled difference Δ_{59}	77

This difference is less than 1/2% of the total gain for the year and can be ascribed to the approximate methods of calculating the individual elements. The correctness of the valuation therefore is confirmed.

N. *Interpreting and Reporting Results*

1) Apportionment of Dividends

The dividend declared under a Deposit Administration group annuity contract reflects the composite of all past years' experience and an analysis of its components is necessarily approximate.

The components of the dividend are:

1. Excess interest earnings,
2. Expense savings,
3. Mortality profits on retired employees.

The excess interest earnings will be a percentage of all reserves under the contract; expense savings are not a function of any one index, but may reasonably be related to current annual contributions; mortality profits will depend on the contingency reserve and mortality pooling practices of the insurer, but may be expressed as a percentage of retired life reserves.

The $955 dividend might be allocated in the following manner:

1. Excess interest—3/4 of 1% of contract reserves, assuming 4.00% was credited to the dividend account. Assume contract reserves to be $80,000	$600
2. Expense savings—3 1/2% of contribution	230
3. Mortality profits—1/4 of 1% of retired life reserves, assuming 100% pooling of mortality for this contract and that actual mortality on pooled experience was 1/4 of 1% below tabular. Assume retired life reserves to be $50,000	125
	$955

2) The simplest and most practical means of treating Δ_{59} is to arbitrarily decrease one or more of the elements of gain. Since the gain due to deaths and nonvested withdrawals is greatest by a wide margin and Δ_{59} is less than 1% of this item, there will be no risk of misinterpretation if the entire $77 is subtracted from this element of gain.

3) The report to the client might include this information:

a) The actual gain for the year was............ $16,015

b) This gain is the net effect of gains and losses from each assumption underlying the actuarial valuation, as tabulated below:

(1) Gain from interest earnings....................	$ 634
(2) Gain from expenses..........................	230
(3) Gain from terminations of employment without vested rights..............................	12,821
(4) Gain from disability retirements................	3,758
(5) Loss from election of retirement and mortality among pensioners............................	(1,294)
(6) Loss from salary increases.....................	(13)
(7) Loss from new entrants........................	(121)
(8) Net gain for year............................	$16,015

c) Expenses for the year were $270, $500 deducted from the Deposit Administration fund and $230 returned through a dividend. The net expenses of administration were 4.1% of contributions.

d) The interest credited to the dividend account was 4.00%.

e) Three employees terminated service; two died and one quit. Refunds of $4,785 in employee contributions were paid, but the liabilities of the pension fund were reduced more than $18,000. The net effect, after allowing for expected terminations, was a gain of $12,821.

f) One employee was retired under the disability retirement provisions of the plan. The liability for benefits to this employee has been reduced approximately $4,500. After adjusting for expected disabilities, the net gain from disability retirements was $3,758.

g) One employee elected to retire shortly before his 65th birthday. The amount deducted from the Deposit Administration fund, in accordance with the purchase rates specified in the group annuity contract, was $1,419 more than the actuarial reserve for this employee's benefits. $130 of the current year's dividend is due to mortality among retired employees. The net effect of these items is an actuarial loss of $1,294.

h) There were negligible losses due to salary increases and addition of new employees with partial benefit credits.

DISCUSSION OF PRECEDING PAPER

J. PERHAM STANLEY:

Mr. Dreher's paper presents a timely analysis of a topic which is becoming of increasing interest to actuaries, particularly those in the consulting field. I should like to comment on one particular aspect of this rather broad field—the question of how gains and losses, having been determined, should be reflected in subsequent contributions to the retirement plan.

On the one hand, it might be argued that gains arising out of a particular year of experience under the plan could be offset in full against current contributions due for the subsequent year. An alternative would be to spread any reduction in contributions over the remainder of the past service funding period, or even over the remaining future working lifetimes of active employees. In general, the latter approach will be of most appeal to the conservative actuary, assuming that the valuation assumptions have been set at a level such that gains (rather than losses) are most likely to arise. Not only does "spreading the gains" tend to result in a more fully funded plan at any given point of time, but also it avoids wide fluctuations in year-to-year contribution requirements under the plan.

The employer, however, will not always be guided solely by considerations of actuarial conservatism. For example, in many union negotiated plans containing a minimum funding requirement which is expressed in actuarial terms rather than in cents-per-hour or like terms, the employer will want to know his contractual minimum contribution, and may even wish to pay this amount regardless of considerations of actuarial soundness. Even where the employer pays a greater amount, he may wish to be advised of the extent of his overpayment (in terms of the contractual plan requirement) in order that he will be in a position to draw upon it in the event of hard times in the future. When actuarial "gains" arise out of favorable plan experience, spreading gains naturally reduces the surplus contributions which might otherwise be available to the employer in time of need.

The question thus presents itself: To what extent is it proper to offset actuarial gains against minimum current contributions, in a plan where the minimum contribution is loosely defined in terms of payment of current service costs and amortization of past service on a level basis over a long period of years?

My personal view is that the answer to this question depends in part on the source of the gain. For example, I believe a good case can be made for offsetting against contributions the full amount of any gains arising from interest earnings in excess of the valuation assumption. On the other hand, if for example a reduction in the work force occurring in the first year of operation of a plan causes a gain from excess withdrawals equal to several years' normal contributions, it hardly seems proper to permit suspension of contributions for several years on this account.

I believe that the various possible sources of gains can be grouped into four categories, wherein varying proportions of the gain might be applied against contributions. These are as follows:

Group I: Gains from expenses
interest
capital appreciation
delayed retirement.

These gains are fully reflected by actual additional cash in the pension fund. A good argument can be made for offsetting such gains in full against current contribution requirements.

Group II: Gains from withdrawals
deaths, both before and after retirement
salary changes
current service credit losses by employees.

Gains from these sources are not reflected in additional cash in the pension fund, but by a reduction in the prospective liabilities of the pension fund. (There may be even a reduction in the cash assets of the fund, if the plan contains lump sum death or severance benefits.) It can be argued that only a portion of such gains might properly be offset against current contribution requirements, such portion not to exceed the ratio that the assets of the fund bear to the gross actuarial deficiency for plan benefits.

Group III: Gains from new entrants.

Under the entry age normal cost method of valuation, if an arbitrary fixed entry age is used, or in fact if any departure is made from the use of exact individual employee entry ages, then substantial actuarial gains or losses will arise if the actual entry ages of new participants into the plan differ from this source in Group II above. On the other hand, it can be argued that no portion of this type of gain should be used as a current offset against contributions.

Group IV: Gains from change in valuation method
change in actuarial assumptions.

The use of even a part of the gains arising from these sources as a current offset against contributions is equivalent to a *retroactive* change in the valuation method or assumptions.

The questions raised here are troublesome ones to the pension consultant. It may well be that no precise answers can be given, but the subject would appear to warrant attention by the profession.

DOUGLAS C. BORTON:

Mr. Dreher's paper is particularly timely because of the recent introduction of electronic data processing machines in consulting offices and insurance companies. A detailed analysis of gain and loss is a logical by-product when an electronic computer is used to prepare the actuarial valuation of a pension fund. In the past, actuaries often have felt the need for such an analysis but have been thwarted by the time and expense involved in making the necessary calculations by desk calculator or conventional tabulating equipment.

The consulting office by which I am employed has done considerable work toward the development of a gain and loss analysis in connection with pension fund valuations on a 650 computer. The formulas we have developed are substantially the same as those presented by Mr. Dreher. However, our procedures differ in a number of ways, the most important of which are as follows:

1. The experience for retired and active members is analyzed separately. This is suggested by Mr. Dreher as a possible alternative to obtaining the gain and loss for the fund as a whole. By isolating the experience after retirement, it is possible to determine the adequacy of the pensioners' mortality tables and to obtain breakdowns of the results by sex and type of benefit. In our office the gain and loss analysis for retired members is calculated as a part of the valuation and is self-checking. For each recipient of a pension, the liability as of the preceding valuation date (or as of the middle of the valuation year for new retirants) is calculated and brought forward by formula to determine the liability as of the current valuation date. The liability so calculated is then compared with the results of an independent calculation of this liability. Pension recipients who have died or returned to active service also are processed, so as to determine the actual liabilities released. In making the calculations the correct formula for each type of case is selected in the computer by utilizing information on the input data cards.

2. For level funding valuations our gain and loss analysis for active members is based on the actuarial reserve at the end of the analysis year, *i.e.*, the current valuation date. This reserve is equal to the present value of future employer-provided benefit payments to the member (employer liability) less the product of the employer's normal contribution rate for the analysis year and the present

value of the member's future compensation. To isolate the effect of the compensation experience during the year, the reserve is determined for each survivor and separation from active service by projecting the member's compensation for the preceding valuation up to the present valuation date by means of the salary scale.

The use of this method necessitates a second valuation for survivors, based on the actual compensation as of the valuation date, which is used to determine the liabilities of the fund for survivors. The effect of actual compensation changes on the normal contribution rate for survivors is shown by the difference between the actual rate for survivors and the rate computed by assuming that compensation follows the salary scale.

The gain or loss arising from each of the various decrements, including retirement, is computed by formulas similar to those used by Mr. Dreher except that the reserves are based on the projected compensation rates. The results usually are expressed in terms of an increase or decrease in the normal contribution rate.

Members joining the plan during the year are valued separately. The effect of new members is shown by the difference between the normal contribution rate for all members and that for survivors only.

3. An analysis of actual and expected separations from active service by each type of separation and of actual and expected deaths among pension recipients also is prepared by number of lives. This information is valuable in reviewing the results of the gain and loss analysis for reasonability.

As Mr. Dreher has suggested, we believe it is more satisfactory to analyze independently the gain or loss from interest earnings and capital gains or losses.

It is important to emphasize that the development of suitable methods requires careful preparation and considerable judgment, even with the use of an electronic computer, if the calculations are not to become unduly burdensome. Where appropriate, approximations may be made or minor benefits ignored. However, even if an exact reconciliation is impossible, a gain and loss analysis of this type can be very useful in checking the underlying actuarial assumptions and verifying the reasonability of the results of a pension fund valuation.

HARRY D. MORGAN:

I would like to express my compliments to Mr. Dreher for having added to our American actuarial literature his treatise on pension fund gain and loss analysis. The paper is well organized and covers the subject matter rather thoroughly.

A critical analysis of this paper, however, leads me to take exception to the proposed formula for determining the gain from salary changes.

His expected increase in accrued liabilities because of anticipated salary increases,

$$S_t^E = (AL)_{t+1}^E \times \frac{s_{y+1} - s_y}{s_{y+1}},$$

is taken as a salary scale function of the expected accrued liability at the year end if all assumptions are exactly realized. This is then compared with

$$S_t^A = (AL)_{t+1}^S \times \frac{s_y}{s_{y+1}} \times \frac{(AS)_{y+1} - (AS)_y}{(AS)_y},$$

which represents the actual increase in liabilities due to the salary changes considering all participants at the beginning of the year. Under this formula, or under another formula

$$S_t^A = (AL)_{t+1}^A \times \frac{(AS)_{y+1} - (AS)_y}{(AS)_{y+1}},$$

which represents the actual increase in liabilities due to the salary changes considering only the active employees remaining at the year end, an error is introduced because S_t^E and S_t^A are based upon different employees. In the case of the formula in the paper, the error results from the fact that S_t^E is based upon expected survivors while S_t^A is based upon all participants at the beginning of the year. In the case of the other formula, the error results from the fact that assumptions in areas other than the salary scale (*i.e.*, deaths and terminations) were not exactly realized. Special care should be exercised in the analysis of this source of gain so that both the expected and actual increases are based upon the same group of employees. I believe that the gain from this source may be better measured by the formula:

$$(AL)_{t+1}^A \times \frac{(AS)_y \times \frac{s_{y+1}}{s_y} - (AS)_{y+1}}{(AS)_{y+1}},$$

i.e., the actual accrued liability at year end multiplied by the percentage of error contained in the salary scale assumption.

Concerning the gain from various decrements, Mr. Dreher offers two alternatives for the calculation of the expected liability releases R_t^{Ev}. The second alternative obtains these releases as a balancing item. Such a calculation method does not always permit a solution to the first paramount problem quoted by the author, namely "testing the mathematical correctness of periodic actuarial valuations." Although in some cases it may be expedient to employ the balancing method with caution, the author's first alternative method should be preferred because of its accuracy and ease of calculation, especially when punch card equipment is in use.

My final comment concerns the calculation of the expected interest earnings I_t^E. Mr. Dreher wisely states that it is simpler to wait until the year end and obtain the actual contributions rather than speculate about the expected contributions. I propose that a similar theory be employed in regard to both benefits and expenses since the client, in measuring his interest gain, is accustomed to thinking in terms of actual income and disbursements rather than those expected by the actuary. The expected interest on the difference between actual and expected benefits and expenses may be included as part of the benefit or expense gain or alternatively included as part of the miscellaneous gains.

SHEPHERD M. HOLCOMBE:

Mr. Dreher has made an outstanding contribution to the technical and practical aspects of pension funding. My main purpose is to compliment him but, incidentally, I should like to make a few comments.

In some gain and loss work that I have done on a unit credit cost basis, I have found it convenient to consider the effect of new entrants as an offset to terminations. This appears logical if it is assumed that the total number of employees should remain constant and, therefore, that every termination is replaced by a new entrant.

In view of the fact that one of the purposes of a gain and loss analysis is to test the mathematical corrections of the valuation, it would seem appropriate to do such an analysis on the present value of normal costs in the case of an entry age normal method of funding with immediate recognition of gains; or such analysis should be performed on the present value of 1% of future compensation where a frozen initial liability method is being used. The development of the proper approach follows very closely the same principles as those presented in the author's paper. To simplify the problem, if we consider a union type plan on the frozen initial basis where the present value is in terms of $1 per employee per year to retirement, the basic reconciliation (which can be modified to a gain and loss with little additional work) is:

Previous year's present value
less Present value for deaths among active lives
less Present value for terminations among active lives
less Present value for retirements
less $1 for each employee who is active at both the beginning and the end of the year
Balance

Balance times $(1+i)\dfrac{^{s}l_x}{^{s}l_x+1}$ (where x is taken at an appropriate average age)

plus Present value for new entrants
plus Corrections to data

Expected Present Value
Actual Present Value
Δ

I agree with the author that under a frozen initial liability method it is desirable to show the effect of gains and losses on the normal cost or on the accrual rate. I also agree that this is a distinct problem. I think an adequate answer can be arrived at by developing a series of normal cost accrual rates, each one showing the successive effects of each element.

The first step in such development is to arrive at the expected accrual rate:

$$(AR)^{E}_{t+1} = \frac{(\text{P.V. of Benefits})_t(1+i) - [(\text{Assets})_t + (\text{Unfunded F.I.L.})_t + (NC)_t](1+i)}{[(\text{P.V. of } 1\% \text{ of Future Compensation})_t - \Sigma(AS)^t_y](1+i)}.$$

It can be shown that

$$(AR)^{E}_{t+1} = (AR)^{A}_{t}.$$

For simplicity we may write the above as:

$$(AR)^{E}_{t+1} = \frac{(\text{Numerator})^{E}_{t+1}}{(\text{Denominator})^{E}_{t+1}}.$$

Also, it can be seen that

$$(AR)^{A}_{t+1} = \frac{(\text{Numerator})^{E}_{t+1} + \Sigma \text{ Gains in Numerator}}{(\text{Denominator})^{E}_{t+1} + \Sigma \text{ Gains in P.V. of } 1\% \text{ of Future Compensation}}.$$

If we define

$(AR)^{E+Ad}_{t+1}$ as Expected accrual rate adjusted for mortality gains,

$(AR)^{E+Ad+Aw}_{t+1}$ as Expected accrual rate adjusted for mortality and withdrawal gains, etc., and RPV as decrease in P.V. of 1% of Future Compensation, then we can arrive at progressive accrual rates:

$$(AR)_{t+1}^{E+Ad} = \frac{(\text{Numerator})_{t+1}^{E} - (R_t^{Ad} - R_t^{Ed})}{(\text{Denominator})_{t+1}^{E} - [(RPV)_t^{Ad} - (RPV)_t^{Ed}]},$$

$$(AR)_{t+1}^{E+Ad+Aw}$$

$$= \frac{(\text{Numerator})_{t+1}^{E} - (R_t^{Ad} - R_t^{Ed}) - (R_t^{Aw} - R_t^{Ew})}{(\text{Denominator})_{t+1}^{E} - [(RPV)_t^{Ad} - (RPV)_t^{Ed}] - [(RPV)_t^{Aw} - R_t^{Ew}]},$$

etc.

Now we can say the gain in accrual rate from mortality gain is:

$$(AR)_{t+1}^{E+Ad} - (AR)_{t+1}^{E}$$

and the gain from withdrawals is:

$$(AR)_{t+1}^{E+Ad+Aw} - (AR)_{t+1}^{E+Ad},$$

and so forth for the other elements of gain until the final actual accrual rate is reached.

It is obvious that a different order of injecting the elements of gain will affect the apparent gain in the accrual rate.

In closing, I should like to compliment Mr. Dreher on the verbal interpretation he has given to some of the formulas. I believe it makes the paper more readable and is something of which we should see more in technical papers.

HARRY M. SARASON:

The importance of actuarial valuations and gain and loss statements is illustrated by the fact that there is a fifteen page form prescribed for valuations for Scotland—Statutory Instrument 1954 No. 1260 (S.123) Pensions Local Government Superannuation Acts (Actuarial Valuations) (Scotland) Regulations, 1954—which is a revision of an earlier form and is available for 6*d* from Her Majesty's Stationery Office, London.

The pension fund gain and loss, as Mr. Dreher indicates, can be broken down into an analysis of increase in reserves and a general statement, as is done in the life insurance statements. It can be broken down by "lines of business": disability, death, retirement, males, females, etc. It can be prepared by regular double entry methods on columnar work sheets such as are described in elementary bookkeeping and accounting textbooks.

Any gain and loss analysis tests numerous assumptions. Thus, in life insurance accounting we expect a first year loss because first year "loading" is not adequate for first year expenses. In pension fund accounting, the opposite is true in many cases. Thus early withdrawal rates which produce gains may be higher than assumed while later withdrawal rates are less than assumed: the deaths immediately after retirement in many plans are likely to show a "selection" the opposite of the selection in life insurance; that is, there is likely to be a gain from annuity mortality immediately after the first retirement date even though the mortality later may be below the tabular. This is due to impaired lives retiring and to other reasons as has been discussed almost every time a retirement plan experience has been published.

Mr. Dreher's paper gives every promise of proving to be useful.

(AUTHOR'S REVIEW OF DISCUSSION)

WILLIAM A. DREHER:

In the course of writing this paper it became apparent to me that the practical problems associated with the daily application of gain and loss theory were so numerous that a book length treatment would be needed if one expected to touch all the bases. Thereupon I decided to set forth the theory with a few examples and look to those who contributed discussions for an elaboration on the practical problems. I am grateful that this expectation was realized so fully; would that the expectation of our actuarial valuations were always so close to the realization!

The appendix to the paper would still be incomplete if I had not had the benefit of thoughtful criticism and prodigious effort by my friend and associate, William A. Ferguson. I am deeply appreciative of his assistance.

I congratulate Mr. Borton for the procedures used in his office to analyze actuarial gains and losses. Theirs is an enviable situation. It should be noted that the appendix of the paper was not available when his discussion was prepared; a review of the examples set out in the appendix will illustrate that we agree on several of the points which might have been thought, from a reading of the paper alone, to involve a different procedure.

We are indebted to Mr. Sarason for bringing the form prescribed for actuarial valuations of local government plans in Scotland to our attention. This form is an excellent check list for anyone preparing a valuation report. It requires, in addition to a detailed balance sheet and statement of assets and security transactions, an analysis of changes in the membership group by age, cause and sex. Without specifically requiring a gain and loss analysis, the regulations state:

9.—(1) The following information shall be furnished in the fourth part of the report—

(a) If a surplus is disclosed in the valuation balance sheet, a statement indicating generally the factors which have caused the surplus and certifying the amount (if any) which may probably be regarded as disposable . . . .

(b) If a deficiency is disclosed, a statement indicating generally the factors which have caused the deficiency and a recommendation of the allocation of the deficiency . . . .

Mr. Stanley poses an intriguing and, as he rightly states, important question. Three points occurred to me during a study of his comments. If benefits in the pension plan are based upon earnings, particularly final average earnings, and the salary scale is presumed to make inadequate allowance for the effect of economic inflation on future salaries, capital gains from equity investments—which are related to the same inflationary forces—will be needed to offset losses due to salary increases in excess of those predicted by the salary scale. If capital gains—either realized or unrealized—are full offsets to current contributions and the loss due to salary increases is only partly reflected in the contribution level, the employer could be fooling himself badly.

Second, if the ratio of assets to gross actuarial liability is high and sizable gains or losses in Group II emerge and if the actuarial assumptions are, in fact, well chosen, the "minimum contributions" from year to year would fly around solely because of chance fluctuations in the experience withdrawal and mortality rates. This could be thoroughly misleading.

Finally, if the experience justified a revision of the valuation assumptions, the effect of this change could be partially incorporated in the "minimum contribution." For instance, if the accrued liability had been estimated at $1,000,000 and the normal cost at $90,000 and the revised valuation basis indicated that the accrued liability was $800,000 and the normal cost $80,000, the minimum contribution could be dropped from $120,000 (assuming a 3% valuation interest rate and a minimum contribution toward the accrued liability) to $104,000. This could be done even though the basis used to determine tax deductions under the Internal Revenue Code was not changed.

Mr. Holcombe's comment that the effect of new entrants is an offset to the effect of terminations in a valuation using the unit credit funding method is well taken. The same is true of the aggregate cost method.

He also outlines a method of relating the various gain and loss elements to the normal cost accrual rate which is more refined than that suggested in the appendix of the paper. This difference in technique is another illustration of the important fact that some clients will understand and appreciate an elaborate analysis while others will be satisfied with less detailed methods. If the gain and loss analysis were programmed on one of the medium or large sized computers, the functions described by Mr. Holcombe could be prepared quite simply and would be at the disposal of the actuary.

Mr. Holcombe states, "It can be shown that $(AR)^E_{t+1} = (AR)^A_t$." Mr. Trowbridge[1] has shown us that this is true in a stationary group, the experience of which follows the service table assumptions, but I wonder whether the equation holds if gains or losses have emerged in the past. In the aggregate cost or frozen initial liability funding method, the effect of past gains or losses on the current accrual rate depends upon an average annuity value related to the present value of future compensation; this annuity value will change from year to year as a function of the current distribution of salaries and ages, even if there are no further gains or losses in the future.

I am particularly pleased by his comment about my verbal interpretation of some formulas in the paper. Most of us who work closely with pension valuations have frequently despaired of explaining satisfactorily to a client what an entry age normal cost or an accrued liability is. One is reminded of the five blind men describing an elephant. But, if we cannot make our point with the public, it is at least gratifying to know that we can still talk intelligibly to one another.

I must apologize for confusing Mr. Morgan. The formula for S^A_t is not clear, although my intention is, I think, clarified by the examples in the appendix. S^E_t is based on all employees in the valuation at the beginning of the year and measures the portion of the expected liability which, according to the salary scale, will be attributable to salary changes during the year. S^A_t is based on survivors to the end of the valuation year and includes the quantity $(AL)^S_{t+1} \cdot s_y/s_{y+1}$, which was computed at the beginning of the year, instead of the quantity, $(AL)^A_{t+1}$, which is calculated at the year end and would have destroyed the independent check of the valuation. It should be noted that this treatment of salary increases is entirely consistent with the handling of gains from withdrawal, death, interest, capital gains, etc.

His comments about I^E_t illustrate again that the valuation report must be written with the client's expectation and understanding well in mind.

Mr. Conrad M. Siegel was unable to prepare a discussion in time for presentation, but has set down some pertinent remarks in a letter which I should like to summarize briefly and comment about.

[1] "Fundamentals of Pension Funding," *TSA* IV, 17.

First, he notes that the formula on page 595, relating to a means of presenting the gain from interest earnings on a Deposit Administration contract, is incorrect because it ignores interest at the valuation rate on the retired life fund. This is quite correct; I_t^A should have been redefined to be the entire amount of interest credited to the dividend fund account for the year in question.

He makes two other points. First, he questions that the gain and loss analysis would have much significance for the average employer and points out that it might, in fact, either confuse him or raise a question in his mind about the competence of the actuary in selecting the valuation assumptions. Second, he is skeptical of the value of gain and loss analysis for purposes of reviewing the appropriateness of the valuation assumptions and feels that a mortality or withdrawal study of the "conventional type" would be a better guide for this purpose.

Mr. Siegel's first point is well taken; not every client will understand or appreciate a summary of actuarial gains and losses. That does not mean that we should not present many clients with a gain and loss analysis. In the first place, some of them are quite astute and respond with interest to such an analysis; this can be a very effective means of educating the client to the abilities and limitations of the actuary and his methods. Furthermore, we consulting actuaries would be unwise to predict the future level of knowledge among clients by projecting our experience of the last 10 or 15 years. Intelligent men are not going to be fooled forever by the actuarial gibberish sometimes ladled up; ask the actuary who has taken over the client of another actuary or who has dealt with experienced labor negotiators. But it is certainly correct that the gain and loss analysis has to be presented with the client in mind. For one, a very elaborate presentation will be necessary; for another, a gain and loss analysis would be restricted to a comment about the change in the normal cost accrual rate and the effect of one or two principal sources of gain and loss.

A mortality or withdrawal study of a "conventional type" does not give proper rates of mortality or withdrawal for a valuation using an aggregate service table. The rates must be based upon the reserve released and these rates are often dramatically different from the unweighted rates of a "conventional" study. The effect of salary increases must be similarly modified. R_t^A and S_t^A—the relevant quantities for this analysis—can be accumulated within quinquennial age groups. Then, for example, the ratio of R_t^{Aw} to $[(AL)_{t+1}^A + R_t^A]$ will give a crude q_t^{aw}; the q_t^{aw} from the service table can then be evaluated. This will remove the necessity for a withdrawal study, at least for the purpose of reviewing valuation assumptions.

I am sympathetic with the views expressed by Mr. Siegel; it is entirely proper to approach cautiously a proposal that much additional work be done as a routine part of the actuary's annual valuation. But the availability of competent and efficient machinery can, as Mr. Borton has described, permit us to make these gain and loss analyses without a fantastic cost or effort. We may find, however, that the slope of our fee schedules will flatten, since this analysis is most likely to be valuable and necessary for the larger groups.

Let me thank again those who offered discussions of this paper. Their remarks will be extremely beneficial to those of us who put into practice the theory of gain and loss analysis.