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# CONTINUANCE FUNCTIONS

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## INTRODUCTION

RECENT developments in the field of accident and sickness insurance have introduced a complexity and variety of claim cost considerations into actuarial research which dearly call for improved tools if any practical degree of standardization and simplification is to be achieved. This is particularly the case in the medical insurance field: complex benefit structures, involving a wide assortment of formulas for covering hospital, surgical, and other medical costs, frequently in combination with a "deductible" concept, can easily lead to problems of claim cost derivation that are almost hopelessly involved.

This basic problem confronting the actuary may be separated into two elements. The first concerns the determination of appropriate assumptions as to claim rates among active lives. The second is the analysis of the claim pattern itself. This latter problem may usually be solved by developing a "continuance" table showing the probabilities of claim continuance to various discrete durations or amounts. For example, disability continuance can be expressed basically as the number remaining disabled at the end of $t$ days (or months or years) out of an assumed initial radix, or number disabled. Hospital confinement may be expressed in the same manner. A major medical continuance table could express the probability that claim expense will equal or exceed $t$ dollars. From such tables, the average size of claim, or "claim annuity" if interest discount is introduced, may be computed which in combination with the rate of claim produces the basic $S_x$ function sought.

This is simple enough where only a single elementary benefit is to be valued, such as income disability, but when benefits are combined in various ways, with a multitude of elimination periods or deductibles and maximum benefit limits, an exceedingly complex problem emerges.

The object of this paper is to present a theoretical basis for the mathematical graduation of continuance data for both elementary and combined benefits, and to demonstrate methods of utilizing the theory to derive claim costs for a wide variety of benefits.

## I. ANALYSIS OF THE BASIC CONTINUANCE FUNCTION

### 1. *The Typical Continuance Pattern*

Let us examine the typical pattern of claim continuance by expressing the continuance table in the form $p_x^{(t)}$, where this symbol denotes the

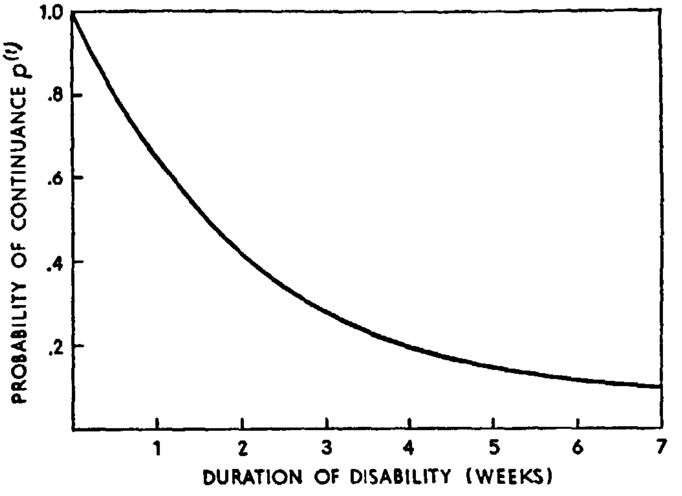


FIG. 1a.—Probability of continuation of disability among 10,000 lives disabled at age 40. (Conference Modification of 1926 Class 3 Disability Table.)

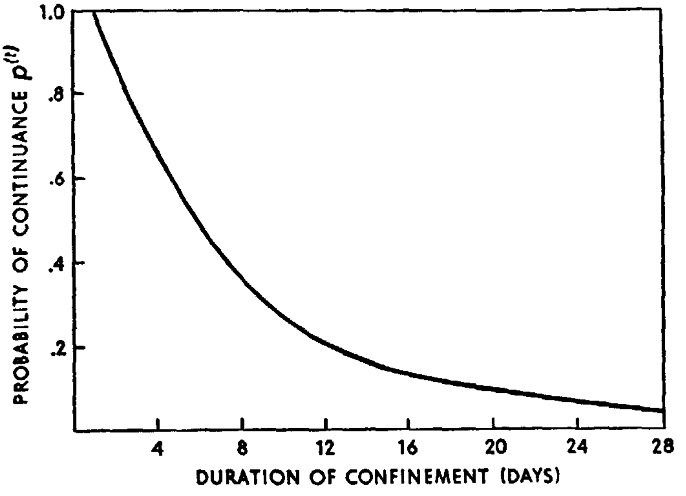


FIG. 1b.—Probability of continuation of hospital confinement among 10,000 lives confined (adult males). (Gingery, "Group Hospital Expense Insurance Experience," *TSA* IV, 87, Table VI-1, 10× Miscellaneous Benefit 31-Day Plans.)

probability that disability incurred at age $x$ will continue to a duration of $t$ units (of either time or money), showing the data in graphical form. Figure 1 shows two such patterns, one for continuance of time loss disability, the other for hospital confinement.

As would be expected, both curves exhibit the same basic pattern: rapid termination over the shorter durations, followed by a sharp bend and leveling out of the curve as the acute disabilities terminate in recovery or death. The remaining lives are the long-term or "permanent" disabilities, the disability function tapering off over a much longer period of time than the confinement function.

## 2. *The Force of Termination*

In order to make a mathematical analysis of these continuance curves, we define the *force of termination*, $\pi_x^{(t)}$:

$$\pi_x^{(t)} = -D_t(\log_e p_x^{(t)}) . \tag{I-1}$$

This definition is precisely parallel to that of the force of mortality, $\mu_x$, except that the variable here is duration, $t$. (In the remainder of the paper, "ln" will denote $\log_e$, and "log" will denote $\log_{10}$.)

Let us assume that a hypothetical continuance experience is subject to a constant force of termination, $k$. Then:

$$D_t(\ln p_x^{(t)}) = -k$$

$$\ln p_x^{(t)} = \int -k dt = -kt + c$$

$$p_x^{(t)} = e^{-kt+c} .$$

Since $p_x^{(0)} = 1$, $c = 0$ and we have:

$$p_x^{(t)} = e^{-kt} . \tag{I-2}$$

If we attempt to fit formula (I-2) to an actual experience of either type shown in Figure 1, we find that a decreasing value of $k$ is obtained as we apply the formula with increasing values of $t$; in other words, the force of termination is a function that decreases with increasing values of $t$. What we require is a function that fulfills this condition but at the same time is simple enough to be easily derived and easily applied in practical use. As will be shown later, the most practical function is one which remains reasonably simple upon being carried through two successive integrations.

A general class of functions which fulfill these requirements is either of the following (dropping, for the moment, the age designation $x$):

$$\text{(a)}\quad \pi^{(t)} = \frac{a}{a + bt^c} \qquad \text{or} \qquad \text{(b)}\quad \pi^{(t)} = \frac{a}{(a + bt)^c} . \tag{I-3}$$

If a completely generalized expression for the integral, $\ln p^{(t)}$, of either of these expressions is worked out, the result is a very cumbersome expression which is inconvenient and, moreover, not integrable a second time in terms of simple elementary functions. Accordingly, we shall confine our investigation to the class of special cases of (a) or (b) obtained by setting $c = 1$:

$$\pi^{(t)} = \frac{a}{a + bt}\,. \tag{I-4}$$

No further loss of generality results by also setting $b = 1$, for if we divide numerator and denominator of (I-4) by $b$, we obtain an expression of the form:

$$\pi^{(t)} = \frac{a'}{a' + t}\,.$$

We therefore investigate the simplified function

$$\pi^{(t)} = \frac{a}{a + t} \tag{I-5}$$

to determine whether this provides a usable basis for expressing continuance data mathematically.

Deriving $p^{(t)}$, we have:

$$D_t(\ln p^{(t)}) = -\frac{a}{a+t}$$

$$\ln p^{(t)} = -\int \frac{a}{a+t}\,dt = -a \ln(a+t) + c\,.$$

Setting $\ln p^{(0)} = 0$, we obtain $c = a \ln a$, and therefore

$$p^{(t)} = \left(\frac{a}{a+t}\right)^a. \tag{I-6}$$

This function (I-6) will be referred to as the "Alpha" function.

A simple method exists of determining both whether a given body of data is likely to be satisfactorily graduated by this formula and, if so, what the approximate values of the constants must be. It is obvious that formula (I-6) cannot work, without some revision, for lifetime disability continuance, since eventually the force of termination must begin to increase because of the advancing death rate at higher ages. We will deal with this problem later, but for the moment we will confine ourselves to continuance in which the termination rate does decline with duration.

### 3. *Graduation of Elementary Data*

In order to proceed with the description of the method of testing for possible graduation and determination of the constants, several definitions will be convenient.

(1) *Elementary Function.* A continuance function defined by a single function of the type I-6.
(2) *Compound Function.* A continuance function requiring two or more elementary functions for its definition.
(3) *Graduation Constants.*
Attenuation—the constant exponent $a$ in formula (I-6).
Range—the constant $\alpha$ in formula (I-6).
(4) *Logarithmic line*, or *logarithmic asymptote.* A curve which appears as a straight line when exhibited on a full-logarithmic graph. A logarithmic asymptote is such a curve defining the limiting position of the curve of an Alpha function, *i.e.*, of a function of the form (I-6).

#### a. The Equation of the Logarithmic Line

In order to appear as a straight line on a logarithmic graph, a curve must have an equation of the form

$$\log y = c \log x + b \qquad \text{or} \qquad y = Bx^c .$$

If $c$ is negative, we may write this expression with a positive exponent as

$$y = \left(\frac{\alpha}{t}\right)^a \tag{I-7}$$

where

$$a = -c, \qquad \alpha = B^{1/a}, \qquad t = x .$$

As a probability function, this is simply formula (I-6) with the substitution $t' = t + \alpha$:

$$p^{(t'-\alpha)} = \left(\frac{\alpha}{t'}\right)^a .$$

The function (I-7) is the limiting curve for (I-6) as $t$ increases, as is readily seen by writing each expression in logarithmic form:

$$1) \ \log p^{(t)} = a[\log \alpha - \log (\alpha + t)]$$

$$2) \ \log y = a \, [\log \alpha - \log t]$$

Subtracting (1) from (2),

$$\log y - \log p^{(t)} = a[\log (\alpha + t) - \log t] .$$

As $t$ increases indefinitely, the expression on the right approaches zero, hence $p^{(t)}$ approaches $y$.

### b. Inspection of Data for Alpha Function Graduation

The fact that the Alpha function possesses a logarithmically linear asymptote is very useful, for it makes possible a simple preliminary inspection of crude or tabular data for the purpose of determining whether an Alpha function is likely to approximately reproduce them. The data are simply graphed in the form $p^{(t)}$ on logarithmic paper and examined to determine whether there is a more or less smooth downward curvature that gradually straightens. If the slope of the curve does not steadily decrease toward a fairly constant negative value, or, in the case of crude data, if the underlying function may not be presumed to do this, the data cannot be successfully graduated by an elementary Alpha function. It may be necessary to perform a preliminary graduation by any of the standard methods in order to smooth crude data containing severe fluctuations.

If the necessary characteristics appear to be satisfied, approximate values of the range and attenuation may be easily obtained by simply drawing in the asymptote in its estimated position.

The range $\alpha$ is the abscissa at the intersection of the line with the ordinate $p = 1$.

The attenuation $a$ is the numerical value of the negative slope of the line. Note that $a$, as used in formula (I-6), is positive. Figure 2 on page 656 illustrates the method.

This quick approximation will not give very accurate results unless the data can be followed through at least 3 and preferably 4 cycles of either $p$ or $t$ on the graph. In particular if the slope is steep the investigator will find it difficult to estimate the position of the asymptote accurately. For this reason, this technique should not usually be relied upon for more than a preliminary test.

### c. 3-Point $p^{(t)}$ Graduation

The Alpha function involves but two constants, $\alpha$ and $a$, and therefore can be fitted exactly to only 2 arbitrary points along a curve, which must be carefully selected to achieve a good average fit. For all $\alpha$ and $a$, if $t$ is set $= 0$, $p^{(t)} = 1$ and hence the point (0, 1) is common to all functions of the form (I-6).

By abandoning this requirement it is possible to introduce a third constant which usually enables the investigator to improve the fit. In fact, the data usually encountered will be found to fit the Alpha function poorly in the early values and by fitting a function through 3 points, one of them a suitable selection in the early range of the curve, a decided improvement may usually be obtained. To get 3 constants into the formula, (I-6) is replaced by the more general expression

$$p^{(t)} = \left(\frac{a}{a'+t}\right)^a, \tag{I-8}$$

which henceforth will be regarded as the general form of the Alpha function. This form of expression is obtainable from (I-6) by the substitution $t' = t + a - a'$ and thus is equivalent to a translation of the vertical axis. Associated with this refined expression is the formula for the force of termination:

$$\pi^{(t)} = \frac{a}{a'+t}. \tag{I-9}$$

The effect of the transformation (I-8) is to introduce the notion of a "minimum duration" into the theory, so that all disabilities are regarded as continuing to a certain duration and only then beginning to terminate.

This minimum duration, $\tau$, has the value $a - a'$, for when $t = a - a'$ is substituted in (I-8) we get $p^{(a-a')} = 1$. A complete definition of the Alpha function must therefore be the following, since $p^{(t)}$ obviously cannot exceed 1:

$$\begin{aligned} &\text{for } 0 \le t \le a - a', \qquad p^{(t)} = 1 \\ &\text{for } t \ge a - a', \qquad p^{(t)} = \left(\frac{a}{a'+t}\right)^a. \end{aligned} \tag{I-10}$$

While we may sometimes refer to the Alpha function as (I-8) only, it is to be understood that (I-10) is actually the complete definition.

The introduction of $a'$ into the function does not alter the asymptotic relation, so that a class of functions which vary only in the value of $a'$ all approach the same asymptote

$$y = \left(\frac{a}{t}\right)^a.$$

Figure 2 has been drawn with $a - a' = 1$, and the curve will be seen to pass through (1, 1), thus having a minimum duration of 1 unit.

The following technique may be used to graduate a body of data by fitting an Alpha function through 3 values of $p^{(t)}$, where these values are chosen to define the smooth curve assumed to underlie the data, and hence are not necessarily points falling on the graph of the crude data itself.

(1) Select $p^{(u)}$, $p^{(v)}$, and $p^{(w)}$ so that a smooth curve through these points will as far as possible lie in an average position among fluctuations in the data, and also so that

$p^{(u)}$ has a value near 1,

$p^{(v)}$ is just beyond the range in which the curve is bending most sharply, $p^{(w)}$ is well out along the curve, near the assumed asymptote if possible.

(2) Solution for the value of $\alpha'$ may be obtained as follows:

Dividing $p^{(u)}$ by $p^{(v)}$, and $p^{(v)}$ by $p^{(w)}$, we obtain:

$$
\begin{aligned}
&1)\quad \frac{p^{(u)}}{p^{(v)}} = X = \left(\frac{\alpha' + v}{\alpha' + u}\right)^a \qquad 2)\quad \frac{p^{(v)}}{p^{(w)}} = Y = \left(\frac{\alpha' + w}{\alpha' + v}\right)^a \\
&3)\quad \log X = a[\log(\alpha' + v) - \log(\alpha' + u)] \\
&4)\quad \log Y = a\,[\log(\alpha' + w) - \log(\alpha' + v)] \qquad\qquad \text{(I-11)} \\
&5)\quad \frac{\log X}{\log Y} = \frac{\log(\alpha' + v) - \log(\alpha' + u)}{\log(\alpha' + w) - \log(\alpha' + v)}\,.
\end{aligned}
$$

This equation may be solved by inserting trial values of $\alpha'$, starting with an estimate obtained from the graph:

The estimated asymptote intersects the ordinate $p = 1$ at $\alpha$.

The estimated smooth curve intersects the ordinate at $\alpha - \alpha'$. Successive test values will usually narrow down the value of $\alpha'$ fairly rapidly.

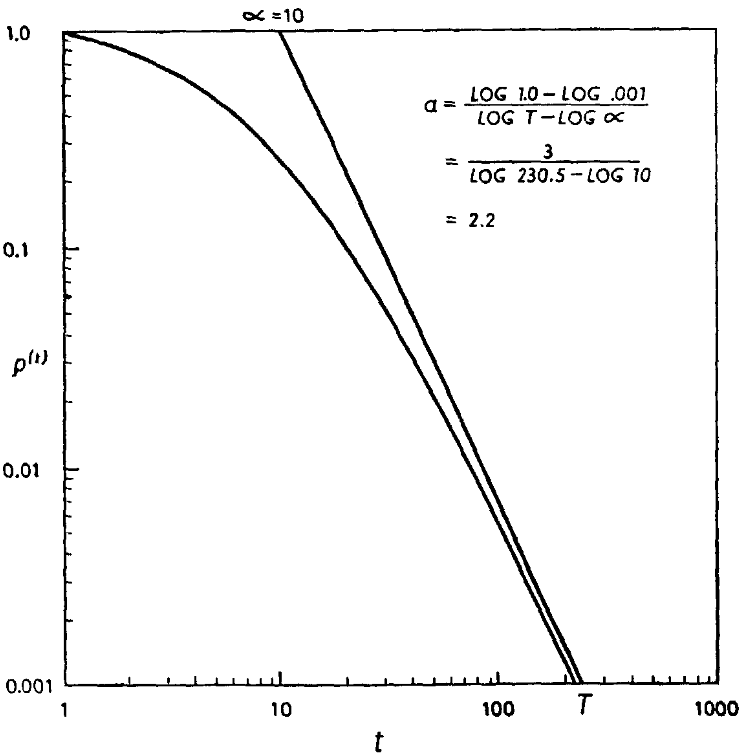


FIG. 2

(3) Having determined $a'$ to an acceptable degree of accuracy, $a$ is obtained from equation 3 or 4 and finally $\alpha$ from any of the original values $p^{(t)}$.

These solutions must then be checked by computing $p^{(u)}$, $p^{(v)}$, and $p^{(w)}$, and tested against the data by computing a series of values $p^{(t)}$. It is usually satisfactory to take $t = 2.5$, 5, and 10 in each of the first 3 cycles or so and check the ratio of the computed value to the corresponding value on the smooth curve originally estimated and from which $p^{(u)}$, $p^{(v)}$, and $p^{(w)}$ were taken.

This check is only partially adequate and a further summation test using values of the continuance integrals, to be discussed in a later section, is also desirable.

Another approach which lends itself conveniently to slide-rule solution proceeds as follows:

We have, as before:

$$1)\quad \frac{p^{(u)}}{p^{(v)}} = X = \left(\frac{a'+v}{a'+u}\right)^a \qquad 2)\quad \frac{p^{(v)}}{p^{(w)}} = Y = \left(\frac{a'+w}{a'+v}\right)^a$$

$$3)\quad \frac{1}{a}\log X = \log\frac{a'+v}{a'+u} \qquad 4)\quad \frac{1}{a}\log Y = \log\frac{a'+w}{a'+v}$$

$$5)\quad \frac{\log X}{\log Y} = \frac{\log\dfrac{a'+v}{a'+u}}{\log\dfrac{a'+w}{a'+v}} \qquad 6)\quad \log\frac{a'+w}{a'+v}\left(\frac{\log X}{\log Y}\right) = \log\frac{a'+v}{a'+u} \qquad \text{(I-12)}$$

$$7)\quad \left(\frac{a'+w}{a'+v}\right)^{\log X/\log Y} = \frac{a'+v}{a'+u}$$

$$8)\quad a'+u = (a'+v)\left(\frac{a'+v}{a'+w}\right)^{\log X/\log Y}.$$

In equation 8, the exponent is a known constant and the solution of $a'$ may be rapidly obtained by successive approximation using a slide rule.

It may happen that equation 5 of (I-11), or 8 of (I-12), is insoluble for $a'$. If this occurs, the selected points violate the condition that $\pi^{(t)}$ be a decreasing function. This is easily tested.

Since $\pi^{(t)} = -D_t(\ln p^{(t)})$ and we are dealing with continuous single valued functions, there exist, by the Theorem of Mean Value, values $\psi_1$ and $\psi_2$ such that (1) $u < \psi_1 < v$ and $v < \psi_2 < w$, and (2) $\pi^{(\psi_1)} = \log X/(v-u)\log e$ and $\pi^{(\psi_2)} = \log Y/(w-v)\log e$. Since $\pi^{(\psi_1)}$ must exceed $\pi^{(\psi_2)}$ for an Alpha function,

$$\frac{\pi^{(\psi_1)}}{\pi^{(\psi_2)}} = \frac{(w-v)\log X}{(v-u)\log Y} > 1\,. \qquad \text{(I-13)}$$

If this test fails, different values of $p$ must be tried which give a successful "$\pi$" test, or else the data under study cannot be graduated by the Alpha function and require a different function, the Lambda function, which will be discussed in paragraph e.

d. Graduation by One Point of $p^{(t)}$ and Two Points of $\pi^{(t)}$

An alternate method of graduation which does not give as much control as the 3-point $p^{(t)}$ method, but which is much quicker, is graduation by equating to one value of $p^{(t)}$ and to two values of $\pi^{(t)}$. The advantage is that this leads directly and easily to an approximate solution without recourse to trial values in obtaining the first constant.

Approximate values of $\pi^{(t)}$ are readily obtained from the logarithmic graph. The slope, $z^{(t)}$, of the $p^{(t)}$ curve at any point is given by

$$z^{(t)} = \frac{d}{d\,(\log t)} \log p^{(t)} = \frac{d}{d\,(\ln t)} \ln p^{(t)}, \tag{I-14}$$

from which we obtain directly

$$\pi^{(t)} = -\frac{z^{(t)}}{t}. \tag{I-15}$$

To obtain $z^{(t)}$ approximately, we carefully strike the logarithmic tangent to $p^{(t)}$ at the desired point and extend it sufficiently to give a good ratio of the logarithmic lengths. Then if the line is defined by the points

$$(p_1, t_1) \qquad \text{and} \qquad (p_2, t_2)$$

we have the relation

$$z^{(t)} = \frac{\log p_2 - \log p_1}{\log t_2 - \log t_1}.$$

Obtaining two such values, $z^{(v)}$ and $z^{(w)}$, we have:

$$\left.\begin{aligned} &1)\quad -\frac{z^{(v)}}{v} = \pi^{(v)} = \frac{a}{a' + v} \\ &2)\quad -\frac{z^{(w)}}{w} = \pi^{(w)} = \frac{a}{a' + w}, \end{aligned}\right. \tag{I-16}$$

from which $a$ and $a'$ are easily obtained. We find $\alpha$ from the third value, $p^u$:

$$p^{(u)} = \left(\frac{a}{a' + u}\right)^{a}$$

by inserting the values of $a$ and $a'$.

It is of the utmost importance to the success of this method that $u$, $v$, and $w$ be carefully chosen. $u$ must be taken so that $p^{(u)}$ is near 1.00, at least .85, say. $v$ and $w$ are taken by first selecting points $v'$ and $w'$ as though using the 3-point $p^{(t)}$ method. Then $v$ and $w$ are selected so that

$$\log v \doteq \frac{\log u + \log v'}{2}$$
$$\log w \doteq \frac{\log v' + \log w'}{2}. \qquad \text{(I-17)}$$

This will ordinarily produce results that equate the values of $p^{(t)}$ to the data at points not too much different from $v'$ and $w'$.

e. The Lambda Function

As discussed in paragraph c, it is not always possible to find a set of 3 points of a decreasing $p$ function that can be fitted by an Alpha function. In order to have a general technique, we therefore introduce the Lambda function, defined by the following expression for $\pi^{(t)}$:

$$\pi^{(t)} = \frac{l}{\lambda' - t}. \qquad \text{(I-18)}$$

Integrating to obtain $p^{(t)}$, we have:

$$\ln p^{(t)} = l \ln (\lambda' - t) + c$$

and, setting $\ln p^{(t)} = 0$ when $t = \lambda' - \lambda$, we have

$$c = -l \ln \lambda,$$

or

$$\begin{aligned} &1)\quad p^{(t)} = \left(\frac{\lambda' - t}{\lambda}\right)^l \quad \text{when} \quad \lambda' - \lambda \le t \le \lambda \\ &2)\quad p^{(t)} = 1 \quad \text{for} \quad 0 \le t \le \lambda' - \lambda \\ &3)\quad p^{(t)} = 0 \quad \text{for} \quad t > \lambda'. \end{aligned} \qquad \text{(I-19)}$$

We will sometimes refer to the Lambda function in terms of formula (1) only, but it is to be understood that equations 1, 2, and 3 of (I-19) constitute the complete definition.

It will be seen that this function has an increasing force of termination and therefore enables us to fit 3 points of a curve for which the $\pi$ test (I-13) is less than 1. It is peculiar in that $p^{(t)}$ reaches zero at the finite value $\lambda'$, and for this reason it can be used only when it appears reasonable that the continuance function should vanish at a definite finite limiting value.

The Lambda function constants may be solved for directly by equations (I-11). However, it is sometimes possible to employ a simple short cut. If the limit value $\lambda'$ can be estimated fairly accurately from the graph, we can adopt this value for the constant. We then require two other equations, and for one of these we may take $p^{(\tau)} = 1$, for which $\tau$ is also usually easy to estimate. The other, $p^{(u)}$, should be taken at the point of sharpest curvature where the slope of the curve begins to turn steeply downward. We then have:

$$A = \lambda' - \tau \qquad B = \lambda' - u \qquad C = p^{(u)}$$

and

$$\left(\frac{A}{\lambda}\right)^l = 1 \qquad \left(\frac{B}{\lambda}\right)^l = C, \tag{I-20}$$

from which the unknowns, $l$ and $\lambda$, are easily obtained.

A casual inspection would suggest that there is no particular relation between the Alpha and Lambda functions. However, a simple algebraic transformation shows that the functions may be expressed in the forms:

$$^{\alpha}p^{(t)} = (b + ct)^{-a} \qquad ^{\lambda}p^{(t)} = (b - ct)^{a}, \tag{I-21}$$

so that each is actually only a restricted case of the more general binomial expression $(b + ct)^a$ in which the constants may take either sign. The form (I-21), while more general, will not usually be used in this paper because, in the author's opinion, it is less convenient numerically than (I-10) and (I-19), and the use of the functions requires clarity as to which of the two is involved in any particular case.

The two functions are also connected by another interesting and useful relationship. The values of the "complete integrals" of either function (to be discussed in detail in a later section) are as follows:

$$^{\alpha}F = \frac{a}{a-1} \qquad ^{\lambda}F = \frac{\lambda}{l+1}.$$

We make use of these equations in the following theorem.

Theorem. The class of Alpha and Lambda functions defined by the conditions that the minimum duration be $\tau$, and that all complete integrals shall have the fixed constant value $F$, possess the same limiting curve, the exponential $e^{-t/F}$, as their constants are allowed to vary and increase without limit.

*Proof:*

Let $\alpha = F(a-1)$ and $\lambda = F(l+1)$; then, translating the $p$ axis through a distance $\tau$ so that $\alpha = \alpha'$, or $\lambda = \lambda'$:

1) $$^{\alpha}p^{(t)} = \left[\frac{F(a-1)}{F(a-1)+t}\right]^a \quad \text{and} \quad ^{\lambda}p^{(t)} = \left[\frac{F(l+1)-t}{F(l+1)}\right]^l$$

2) $$\ln {}^{\alpha}p^{(t)} = a\,[\ln F(a-1) - \ln(Fa - F + t)]$$

$$\ln {}^{\lambda}p^{(t)} = l\,[\ln (Fl + F - t) - \ln F(l+1)].$$

When $a$ and $l$ increase without limit, these become indeterminate. We therefore write them as fractions of the form $\frac{0}{0}$ and differentiate numerator and denominator with respect to $a$ and to $l$:

$$3)\quad \lim_{a\to\infty} \ln {}^{a}p^{(t)} = \lim \frac{\dfrac{1}{a-1} - \dfrac{F}{Fa-F+t}}{-a^{-2}}$$

$$\lim_{l\to\infty} \ln {}^{\lambda}p^{(t)} = \ \mathrm{m} \frac{\dfrac{F}{Fl+F-t} - \dfrac{1}{l+1}}{-l^{-2}}$$

$$4)\quad \lim \ln {}^{a}p^{(t)} = \lim\left[-\frac{1}{a^{-2}}\left(\frac{Fa-F+t-Fa+F}{Fa^2-2Fa+ta+F-t}\right)\right]$$

$$\lim \ln {}^{\lambda}p^{(t)} = \lim\left[-\frac{1}{l^{-2}}\left(\frac{Fl+F-Fl-F+t}{Fl^2+2Fl-tl+F-t}\right)\right]$$

$$5)\quad \lim \ln {}^{a}p^{(t)} = -t/F \qquad \lim \ln {}^{\lambda}p^{(t)} = -t/F$$

or

$${}^{a}p^{(t)} = {}^{\lambda}p^{(t)} = e^{-t/F}$$

in the limit.

This relationship will be used later in developing techniques for evaluating "composite" functions.

## II. COMPOUND GRADUATION

More often than not, it will be found that a continuance experience is too complex to be satisfactorily graduated by a single elementary Alpha or Lambda curve. If it is not essential to have a graduation that is reasonably accurate at nearly all durations, an elementary graduation may suffice, but the usual experience encountered will contain inflection points or sharp bends that defy elementary graduation of any accuracy.

It is usually possible to handle such cases with excellent success by means of a "compound" graduation, which is simply based on the assumption that the curve is the sum of two or more components, each of which is itself an elementary function. The rationale involved here is that the total continuance experience under investigation is composed of more than one class of disabilities, each class being subject to its own characteristic force of termination. It will usually be found in a two-element compound experience that one element fits well what would be expected from a group of acute disabilities out of which rapid termination occurs through recovery or death but at a decreasing rate (*i.e.*, an Alpha function with a high attenuation), while the second element consists of long-term or permanently disabled lives from which termination occurs primarily by death and at an increasing rate (Lambda function with large value of range, $\lambda'$). Neither concept will be entirely precise at all durations, to be sure, but the combination usually produces highly satisfactory results, and the advantages of the type of mathematical graduation we are investigating make it well worth the attempt.

1. *Two-Element Compound Curves ("6-point" graduation)*

One means of working up a two-element graduation is by the "6-point graphic" method. The method consists in inspecting the appearance of the continuance data as displayed on full-logarithmic paper, in order to estimate the general position and nature of the underlying elements. From this graphic inspection, 3 points are adopted for the element that appears to control the extreme portion of the curve, and the constants are obtained by the 3-point $p^{(t)}$ method (section I-3c) or by the method of section I-3d. Values of this element are then plotted and the curve subtracted from the data curve, thus exposing the values of the other element which is then likewise plotted. If it appears to be a good candidate

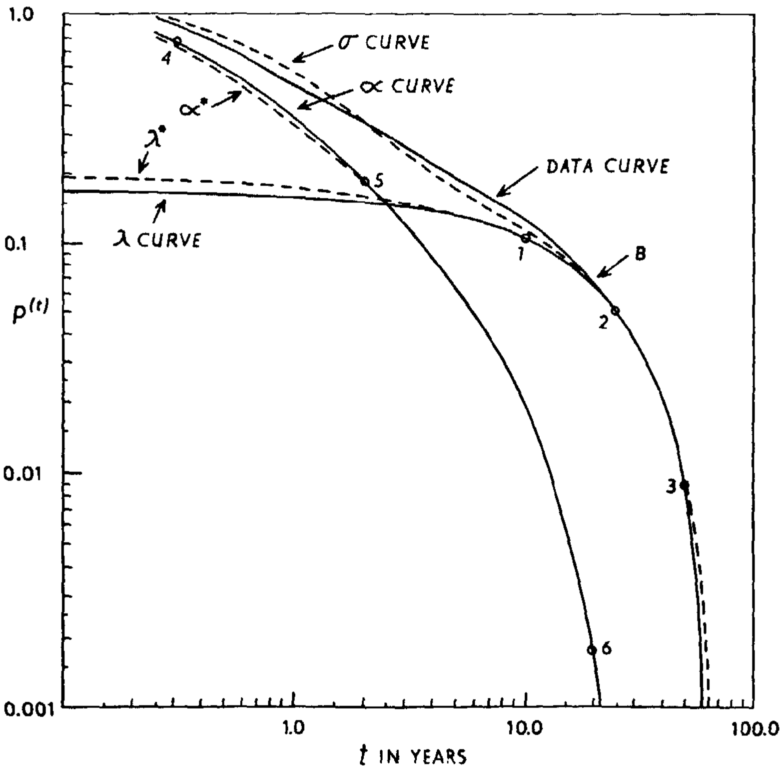


FIG. 3.—1952 Disability Table, Age 27½—Benefit 2. 6-point Graduation

for a final elementary graduation, it too is solved by the 3-point $p^{(t)}$ method. The results, of course, must be checked by comparing sums of elementary values at the same duration with the values of $p^{(t)}$ for the original compound data curve.

We will illustrate the method by converting the 1952 Disability Table at central age $27\frac{1}{2}$ into a two-element mathematical graduation. Figure 3 shows the plotting, setting $p^{(t)} = 1$ at 3 months, where the table begins.

The steps:

1. By referring to the general shape of the data curve and the location of the sharp bend $B$, we decide upon a $\lambda$ curve estimate: the dotted $\lambda^*$ curve, which is a carefully drawn but merely preliminary estimate.

2. Subtracting this curve from the data curve, we derive the estimated curve $\alpha^*$ in order to determine whether its values should have any effect at the extreme right-hand part of the curve that we assume to be controlled almost entirely by the $\lambda$ element, and also to determine whether it appears to have the approximate shape of an elementary function. If it does not, we attempt adjustments in $\lambda^*$ to improve the shape of $\alpha^*$.

3. From the $\lambda^*$ and $\alpha^*$ estimates in Figure 3, we conclude that reasonable values for the 3 $\lambda$ points are (with $t$ in years):

$$\begin{aligned} &1.\ p^{(10)} = .11 \\ &2.\ p^{(25)} = .0528 \\ &3.\ p^{(50)} = .009\,. \end{aligned}$$

Solving by the "3 points of $p$" method, we obtain:

$${}^{\lambda}p^{(t)} = \left(\frac{86.2 - t}{147.4}\right)^{3.35}.$$

(This solution gives an $\omega$ of $27.5 + 86.2 = 113.7$, somewhat high.)

4. This curve is graphed as the $\lambda$ curve, and subtracted from the data curve to obtain the $\alpha$ curve. To solve this we take:

$$\begin{aligned} &4.\ p^{(0.3)} = .78 \\ &5.\ p^{(2)} \ \ = .193 \\ &6.\ p^{(20)} = .00174 \end{aligned}$$

and solving, obtain:

$${}^{\alpha}p^{(t)} = \left(\frac{2.54}{2.465 + t}\right)^{2.914}.$$

5. A final test gives these ratios of the mathematical compound, ${}^{\sigma}p^{(t)} = {}^{\alpha}p^{(t)} + {}^{\lambda}p^{(t)}$, to the original data curve:

| $t$ (Years) | Ratio | $t$ (Years) | Ratio |
|---|---|---|---|
| 0.25 | 98.9% | 5.0 | 90.1% |
| 0.5 | 107.6 | 10.0 | 92.0 |
| 1.0 | 111.3 | 25.0 | 101.7 |
| 2.0 | 97.0 | 50.0 | 100.5 |

In step (4), if the subtracted $\lambda$ curve does not reveal a likely $a$ curve, it may be necessary to modify the points in step (3) to achieve a better solution. Alternatively, the balance curve may sometimes be broken down itself into elements, yielding a three-element compound. In fact, the 6-point graduation illustrated in Figure 3 may be considerably improved by moving to a 3-element (9-point) graduation. In such a case, the solution points of the 3rd element are obtained by deducting the first two elements from the data curve. In Figure 3, the $\lambda$ curve would have been solved by the same equations at $t = 10, 25$ and $50$. The second curve, however, would then be solved by points closer together, such as $t = 1$, 5, and 20. The 3rd element would be used to obtain the fit in the short ranges $t < 1$, and will generally prove to be another $\lambda$ curve. There is seldom any justification for going beyond three, unless the investigator is aiming at a mathematical set of curves intended to reproduce the data very precisely.

The formulas obtained for the age 27.5 curve may be employed to extend the data backward to $t = 0$, although this leaves unanswered any question of unknown peculiarities in the short range. The reasonability of the values in the duration 0 to 3 months should therefore be compared to short-range experience from other sources. Such an extrapolation should not be used, in particular, unless the slope of the $\sigma$ curve is nearly equal to that of the data curve at 3 months. In the Figure 3 example, this condition can only be met with a 3-element graduation. The method of extrapolation is simply that of computing $p^{(0)}$ (which will exceed 1.0, this being the value for $p^{(0.25)}$) and modifying the range constants so as to introduce the factor $1/p^{(0)}$ into all values of $p^{(t)}$, thus setting $p^{(0)} = 1.0$:

$$a'' = a(p^{(0)})^{-1/a}, \qquad \lambda'' = \lambda(p^{(0)})^{1/l}, \qquad \mu'' = \mu(p^{(0)})^{1/m}, \quad \text{(II-1)}$$

where $\mu$ pertains to the third element.

These values $a''$, $\lambda''$, and $\mu''$, are then adopted as the final values of $a$, $\lambda$, and $\mu$ to be used in the mathematical formulas. It is important to keep the rate of claim, $r$, consistent with these adjustments. If $r$ is the rate of claim showing the rate of disability lasting at least 3 months out of the exposure, then $r$ must likewise be modified:

$$r'' = rp^{(0)}, \qquad \text{(II-2)}$$

where $p^{(0)}$ is the original value $>1$.

Note that under these assumptions there is no "minimum duration" (see I-3c). Attenuation sets in immediately at duration zero.

In cases where this method is used with data that can actually be followed from duration zero, it will be necessary to determine the value of the "minimum duration," ${}^{\sigma}\tau$. This will be the value of $t$ at which ${}^{\sigma}p^{(t)} =$

1.0 and is most easily determined by graph, or alternatively by trial and error solution for $\tau$ in the equation (assuming an $\alpha$ and a $\lambda$ element):

$$^{\alpha}p^{(\tau)} + {}^{\lambda}p^{(\tau)} = 1 . \qquad \text{(II-3)}$$

From this, we establish the complete definition of ${}^{\sigma}p^{(t)}$:

$$\begin{aligned} &1)\ {}^{\sigma}p^{(t)} = 1.0 && (0 \leq t \leq {}^{\sigma}\tau) \\ &2)\ {}^{\sigma}p^{(t)} = {}^{\alpha}p^{(t)} + {}^{\lambda}p^{(t)} && ({}^{\sigma}\tau \leq t \leq \lambda') \quad \text{(II-4)} \\ &3)\ {}^{\sigma}p^{(t)} = {}^{\alpha}p^{(t)} && (t \geq \lambda') . \end{aligned}$$

2. *Select and Ultimate Graduation*

Instead of using the two-element compound method described in the previous section, it is often possible to achieve excellent results by graduating disability data in two segments, one covering the select period and the other the ultimate durations. This approach is also easier to work out, leads to simpler evaluation of final values, and corresponds better to the usual type of experience study which merges all data at durations beyond some reasonable select period into a single body of ultimate data.

In order to achieve a smooth transition from the select to the ultimate period, it is convenient to use the one point of $p$, two points of $\pi$, technique of section I-3d.

We first solve for the element that approximates the ultimate period, which will ordinarily be a Lambda function. To illustrate, we will turn again to the 1952 Disability Table.

The ultimate portion of this table begins after 15 years. We obtain $\lambda'$ and $l$ for the ultimate function by graphically deriving two $\pi$ values, using ultimate attained age less 40 as the age variable, $x$, in order to spread out the data for accurate determination of the logarithmic tangent:

$$^{(u)\lambda}\pi_{50} = -\frac{z_{10}}{10} = \frac{l}{\lambda' - 50} = .04587$$

$$^{(u)\lambda}\pi_{80} = -\frac{z_{40}}{40} = \frac{l}{\lambda' - 80} = .12797 .$$

Solving, we obtain:

$$l = 2.145 \qquad \lambda' = 96.76 .$$

$\lambda$ is left unsolved, since this constant will be used to equate the select and ultimate values of $p^{(t)}$ at duration $t = 15$ for each disability age.

We then set up equations for the select curve, using central age 27.5 again for illustration:

$$^{(s)\alpha}p_{27.5}^{(0.25)} = 1 \quad {}^{(s)\alpha}\pi_{27.5}^{(1)} = -\frac{z^{(1)}}{1} = .5039 \quad {}^{(s)\alpha}\pi_{27.5}^{(15)} = {}^{(u)\lambda}\pi_{42.5} = .03953 .$$

${}^{(s)}\pi^{(15)}$ is equated to ${}^{(u)}\pi$ for smooth transition into the ultimate curve. Solving these equations we get:

$$a = .4418 \qquad a' = .1918 \qquad a = .6005 .$$

To obtain $\lambda$ for disability age 27.5, we then set

$$ {}^{(s)a}p_{27.5}^{(15)} = {}^{(u)\lambda}p_{42.5} = \left(\frac{\lambda' - 42.5}{\lambda}\right)^{l} = .119 ,$$

which gives $\lambda = 146.7$.

The same values of $\lambda'$ and $l$ will thus apply for all disability ages, but $\lambda$ must be solved for each age to join the select and ultimate portions of the curve, which will make a smooth juncture since the juncture values of $\pi$ have also been equated. In the select period, $t$ is in terms of duration, and thereafter in terms of attained age, unless $\lambda'$ be adjusted to give terminal duration rather than terminal age $\omega$.

We have the following ratio test of the mathematical values of $p^{(t)}$ to those of the data:

| $t$ (Years of Duration) | Ratio | $t$ (Years of Duration) | Ratio |
|---|---|---|---|
| 0.25 | 100 % | 5.0 | 114.5% |
| 0.5 | 102.6 | 10.0 | 116.1 |
| 1.0 | 108.4 | 25.0 | 145.5 |
| 2.0 | 106.8 | 50.0 | 160.0 |

These ratios show a considerably poorer fit than those of the two-element graduation of the previous section. The reason is that we have equated the mathematical curve to the absolute value of the data curve only once: at $p^{(0.25)} = 1$, relying only on $\pi$ values thereafter. By sacrificing the requirement that the select and ultimate forces of termination be equal at 15 years duration, we could have materially improved the fit by solving for the select curve entirely on select values. The curves may also be determined by 3-point $p^{(t)}$ graduation, which, with its better control of the absolute $p$ values, would produce a better fit throughout. In addition, compound graduation may be applied to either select or ultimate periods, or both.

Before leaving this illustration, it should also be mentioned that the Lambda constants will often be materially affected by the selection of the points, depending on how well the ultimate data really fit a Lambda curve. The same, of course, is true of the Alpha select curve. Some experimentation with the equation points may therefore lead to great improvement in the over-all fit. The above example is only illustrative of method. These particular results would need to be improved in actual practice.

### 3. *General Methods of Compound Graduation*

The graphic method described in section 2 is a powerful graduating tool and has wide application, and can be used for compounds of more than two elements by simply extending the technique as far as required. The principal talent that must be acquired is sufficient familiarity with elementary and compound curve-forms so as to be able, without too much trouble, to break the data into the acceptable number and types of elements needed. Steps 1 and 2 are 90% of the job in the graphic method.

The 1952 Disability Table can be handled quite well with 3-element $\alpha, \lambda, \mu$ compounds. The Group Hospital Experience developed by Gingery (*TSA* IV) and the closely similar hospital continuance data summarized by Bartleson and Olsen (*TSA* IX) are well represented by means of compounds involving two $\alpha$ elements.

Tables of values for these and other experiences are shown in the Appendixes. Some have been refined by further techniques yet to be discussed, but all are basically derived by the graphic technique of section 1, or that of section 2.

## III. CONTINUANCE INTEGRALS

We now reach the most important objective of our investigation, which is the evaluation of the average size of claim, or the "claim annuity" in disability continuance where interest discount becomes significant. In discussing the basic nature of continuance functions, we stated that a practical mathematical function should produce reasonably simple expressions when carried through two successive integrations. The first of these was carried out in the derivation of the formula for $p^{(t)}$. The second is now required in obtaining the average size of claim, for which we will adopt the symbol $F$.

### 1. *Incomplete Continuance Integrals*

The average size of claim for a continuance experience truncated at duration $T$ is given by the relation:

$$F^T = \int_0^T p^{(t)} dt = \int_0^\tau dt + \int_\tau^T p^{(t)} dt , \qquad \text{(III-1)}$$

since $p^{(t)} = 1$ for $t < \tau$. In the case of an Alpha function, we have:

$$1) \quad {}^{\alpha}F^T = T , \qquad \text{for } 0 \leq T \leq \tau$$

and

$$2) \quad {}^{\alpha}F^T = \tau + \int_\tau^T \left( \frac{a}{a' + t} \right)^{\alpha} dt , \qquad \text{for } T > \tau$$

$$\int_{\tau}^{T}\left(\frac{a}{a'+t}\right)^{a} dt = \frac{a^{a}}{1-a}\Big[(a'+t)^{1-a}\Big]_{a-a'}^{T}$$

$$= \frac{a^{a}}{a-1}\left[\frac{1}{a^{a-1}} - \frac{1}{(a'+T)^{a-1}}\right]$$

$$= \frac{a}{a-1}\left[1-\left(\frac{a}{a'+T}\right)^{a-1}\right] = {}^{a}F'^{\,T}.$$

This development does not, of course, apply when $a = 1$, a situation which would almost never occur in practice. In this case, we get:

$${}^{a}F'^{\,T} = \int_{\tau}^{T}\frac{a}{a'+t}\,dt = a\,[\ln(a'+t)]_{a-a'}^{T}$$

$$= a\,[\ln(a'+T) - \ln a] \qquad = \ln\left(\frac{a'+T}{a}\right)^{a}.$$

${}^{a}F'^{T}$, the value of the definite integral alone, is called the "incomplete" integral. Allowing $T$ to increase without limit in the expression for ${}^{a}F'^{T}$, we obtain the "complete" integral, ${}^{a}F'$ (for $a > 1$):

$${}^{a}F' \;(= {}^{a}F'^{\infty}) \;= \frac{a}{a-1},$$

a delightfully simple expression. It is thus quite easy to describe the average size of claim for Alpha functions. The complete definitions are:

$$1)\;\; {}^{a}F' = \frac{a}{a-1} \qquad 2)\;\; {}^{a}F = \frac{a}{a-1} + \tau \tag{III-2}$$

$${}^{a}F'^{\,T} = {}^{a}F'\left[1-\left(\frac{a}{a'+T}\right)^{a-1}\right], \quad \text{for } T > \tau \tag{III-3}$$

$$\begin{aligned} &1)\;\; {}^{a}F^{T} = T, && \text{for } 0 \le T \le \tau \\ &2)\;\; {}^{a}F^{T} = {}^{a}F'^{T} + \tau, && \text{for } T \ge \tau. \end{aligned} \tag{III-4}$$

In (III-3), the fractional quantity, called the "root," is denoted by the symbol ${}^{a}R^{T}$, and this quantity raised to the power $a - 1$ is denoted by $f'^{T}$. We thus have the convenient expressions:

$${}^{a}R^{T} = \frac{a}{a'+T} \tag{III-5}$$

$${}^{a}p^{(T)} = ({}^{a}R^{T})^{a} \tag{III-6}$$

$${}^{a}f'^{T} = ({}^{a}R^{T})^{a-1} \tag{III-7}$$

$${}^{a}f^{T} = 1 - {}^{a}f'^{T} \tag{III-8}$$

$${}^{a}F'^{T} = {}^{a}F' \cdot {}^{a}f^{T}. \tag{III-9}$$

For Lambda functions, we have:

$$F'^{T}=\int_{\tau}^{T}\left(\frac{\lambda'-t}{\lambda}\right)^{l}dt=-\frac{1}{\lambda^{l}(l+1)}\left[(\lambda'-t)^{l+1}\right]_{\lambda'-\lambda}^{T}$$

$$=\frac{1}{\lambda^{l}(l+1)}\left[\lambda^{l+1}-(\lambda'-T)^{l+1}\right]=\frac{\lambda}{l+1}\left[1-\left(\frac{\lambda'-T}{\lambda}\right)^{l+1}\right].$$

From this we have the complete definition:

$$1)\ {}^{\lambda}F'={}^{\lambda}F'^{\lambda'}=\frac{\lambda}{l+1}\qquad 2)\ {}^{\lambda}F=\frac{\lambda}{l+1}+\tau \qquad \text{(III-10)}$$

$${}^{\lambda}F'^{T}={}^{\lambda}F'\left[1-\left(\frac{\lambda'-T}{\lambda}\right)^{l+1}\right],\qquad \text{for } \tau\leq T\leq\lambda' \quad \text{(III-11)}$$

$$\begin{aligned}
&1)\ {}^{\lambda}F^{T}=T\,, && \text{for } 0\leq T\leq\tau\\
&2)\ {}^{\lambda}F^{T}={}^{\lambda}F'^{T}+\tau\,, && \text{for } \tau\leq T\leq\lambda' \quad \text{(III-12)}\\
&3)\ {}^{\lambda}F^{T}={}^{\lambda}F\,, && \text{for } T>\lambda'\,.
\end{aligned}$$

It should be noted that ${}^{\lambda}F'={}^{\lambda}F'^{\lambda'}$, while ${}^{\alpha}F'={}^{\alpha}F'^{\infty}$.

The subsidiary expressions are:

$${}^{\lambda}R^{T}=\frac{\lambda'-T}{\lambda} \qquad \text{(III-13)}$$

$${}^{\lambda}p^{(T)}=({}^{\lambda}R^{T})^{l} \qquad \text{(III-14)}$$

$${}^{\lambda}f'^{T}=({}^{\lambda}R^{T})^{l+1} \qquad \text{(III-15)}$$

$${}^{\lambda}f^{T}=1-{}^{\lambda}f'^{T} \qquad \text{(III-16)}$$

$${}^{\lambda}F'^{T}={}^{\lambda}F'\cdot{}^{\lambda}f^{T}\,, \qquad \text{(III-17)}$$

setting ${}^{\lambda}f^{T}=1$ for $T>\lambda'$.

We are often interested in the average size of claim included between two values of $t$, as, for example, when we desire the value of a disability benefit with a 30 day elimination period payable for a maximum duration of 1 year. The desired expression, called a "bounded" continuance integral, is given by:

$$F^{T_1:T_2}=F^{T_2}-F^{T_1}=F'(f'^{T_1}-f'^{T_2})\,. \qquad \text{(III-18)}$$

The pre-superscripts are not shown because the formula as expressed here is identical for Alpha and Lambda functions. In the bounded integral, there is no distinction between $F$ and $F'$ so long as the limits lie between $\tau$ and $\infty$ for Alpha functions, and between $\tau$ and $\lambda'$ for Lambda functions. If either limit lies outside of these boundaries, the proper adjustment must be made.

From the desired $F$ value, the value of $S$ follows immediately:

$$S_x^{T_1;T_2} = r_x F_x^{T_1;T_2} \qquad \text{(III-19)}$$

## 2. *Adjustment for Discrete Experience*

The expressions of section 1 assume that the benefit involved is of a continuous nature. This is not in general the case, since in the actual situation a person will be "disabled," as a rule, for a complete day or none, or be charged a full day's hospital room and board charge, or none. It is easy to approximate the necessary adjustment. In Figure 4, the value

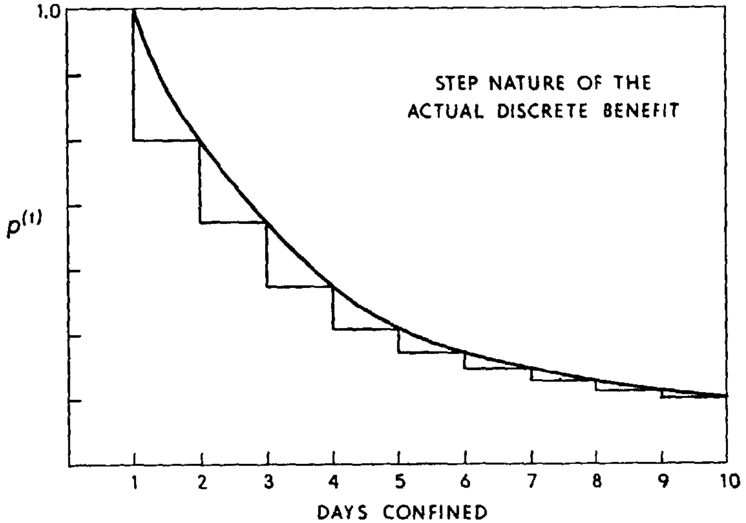


FIG. 4

of the discrete benefit is represented graphically, using here, for the sake of clarity, rectangular units. A hospital confinement function is assumed.

Assuming that the mathematical function was developed from crude data showing the number confined at the beginning of each $t$th day, we have the situation shown in the graph, where the values of successive days' benefits are represented by a series of diminishing rectangles.

It may be seen that the area (the continuance integral) under the curve exceeds the actual benefit by the value of a series of approximate triangles, whose combined area out to any day $T$ is given by the expression:

$$A \doteq \tfrac{1}{2}(1 - p^{(T)}) ,$$

since the base of each triangle equals 1 unit.

Hence we have the adjusted "discrete" approximations:

$$F^{T} \fallingdotseq F'^{T} + \tau - \tfrac{1}{2}q^{(T)} \tag{III-20}$$

$$F^{T_1;T_2} \fallingdotseq F'(f'^{T_1} - f'^{T_2}) - \tfrac{1}{2}(p^{(T_1)} - p^{(T_2)}). \tag{III-21}$$

It is important to bear in mind here that the discrete unit must be the same as the unit of the variable $t$ in the formulas. In disability continuance, where the unit of the variable may be months or years whereas the discrete breaks ordinarily occur daily, this adjustment may usually be ignored. It may be of consequence in a Waiver of Premium benefit, however.

3. *Expressions for Compound Functions*

The $F$ function for a compound function is simply the sum of the appropriate elementary integrals:

$${}^{\sigma}F^{T} = {}^{\alpha}F'^{\tau;T} + {}^{\beta}F'^{\tau;T} + \ldots + {}^{\lambda}F'^{\tau;T} + {}^{\mu}F'^{\tau;T} + \ldots + {}^{\sigma}\tau\,. \tag{III-22}$$

It is of importance here to recognize the possible differences in the limits of integration of each element. The lower limit is always ${}^{\sigma}\tau$, which gives a bounded, not an incomplete integral. The upper limit depends, in the case of Lambda functions, upon whether or not $T$ exceeds the $\lambda'$ constant. For example, in a two-element $\alpha$, $\lambda$ compound,

$${}^{\sigma}F^{T} = {}^{\alpha}F'^{\tau;T} + {}^{\lambda}F'^{\tau;\lambda'} + {}^{\sigma}\tau\,, \qquad \text{for } T > \lambda'\,. \tag{III-23}$$

4. *The Disabled-Life Annuity*

Where we wish to take account of interest, the value of the claim annuity is given by:

$$F^{T} = \int_{0}^{T} p^{(t)}\, v^{t} dt\,. \tag{III-24}$$

For an Alpha function this becomes:

$${}^{\alpha}F^{T} = \int_{0}^{\tau} v^{t} dt + \int_{\tau}^{T} \left(\frac{a}{a'+t}\right)^{a} v^{t} dt\,. \tag{III-25}$$

The second term of this expression cannot be evaluated in terms of any elementary mathematical functions, so that we must resort to approximate methods if we are to avoid carrying out an entire summation of commuted values in order to get the annuity value—labor that our methods are intended to avoid.

The function we are dealing with here, $p^{(t)}v^{t}$, can usually be approximated quite well by simply using elementary Alpha or Lambda functions to approximate discounted elementary functions. This approach gives excellent accuracy so long as the attenuation of $p^{(t)}$ is greater than about

1.2, which is almost always the case, and so long as $i$ is less than about .05, again usually the case. The ratios of the approximated to the true values can be made very close to unity over nearly all of the significant range of the curve, with the approximation curve beginning to overstate toward the extreme range of the curve when an Alpha function is used, and slightly understating with a Lambda function.

It is usually quite satisfactory, and very quick, to employ the method of section I-3d, *i.e.*, one point of $p$ and two points of $\pi$, to obtain the constants of an elementary function approximating the discounted curve. We have these relations, assuming an Alpha function for the undiscounted continuance:

$$\begin{aligned} {}_{a}\overset{(i)}{\pi}{}^{t} &= -D_t \ln p^{(t)} v^t = -D_t[t \ln v + a(\ln a - \ln \overline{a' + t})] \\ &= {}^{a}\pi^{(t)} + \ln(1+i) = {}^{a}\pi^{(t)} + \delta\,. \end{aligned} \tag{III-26}$$

For a Lambda function, we have the similar result

$$ {}^{\lambda}\overset{(i)}{\pi}{}^{(t)} = {}^{\lambda}\pi^{(t)} + \delta\,. \tag{III-27}$$

It will be evident that the force of termination for a discounted Alpha function decreases with $t$, and for a Lambda function increases. Hence a discounted Alpha function will be approximated by an Alpha function, and a discounted Lambda function by a Lambda function. The constants are solved by taking, assuming an Alpha function,

$$\begin{aligned} &1)\quad {}^{(i)a}p^{(u)} = {}^{a}p^{(u)}\,v^u \\ &2)\quad {}^{(i)a}\pi^{(v)} = {}^{a}\pi^{(v)} + \delta \\ &3)\quad {}^{(i)a}\pi^{(w)} = {}^{a}\pi^{(w)} + \delta\,. \end{aligned}$$

The selection of $u$, $v$, and $w$ and the solution of the function are accomplished by the methods of I-3d. It is important to remember that $\delta$ must correspond to the effective rate of interest over the same unit period as the time unit of the continuance function.

We then have for the claim annuity,

$$\begin{aligned} {}_{a}\overset{(i)}{F}{}^{T} \doteq {}^{(i)a}F^{T} &= \int_0^{\tau} v^t\,dt + \int_{\tau}^{T} {}^{(i)a}p^{(t)}\,dt \\ &\doteq \bar{a}_{\overline{\tau}|} + {}^{(i)}\left\{\frac{a}{a-1}\left[1 - \left(\frac{a}{a'+t}\right)^{a-1}\right]\right\} \end{aligned} \tag{III-28}$$

where the $(i)$ pre-superscript indicates the altered constants of the approximate discounted Alpha function.

If $\tau$ is small, as will usually be the case, the first term may be taken simply as $\tau$. Also, if the time-unit is years, or even months, any adjustment for the discrete nature of the experience by days may be ignored. With waiver of premium, the adjustment may become significant. (See section III-2.)

In the case of lifetime benefits, where some terminal age $\omega$ is usually assumed, the lifetime annuity may be evaluated with the complete integral or with an incomplete integral with upper limit $\omega$. The difference will usually be negligible, and it is ordinarily preferable (and simpler) to use the complete integral, which in effect means that no terminal duration or limiting age of the disabled life table is assumed for Alpha functions. It should be noted that using continuance integrals for the evaluation of the disability claim annuity is equivalent to assuming disabled life values that are select at all durations unless select and ultimate segments are employed. This is no problem when continuance integrals are used, since the functions are so concisely defined.

Where greater precision is desired in evaluating discounted functions, as may occasionally be necessary in the evaluation of reserves on disability claims, recourse may be had to approximation of the discounted curve in segments, using two or more elementary functions. Where the original undiscounted curve is an Alpha function, this is often best accomplished by employing an exponential curve (with constant $\pi^{(t)} = \delta + \epsilon$) for the long range, and an Alpha function for the short range. In such a case, the resulting exponential function will be of the form

$$^{e}p^{(t)} = e^{-[(\delta+\epsilon)t+k]} , \qquad \text{(III-29)}$$

where $\epsilon =$ some $\pi^{(T)}$ within the long range segment. Where a select and ultimate curve (see section II-2) is discounted, each segment must be separately approximated.

## 5. *Error of Approximation of Discounted Functions*

In approximating discounted elementary functions by means of elementary functions, it is important to have some idea of the degree of error involved. This may be done by fitting a series of functions to succeeding segments of the curve and comparing the results to those obtained with a simple elementary approximating function. Two or three successive segments will generally provide very high precision, and the sum of the integrals of each of these, between the proper limits, will give a highly accurate approximation. In most cases it will be found that the single element approximation is quite adequate.

When long term disability continuance is required, the selection of the equating points $u$, $v$, and $w$ must be such that the approximating curve will not begin to diverge materially for a considerable duration. For example, in approximating the 3% discount of the $a$ element of the 1952 Disability curve for age 27.5 (see section II-1), if we take

$$u = \ .2$$

$$v = \ 1.2$$

$$w = 10$$

the resulting approximation will not begin to diverge greatly before about 40 years of duration, which is enough to insure fair accuracy even in evaluating lifetime benefits. This curve will serve as a good illustration of the error of approximation.

The function, discounted, is:

$${}_a^{(.03)}p_{27.5}^{(t)} = 1.03^{-t}\left(\frac{2.54}{2.465+t}\right)^{2.914} .$$

First, obtaining the single element approximation, we have:

$${}_a^{(.03)}p(0.2) = .864 \qquad {}_a^{(.03)}\pi(1.2) = .795 + .0296 = .8246$$

$${}_a^{(.03)}\pi(10) = .234 + .0296 = .2636$$

These relations yield the single function[1]

$${}^{(.03)}a(3.00,\ 2.93,\ 3.41).$$

Evaluation of the complete integral gives:

$${}^{(.03)}F \doteq .07 + \frac{3}{2.41} = 1.315 .$$

Next, making a 3-segment approximation over the intervals

(1) $<0, 2>$ (2) $<2, 15>$ (3) $<15, \infty>$

we have, for (1):

$${}_a^{(.03)}p(0.2) = .864 \ , \qquad {}_a^{(.03)}\pi(0.4) = 1.0171 + .0296 = 1.0467 \ ,$$

$${}_a^{(.03)}\pi(1.3) = .7740 + .0296 = .8036$$

yielding

$${}^{(.03)}a\,(2.648,\ 2.575,\ 3.114)\ ;$$

for (2):

$${}_a^{(.03)}p(2.0) = .1829 \ , \qquad {}_a^{(.03)}\pi(4.0) = .4507 + .0296 = .4803 \ ,$$

$${}_a^{(.03)}\pi(10.0) = .2338 + .0296 = .2634$$

[1] See Appendix A, *System of Notation*, 2 f (3).

yielding

$$^{(.03)}a(3.255,\ 3.286,\ 3.499)\ ;$$

for (3):

$$a\overset{(.03)}{p}(15.0) = .00233\ , \qquad a\overset{(.03)}{\pi}(30.0) = .0898 + .0296\ = .1194\ ,$$

$$a\overset{(.03)}{\pi}(100.0) = .0284 + .0296 = .0580\ ,$$

yielding

$$^{(.03)}a(23.72,\ 36.12,\ 7.895)\ .$$

The evaluation of $F$ is then:

$$F = .073 + \frac{2.648}{2.114}\left[1 - \left(\frac{2.648}{4.575}\right)^{2.114}\right] \qquad \text{over } \langle 0,\ 2 \rangle$$

$$+ \frac{3.255}{2.499}\left[\left(\frac{3.255}{5.286}\right)^{2.499} - \left(\frac{3.255}{18.286}\right)^{2.499}\right] \quad \text{over } \langle 2,\ 15 \rangle$$

$$+ \frac{23.72}{6.895}\left[\left(\frac{23.72}{51.12}\right)^{6.895}\right] \qquad \text{over } \langle 15,\ \infty \rangle$$

$$= .073 + .8706 + .3707 + .0172 = 1.332\ ,$$

so that the difference here from the single element approximation is $+.017$. Since the single element overstates in the extreme range, the error will be slightly greater for evaluations of the incomplete integral.

## 6. *Time Units, and Conversion of Units*

Thus far, we have dealt with disability continuance and discount using a time unit of one year. The numerical values of the functions given are for annuities of \$1 annually payable continuously. Any unit may be used, of course, and it is of value to consider how a given function may be transformed to convert units.

Given the constants for a function in units of, say, one year, we must have the following relation for the equivalent function where $t'$ is in months:

$$\left[\left(\frac{a}{a'+t}\right)^{a}\right]^{\text{yr.}} = \left[\left(\frac{\beta}{\beta'+t'}\right)^{a}\right]^{\text{mo.}},$$

where $t' = 12t$.

The equality will obviously hold for all $t'$ if we have:

$$\beta = 12a$$

$$\beta' = 12a'\ ,$$

so that we have the very simple rule that to convert from a unit $t$ to a new unit $t/x$, the range constants are all multiplied by the factor $x$, the attenuation remaining unchanged.

In the settlement of accident and sickness disability claims, the typical procedure where income is payable monthly is to regard a day as 1/30 of a month, and, hence, as 1/360 of a year, so that the indemnity for a period equal to a fraction of a month at the end of a disability is equal to

$$\frac{x}{30} \times \text{Monthly Income}\,,$$

where the fraction is $x$ days.

This is the practice followed in my Company for reasons of simplicity, and it is evident that we are dealing here with a "day" that is not quite the same as a calendar day, and is, in fact, a variable unit depending on the dates of the year over which claim is paid. To distinguish this unit from a calendar day, we record our claim experience in units of one marshall, defined thus:

$$\begin{aligned} 1 \text{ marshall} &= 1 \text{ day, for fractions of a month}\,. \\ 30 \text{ marshalls} &= 1 \text{ month, for any month of the year}\,. \\ 360 \text{ marshalls} &= 1 \text{ year, either ordinary or leap year}\,. \end{aligned}$$

This provides a convenient and consistent record of experience which is readily summarized and converted to monthly or yearly units in morbidity investigations, from which experience continuance functions are derived.

## 7. *F-Function Testing of Graduation*

In discussing the testing of a graduation of the $p^{(t)}$ function (section I-3), mention was made of the necessity of testing the continuance integral. This test consists in comparing values of $F^T$ computed by

| Col. 1<br>End of Interval $(=t)$ | Col. 2<br>Number of Claims Terminating in Interval | Col. 3<br>Amount of Claims in Col. 2 | Col. 4<br>$\sum_{250}^{t}$ of Col. 2 | Col. 5<br>$\sum_{25}^{t}$ of Col. 3 | Col. 6<br>Col. 5 + $t\times$(Col. 4)$_{+1}$ | Col. 7<br>(Col. 4) $\div 978$ $(=(p^{(t)})_{-1})$ | Col. 8<br>(Col. 6) $\div 978$ $(=F^t)$ |
|---|---|---|---|---|---|---|---|
| $ 25 | 84 | $ 1,596 | 978 | $ 1,596 | $23,946 | 1.000 | $24.48 |
| 50 | 233 | 8,621 | 894 | 10,217 | 43,267 | .914 | 44.24 |
| 75 | 210 | 12,810 | 661 | 23,027 | 56,852 | .676 | 58.13 |
| 100 | 186 | 15,252 | 451 | 38,279 | 64,779 | .461 | 66.24 |
| 125 | 124 | 13,640 | 265 | 51,919 | 69,544 | .271 | 71.11 |
| 150 | 83 | 11,039 | 141 | 62,959 | 71,659 | .144 | 73.27 |
| 200 | 41 | 6,806 | 58 | 69,764 | 73,164 | .059 | 74.81 |
| 250 | 17 | 3,672 | 17 | 73,436 | 73,436 | .017 | 75.09 |
| Total | 978 | $73,436 | ...... | .......... | .......... | .......... | ......... |

formulas against the crude values obtained by summation of the crude data. The crude data may be conveniently processed as shown in the table on the opposite page, assuming that the experience as first summarized is of the form exhibited in columns 1–3. This hypothetical example might be the means of dealing with a body of data giving miscellaneous hospital expense experience by size of claim. For the sake of brevity, we will assume that the experience expires at \$250.

The columns are self-explanatory with the possible exception of column 6, which gives the total amount of claim included within the first \$$t$, in the experience sample. Column 8 then gives the crude values of $F^t$. Column 7 gives $p^{(t)}$, offset one line. The function obtained by graduating column 7, having been tested against column 7 or a preliminary graduation of this column, must then be integrated using formula (III-4) or (III-12) and ratioed to column 8. These ratios are a further test of the graduation; and if any values are much under 100%, or if the trend in the ratios moves consistently upward or downward with increasing $t$, the graduation is defective. Short of a perfect fit, the best results are values running between 100% and a slightly higher value.

The $F$-function may be used itself as the basis of graduation, and this approach has the double advantage of (1) employing a crude function that contains a measure of "natural" graduation, being a summation function, and (2) directly assuring more reasonable $F$-function tests. It can have the disadvantage of being less faithful to the values of $p^{(t)}$, which is not too important unless the derived function is to be used as a basis of computing open claim reserves, to be referred to later, or used to derive a wide variation of bounded values, which requires that the function give acceptable accuracy over all intervals needed.

## 8. *Graduation on the F-Function*

Solution of the constants by using the $F$-function is a more difficult problem than is the case with the $p$-function, since the unknowns cannot be readily eliminated from the equations. Probably the easiest technique is the following, using the Alpha function as illustration:

We have:

$$F^T = \frac{a}{a-1}\left[1-\left(\frac{a}{a'+T}\right)^{a-1}\right]+\tau$$

and

$$F^{T_1;T_2} = \frac{a}{a-1}\left[\left(\frac{a}{a'+T_1}\right)^{a-1}-\left(\frac{a}{a'+T_2}\right)^{a-1}\right]$$

$$= \frac{a^a}{a-1}\left[\left(\frac{1}{a'+T_1}\right)^{a-1}-\left(\frac{1}{a'+T_2}\right)^{a-1}\right].$$

We obtain data values of $F^{T_1;T_2}$, $F^{T_1;T_3}$, and $F^{T_1;T_4}$, and set up the following equations in $\alpha'$ and $a$:

$$\frac{F^{T_1;T_2}}{F^{T_1;T_3}}=\frac{\left(\frac{1}{\alpha'+T_1}\right)^{a-1}-\left(\frac{1}{\alpha'+T_2}\right)^{a-1}}{\left(\frac{1}{\alpha'+T_1}\right)^{a-1}-\left(\frac{1}{\alpha'+T_3}\right)^{a-1}}$$

$$\frac{F^{T_1;T_2}}{F^{T_1;T_4}}=\frac{\left(\frac{1}{\alpha'+T_1}\right)^{a-1}-\left(\frac{1}{\alpha'+T_2}\right)^{a-1}}{\left(\frac{1}{\alpha'+T_1}\right)^{a-1}-\left(\frac{1}{\alpha'+T_4}\right)^{a-1}}. \qquad \text{(III-30)}$$

This yields two equations in two unknowns, but the expressions are not in a form such that one unknown may be readily eliminated, and we have a situation where simultaneous trial and error in two unknowns confronts us. To simplify the problem, we obtain a preliminary solution of the constants by means of the data $p$ function. This may be done either by the 3-point $p^{(t)}$ method using the $p$ data, or else by graphically constructing tangents to a graphically smoothed $F$ curve, employing the proper adjustments to the value of the slope if the work is done on logarithmic or semilogarithmic paper, from which approximate values of $p$ may be easily obtained. The $p$ values so determined are also useful in determining whether an Alpha or a Lambda function is required, by employing the $\pi$-test (formula (I-13)).

From this preliminary solution, we take the constant $a$ and solve for $\alpha'$ in the first of equations (III-30), using the preliminary $\alpha'$ value as an initial test. Let this result be $\alpha^*$.

Substituting $a$ and $\alpha^*$ in the second of equations (III-30), we inspect the resultant inequality. If the substitution yields a value less than the data ratio, $a$ is too large. The opposite inequality means that $a$ is too small. A new trial value of $a$ must then be adopted from these results, and equations (III-30) used again, the process being repeated, with the aid of interpolation when possible to get improved trial values, until acceptable approximation is achieved.

The constant $\alpha$ may then be easily obtained from the absolute value of $F^{T_1;T_2}$ (or either of the other two bounded values).

By abandoning the requirement that the constant addition to $F'^{T_1}$ to give $F^{T_1}$ be $\tau = \alpha - \alpha'$, it is possible to obtain a formula of 4 constants for $F^T$ so that we can equate to all four values, at $T_1$, $T_2$, $T_3$, and $T_4$, thus employing a 4-point $F$ graduation. We simply solve for the constant $\epsilon$ in the following formula equated to the data value of $F^{T_1}$ (or any of the other data values):

$$F^T = \frac{a}{a-1}\left[1-\left(\frac{a}{a'+T}\right)^{a-1}\right]+\epsilon. \qquad \text{(III-31)}$$

This is then used as the $F$ formula, which should be checked to verify that it reproduces the data values at all 4 values of $T$. It must be remembered that this, in effect, may introduce a slight inconsistency into the underlying $p$ curve of which $F$ is the integral. To minimize this inconsistency and to obtain a proper fit over most ranges of $F$, $T_1$ should be taken near the value of $\tau$, $T_2$ and $T_3$ at moderate to intermediate durations, and $T_4$ at the longest duration for which any significant data remain.

When a compound graduation appears to be desirable, the easiest procedure is to graduate one element on the more easily handled $p$ curve, and deduct the integral of this curve from the data to get the $F$ values for the second, which can then be solved on its $F$-function if this is desired. More laborious and refined methods can be devised to yield equality at more than 4 values of $T$, but this is not usually necessary. Compound $p$-graduations generally produce acceptable results without laborious solution of multiple $F$-values.

## IV. COMPOSITE FUNCTIONS

We will now turn to what is perhaps the most complex basic problem confronting the accident and sickness actuary: the evaluation of claim costs for multiple medical benefits subject to common deductibles and maximums. These are features usually found in modern medical expense policies, especially the major medical type.

The great variety of possible benefit combinations inherent in medical coverage makes the combined problems of reduction of experience data, consistency, standardization, and equitable evaluation of costs enormously difficult. Conceivably, we could analyze each of the basic plans in force as though each involved an independent body of experience. It is, however, far better if some practical interrelationship can be set up, so that experience on a variety of plans can be reduced to a fairly simple assortment of basic costs and rates from which the basic values for any desired multiple benefit plan may be constructed with reasonable simplicity and accuracy.

We therefore seek methods which will be of help in meeting each of the following criteria:

(1) *Reduction of data.* If possible, we want to be able to combine experience from different plans into a single basic body of experience values, or at least a limited assortment of basic values, in order to provide the advantages of volume experience and of simple basic tables.

(2) *Consistency.* A corollary of criterion (1) is consistency among plans. We can only be vaguely assured of consistency between plans that cannot be related mathematically to the same basic measures.

(3) *Standardization.* A further corollary is the standardization of basic values, both for premium computation and, especially, for reserve valuation purposes, where it is desirable to achieve uniform basic minimum standards for intercompany use, without resorting to voluminous and complex published tables.

(4) *New requirements.* It is obviously helpful if the cost of new benefit combinations can be constructed with some confidence from existing basic values, which may in themselves be derived from experience on plans not entirely identical with what is sought.

The techniques now to be described are aimed at answering each of these considerations. We will not attempt to develop mathematically exact formulas, but resort instead to several convenient approximate techniques which are easily derived and fairly simple to use.

1. *System of Notation*

In the several illustrative examples that will be described, the System of Notation of Appendix A will be employed. It is recommended that some familiarity with this be acquired, so that notational expressions may be readily interpreted and manipulated.

2. *Terminology*

Several new definitions are convenient.

*a*) *Coincidence.* Two benefits are coincident if they occur simultaneously, that is, on the same disability.

*b*) *Exclusion.* Benefits that are not "coincident" are "exclusive." Thus we may have surgery requiring hospitalization, and surgery that does not involve hospitalization, assuming that both do not occur together in one disability.

*c*) *Correlation.* Benefits that are coincident may be correlated to varying degrees. By a correlation of 1.0 between two benefits, we will mean that continuances of equal probability are everywhere coincident. In other words, if, say, continuation of \$100 under one benefit is associated with a probability of 0.1, and a second, coincident continuation of \$50 under another benefit has a probability of 0.1, then the second \$50 continuance never occurs except in conjunction with \$100 continuance under the other benefit. While purely hypothetical, this completely dependent relationship will be shown to have very practical value.

By a correlation of zero, we will mean that any continuance under one benefit may occur coincidentally with any continuance whatever under a second coincident benefit. Thus all possible coincident conditions involve correlation ranging from zero to one. We will assume that negative correlation is nonexistent in coincident medical benefits, although instances of this doubtless exist.

*d*) *Relative disability rate*, ${}^{\theta}\rho_x$. The relative disability rate is the ratio of incidence of any particular coincident set of benefits to the total rate for all benefits. Thus the disability rate for a particular coincident set is equal to ${}^{\theta}\rho_x r_x$.

*e*) *Composite function*. A "composite" function is a function composed of two or more coincident elements. This is not the same as the compound function discussed in Part II. That function involves exclusive elements, and the two are treated by completely different techniques.

3. *Analysis of the Incidence of the Elements*

The first problem presented by a multiple benefit risk where the benefits are subject to common limits is the analysis of the incidence of the various elements included. All possible coincident sets must be isolated, so that the distribution of expected claims is reduced to mutually exclusive sets of coincident combinations. For example, if we are dealing with benefits providing for hospital room and board, miscellaneous hospital expense, and surgery, there will be three exclusive sets, assuming the two types of hospital expense to be invariably coincident:

*a*) Hospitalization without surgery.
*b*) Surgery without hospitalization.
*c*) Hospitalization with surgery.

Later in the chapter, an approximation that ignores this analysis will be described, but we will begin by considering the effect of the incidence of benefits.

Having determined or assumed the number of exclusive sets, we must assign to each its relative rate of claim either by analysis of existing experience or by assumption, and knowing the total rate of claim, we then have the rate for each set. Thus a general formula for the average amount of cost for multiple benefits is:

$$ {}^{\sigma}F = {}^{\theta_1}(\rho F) + {}^{\theta_2}(\rho F) + \ldots + {}^{\theta_n}(\rho F) \qquad \text{(IV-1)} $$

and for the annual cost of disability:

$$ {}^{\sigma}S_x = {}^{\sigma}(rF)_x \, . \qquad \text{(IV-2)} $$

We have thus expressed the multiple average cost as a sum of terms each representing the average cost for one exclusive set. It is neither necessary nor practical to attempt to distinguish every conceivable exclusive set. For example, a multiple benefit including private nursing care might be evaluated by loading the hospital room and board cost to recognize the added cost of nursing care, without regarding the latter as a separate benefit element. A doctor benefit may sometimes be treated in the same way. The cost of out-of-hospital drugs and supplies could be loaded onto out-of-hospital examination costs, and so forth.

It is also possible, in certain cases, to assume that a certain benefit appearing in several different sets is defined in each case by the same constants. This must be done, however, with caution. For example, it would be most unwise to assume that the same continuance curve defines both hospitalized and nonhospitalized surgery, since the latter, involving less serious cases, will obviously exhibit a much smaller average amount of cost.

Considerable research and experience is still needed to adequately solve this question of relative incidence. It would appear, however, that these four exclusive groups may provide an adequate basis for reasonably accurate evaluation of multiple costs:

(1) Hospitalization, surgery not involved.
(2) Hospitalization, surgery involved.
(3) Surgery and treatment without hospitalization.
(4) Diagnosis and treatment without surgery or hospitalization.

In our examples, we will assume this to be a sufficient breakdown. In using these, the fact must not be overlooked that sets (1) and (2) will involve some expense of the same type as set (4) when this expense is incurred on the same disability as the expense classified in sets (1) or (2). We will meet this consideration by loading the elements used in (1) and (2).

## 4. *Adjustment for Unit Rates*

When working with a single element, we need not be concerned with the unit rate of benefit until the final step of calculation. In fact, this need not be introduced until the $K_x$ function is evaluated.

When working with composite functions, however, all elements must be reduced to a common monetary measure of continuance. Thus, when working with hospital continuance alone, we may take days as the continuance unit, but when this is taken into combination with other continuance elements, it must, along with all other elements, be converted to monetary units.

This is very simply done. Let a hospital continuance element be defined as follows, where 1 day is the unit of duration:

$$^{h}p^{(t)} = \left(\frac{a}{a'+t}\right)^{a}.$$

If now the daily room and board rate is $x$, we have

$$ {}^{xh}p^{(t)} = \left(\frac{xa}{xa' + xt}\right)^{a} = \left(\frac{xa}{xa' + t'}\right)^{a}, $$

where $t' = xt$ and $t'$ is then the monetary unit. Thus the conversion is exactly the same as that required for converting time units, discussed in section III-2. There is one important difference, however, when it is desirable to adjust for discrete experience. The change in units here carries with it a corresponding change in the discrete interval, since the steps still occur at a daily rate. The discrete adjustment therefore is given, approximately, by:

$$ A \doteq \frac{x}{2}\left(p^{(T_1)} - p^{(T_2)}\right), $$

so that we have the formula

$$ F^{T_1;T_2} \doteq \frac{\beta}{a-1}\left[\left(\frac{\beta}{\beta' + T_1}\right)^{a-1} - \left(\frac{\beta}{\beta' + T_2}\right)^{a-1}\right] - \frac{x}{2}\left(p^{(T_1)} - p^{(T_2)}\right); \qquad \beta = xa, \qquad \beta' = xa'. \tag{IV-3} $$

When the minimum duration must be taken into account, this is:

$$ \tau' = x(a - a'). \tag{IV-4} $$

Corresponding adjustment of the range constants is required with Lambda functions.

5. *Coincident Benefits with Correlation 1.0*

The evaluation of completely correlated benefits is quite simple. We have these relations by definition:

$$ {}^{\sigma}p^{(T)} = {}^{\theta_1}p^{(u)} = {}^{\theta_2}p^{(v)}, \tag{IV-5} $$

where $T = u + v$.

The algebraic determination of $u$ and $v$ corresponding to any given duration $T$ is somewhat troublesome, so it is usually better, and quite satisfactory, to obtain the values graphically.

We merely construct the curves ${}^{\theta_1}p^{(t)}$ and ${}^{\theta_2}p^{(t)}$ on the same graph and add values of $t$ for equal values of ${}^{\theta_1}p$ and ${}^{\theta_2}p$ to plot the composite curve. The desired $u$ and $v$ for any given composite duration $T$ are then easily read off and we have:

$$ {}^{\sigma}F^{T_1;T_2} = {}^{\theta_1}F^{u_1;u_2} + {}^{\theta_2}F^{v_1;v_2}. \tag{IV-6} $$

Either benefit itself may be a compound curve. Figure 5 illustrates such a situation, where Benefit $h$ is compounded of two Alpha elements, without truncation, and Benefit $m$ is an Alpha element truncated at duration $k$. Thus we have:

$$ {}^{\sigma}F^{T} = {}^{\alpha}F^{u} + {}^{\beta}F^{u} + {}^{m}F^{v} \,, \tag{IV-7} $$

a result quite readily obtained.

The truncation of $m$ presents no problem. If we wish to evaluate the average size for duration $W$, beyond the point of truncation, we have simply:

$$ {}^{\sigma}F^{W} = {}^{\alpha}F^{y} + {}^{\beta}F^{y} + {}^{m}F^{k} \,; \qquad W = y + k \,. $$

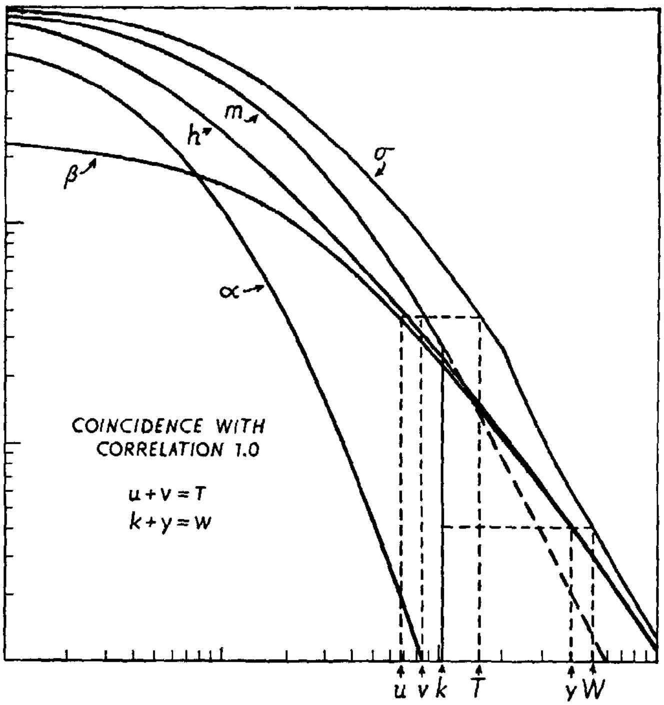


FIG. 5

## 6. *Coincident Benefits with Correlation Zero*

If all coincident sets were fully correlated, we would have little to worry about. Unfortunately, this is not the case, and we must consider correlations other than 1.

Let us assume that two coincident benefits have a zero correlation. The composite probability may then be given by:

$$ {}^{\sigma}p^{(t)} = {}^{\theta_1}p^{(t)} + \int_0^t {}^{\theta_1}(\pi^{(s)}p^{(s)}) \cdot {}^{\theta_2}p^{(t-s)} d\,s \,, \tag{IV-8} $$

or any of several alternative expressions. It is not possible to express ${}^{\sigma}p^{(t)}$ in a form that may be directly evaluated by any elementary methods of integration—except in the unlikely case where the attenuation constants are integers, when standard binomial reduction formulas may be applied. Probably the simplest means of approximate integration of the second term of (IV-8) is by Simpson's rule.

If several examples of zero correlated benefits are worked out, it will be found that the composite probability exceeds that which would be obtained for correlation 1 over the lower range of the curve, and eventually becomes and remains less, a result which one might expect. We can easily prove that the initial portion of the noncorrelated curve will always exceed the correlated curve by means of the following existence theorem.

THEOREM. If two coincident continuance functions are uncorrelated, then there exists some finite value $T$, such that for all continuance less than $T$ the composite probability associated with the uncorrelated benefits exceeds the composite probability of the same continuance were the same functions fully correlated.

*Proof*:

Let the uncorrelated case be U, the correlated case be C. Then, expressing the probabilities by means of the $q$-function, we have:

$$^{(U)\sigma}q^{(t)} = \int_0^t {}^{\theta_1}q^{(t-s)} \cdot {}^{\theta_2}(\pi^{(s)}p^{(s)})\, d\,s\,, \qquad ^{(C)\sigma}q^{(t)} = \int_0^u {}^{\theta_2}(\pi^{(s)}p^{(s)})\, d\,s\,.$$

Since we have made no restriction in these expressions concerning the identity of $\theta_1$ or $\theta_2$, let $\theta_2$ be the function which, in Case C, involves the initially higher duration unless the durations be equal: that is, $u \geq t/2$.

Now the function ${}^{\theta_2}(\pi^{(s)}p^{(s)})$ is a decreasing function (except for $l \leq 1$ of a Lambda function), since we have:

$$^{a}(\pi^{(s)}p^{(s)}) = \frac{a\,a^{a}}{(a'+s)^{a+1}}$$

and

$$^{\lambda}(\pi^{(s)}p^{(s)}) = \frac{l\,(\lambda'-s)^{l-1}}{\lambda^{l}}\,;$$

when $l = 1$, ${}^{\lambda}(\pi^{(s)}p^{(s)})$ is constant for all $s < \lambda'$.

Hence in all cases except $l < 1$, we have for Case C:

$$^{(C)\sigma}q^{(t)} > \frac{1}{2}\int_0^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\, d\,s\,,$$

since

$$\int_0^u {}^{\theta_2}(\pi^{(s)}p^{(s)})\, d\,s > \int_u^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\, d\,s\,.$$

When $l$ is less than 1, we have:

$$\int_0^u {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s > \int_0^u \frac{l}{\lambda^l \lambda'^{1-l}}\,d\,s$$

and

$$\int_0^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s < \int_0^t \frac{l}{\lambda^l(\lambda'-t)^{1-l}}\,d\,s\,,$$

so that

$$\int_0^u {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s > \frac{1}{2}\left(\frac{\lambda'-t}{\lambda'}\right)^{1-l}\int_0^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s$$

$$= \tfrac{1}{2}k\int_0^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s\,,$$

where $k$ is a positive finite fraction provided that $t$ does not approach $\lambda'$. Thus for all correlated cases we must have

$${}^{(C)\sigma}q^{(t)} > \tfrac{1}{2}k\int_0^t {}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s\,.$$

For Case U, we have:

$${}^{(U)\sigma}q^{(t)} < \int_0^t {}^{\theta_1}q^{(t)}\cdot{}^{\theta_2}(\pi^{(s)}p^{(s)})\,d\,s\,.$$

Since continuance functions are continuous and single valued over a finite interval above $t = 0$, and since ${}^{\theta_1}q^{(0)} = 0$, we can always find some finite value $T$ of $t$ such that

$${}^{\theta_1}q^{(T)} < \tfrac{1}{2}k\,.$$

Hence

$${}^{(U)\sigma}q^{(t)} < {}^{(C)\sigma}q^{(t)} \qquad \text{for} \qquad t < T\,,$$

and the theorem is proved.

Since ${}^{(U)\sigma}F^{\infty}$ must equal ${}^{(C)\sigma}F^{\infty}$, there must also be values ${}^{(U)\sigma}p^{(t)} < {}^{(C)\sigma}p^{(t)}$, and it is evident that there will be one intersection point, all values to the right giving

$${}^{(U)\sigma}p^{(t)} < {}^{(C)\sigma}p^{(t)}\,,$$

although we will not give here a rigorous proof of this.

These relations suggest the possibility of attempting to approximate ${}^{(U)\sigma}p^{(t)}$ by some distortion of the elementary curves, the distorted curves being treated as correlated. This can be done, in fact, with rather good results, and it is evident that the very great simplification of labor in the computation of such an approximate equivalence is a desirable end, for even if values of ${}^{\sigma}p^{(t)}$ be computed by a method such as Simpson's rule, we have some further laborious approximate process to go through to compute the $F$-functions required.

If a number of examples of ${}^{(U)\sigma}p^{(t)}$ be worked out, and we then deduct the abscissas of the element of lesser attenuation, the resulting "correlated" curve will turn out to be a curve approximating a distortion of the other original curve obtained by increasing the attenuation and holding $F^{\infty}$ constant. What is needed, then, is a means of estimating the change in the original constants needed to give the best approximation to this distortion. For this purpose, we will discuss measurement of the attenuation of continuance functions.

## 7. *The Index of Attenuation, $f'^a$ or $f'^l$*

While the constant $a$ or $l$ is called the "attenuation," it is not in itself a good measure of attenuation, *i.e.*, the rapidity with which the continuance function diminishes into insignificant values. For this purpose we use a particular value of the function $f'$.

As defined in section III-1, $f'^t$ is given by the relations:

$$ {}^{a}f'^{t} = \left(\frac{a}{a'+t}\right)^{a-1}; \qquad {}^{e}f'^{t} = e^{-(t-a')/F'}; \qquad {}^{\lambda}f'^{t} = \left(\frac{\lambda'-t}{\lambda}\right)^{l+1}. $$

For each of these three fundamental functions, let us evaluate $f'^F$, *i.e.*, the value for which $t$ is equal numerically to $F^{\infty}$.

We have:

$$ {}^{a}f'^{F} = \left(\frac{a}{a'+\dfrac{a}{a-1}+a-a'}\right)^{a-1} = \left(\frac{1}{\dfrac{1}{a-1}+1}\right)^{a-1} = \left(\frac{a-1}{a}\right)^{a-1} $$

$$ {}^{e}f'^{F} = e^{-(F'+a'-a')/F'} = e^{-1} = .368 \,. \qquad \text{(IV-9)} $$

$$ {}^{\lambda}f'^{F} = \left(\frac{\lambda'-\dfrac{\lambda}{l+1}-\lambda'+\lambda}{\lambda}\right)^{l+1} = \left(1-\frac{1}{l+1}\right)^{l+1} = \left(\frac{l}{l+1}\right)^{l+1}. $$

These results are functions of the attenuation only. Moreover, recalling the Theorem in section I-3e, values for Alpha functions decrease from 1 at $a = 1$ down to the limit of .368 at $a = \infty$; values for Lambda functions decrease from the limit of .368 when $l = \infty$ down to the extreme limit of 0 when $l = 0$. Thus we have a convenient measure varying continuously from 1 to 0 through the Alpha function, the exponential and finally the Lambda function. We call this measure the "index of attenuation," and use the symbols $f'^a, f'^e$, or $f'^l, f'^e$ having the constant value .368 to 3 decimals.

8. *The Index of Compression, I*

In section IV-6 we discussed the idea of finding a "distortion" of one of the original elements that will provide a "correlated" curve to approximate the actual composite curve produced by uncorrelated coincidence. We will use the index of attenuation $f'^{a}$ as a basis for measuring the amount of distortion. This distortion is always such as to decrease the value of $f'^{a}$ while holding $F'^{\infty}$ constant, and will be called "compression" of the curve, the resulting "compressed" curve being symbolized by

$$^{(I)}\theta_1$$

where the original is

$$\theta_1 .$$

No precise and explicit means of measurement has been developed, but the following empirical formula gives results of reasonable accuracy:

$$\frac{^{(I)\theta_1}f'^{a}}{^{\theta_1}f'^{a}} = {}^{(\theta_1,\theta_2)}I \doteq 1.185\left[\frac{\left({}^{\theta_1}F'^{T}\right)^{.31-.039a}}{a^{.0621}\left({}^{\theta_2}F'^{T} + {}^{\theta_2}\tau - {}^{\theta_1}\tau\right)^{.31}}\right] \quad \text{(IV-10)}$$

where ${}^{\theta_2}\tau > {}^{\theta_1}\tau$,
or

$${}^{(\theta_1,\theta_2)}I \doteq 1.185\left[\frac{\left({}^{\theta_1}F'^{T} + {}^{\theta_1}\tau - {}^{\theta_2}\tau\right)^{.31-.039a}}{a^{.0621}\left({}^{\theta_2}F'^{T}\right)^{.31}}\right]$$

where ${}^{\theta_1}\tau > {}^{\theta_2}\tau$.

In this formula $T$ is chosen such that ${}^{\theta_1}f'^{T} = .01$, and the constant $a$ is the attenuation of the function $\theta_1$, *i.e.*, the function of greater attenuation, which is the function being compressed.

This value ${}^{(\theta_1,\theta_2)}I$, called the "index of compression," may be computed directly from known values determined by the constants of the two original elements. The formula was derived by Mr. William Hoop.

Having thus determined approximately a measure of the "compression," we are now able to solve for the constants of the compressed curve. We have:

$$\frac{^{(I)}f'^{a}}{f'^{a}} \doteq I$$

or (IV-11)

$$^{(I)}f'^{a} \doteq I\, f'^{a},$$

from which the nature of the compressed curve as an Alpha or a Lambda function may be determined and the constant of attenuation obtained. Table 7 may be used for this purpose, first difference interpolation being sufficient. Lambda functions always compress into Lambda functions, of course. Alpha functions may compress into Lambda functions under some combinations.

For the other two constants, we have:

$$^{(I)}\alpha = [^{(I)}a - 1]F' \tag{IV-12}$$

or

$$^{(I)}\lambda = [^{(I)}l + 1]F'$$

since $F'$ is unchanged, and

$$^{(I)}a' = {}^{(I)}a - \tau \tag{IV-13}$$

or

$$^{(I)}\lambda' = {}^{(I)}\lambda + \tau$$

since $\tau$ also remains unchanged.

The element to be compressed is always the element with the greater original index of attenuation. This is because the composite curve will always approach the curve with the lesser index of attentuation as $t$ increases, so there is no need to tamper with this element.

The compression technique may be used repeatedly where more than two elements are involved, although the accuracy of the procedure will diminish with repeated application. It is nevertheless to be preferred to the enormous labor of more straightforward methods of approximate evaluation. With more than two elements, we first take the pair of lowest attenuation. After determining the first compression, we move to the curve of next higher attenuation, computing the compression index on the sum of the appropriate $F$ values of the first two curves, and so on.

Finally, values of ${}^{\sigma}F$ are computed from the compressed curves assuming them to possess a correlation of 1.0, using the method described in section IV-5.

## 9. *Coincident Curves of Intermediate Correlation*

In actual practice, practically every coincident set will involve correlation intermediate to the extremes of zero and one. We therefore use such experience as we have available, or else make assumptions as to the fractional value of ${}^{(\theta_1,\theta_2)}I$ to be used, since such situations will call for compression lying somewhere between that associated with zero correlation, and no compression at all with correlation 1. If practically nothing is known of the correlation, the best guess is probably to take from 33% to 50% compression. Thus, for 50% compression, we have ${}^{50\%}I = .5(1 + I)$. ${}^{(I)}f'^{a}$ computed from this relation then gives an intermediate index of attenuation. An example of numerical evaluation, using the tables in Appendix C, will be given in section 11.

## 10. *Simplified Approximation*

While the compression technique previously described is not difficult, it is evident that it can become prolonged in complex situations involving multiple elements and several exclusive sets where claim costs must be derived for several plans at a number of ages. Where costs for a wide range of ages are needed, it is usually sufficient to compute the composite values at decennial or quinquennial ages and interpolate for annual values.

In some cases, the careful and detailed evaluation involved in the compression-exclusive set technique may not appear to be necessary, and it is useful to have some quick alternative method of approximate evaluation available.

a. Single element approximation

When the benefit structure is not so complex as to render the method altogether inaccurate, several elements may sometimes be replaced by a single substitute element approximating a composite function or even several exclusive functions. The method can sometimes give very misleading results, however, when a wide range of bounded values are required, and is not to be resorted to rashly.

The technique is to assume that at some fairly large value of $t$, the value $f'^{t}$, for the curve involving the higher attenuation of a pair, becomes effectively zero. It is usually sufficient for the purpose to take any value $T$ for which $f'^{T}$ is actually less than .01. Correlation less than 1.0 may be roughly accounted for by taking $T$ somewhat smaller than that for which $f' = .01$.

Thus we have, for a coincident set:

$$\begin{aligned} {}^{\sigma}F'^{T} &\doteqdot {}^{\theta_1}F' + {}^{\theta_2}F'^{T} \\ {}^{\sigma}F' &= {}^{\theta_1}F' + {}^{\theta_2}F' \\ {}^{\sigma}\tau &= {}^{\theta_1}\tau + {}^{\theta_2}\tau\,. \end{aligned} \tag{IV-14}$$

From these equations, the constants for the approximating element are derived with little difficulty.

For an exclusive set:

$$\begin{aligned} {}^{\sigma}F'^{T} &\doteqdot {}^{\theta_1}(\rho F') + {}^{\theta_2}(\rho F'^{T}) \\ {}^{\sigma}F' &= {}^{\theta_1}(\rho F') + {}^{\theta_2}(\rho F') \\ {}^{\sigma}\tau &\doteqdot {}^{\theta_1}(\rho\tau) + {}^{\theta_2}(\rho\tau)\,, \end{aligned} \tag{IV-15}$$

where ${}^{\theta_1}\rho + {}^{\theta_2}\rho = 1.0$.

A little reflection will show that the approximation resulting from this gives the correct value for ${}^{\sigma}F^{\infty}$.

In the case of exclusive sets, the compound curve is so easy to evaluate that there is seldom any need to derive a single element approximation.

It will be apparent how the technique is extended to more than two elements, since each succeeding pair is reduced to one. The process should begin with the lowest attenuations.

b. Approximation by curve segments

Another, more accurate approximation begins in the same manner as the previous method. We assume that at point $T$, ${}^{\theta_1}f'^{T} = 0$; we then evaluate the remainder of the $\theta_2$ curve:

$$ {}^{\theta_2}F^{T;\infty} = {}^{\theta_2}[F' \cdot f'^{T}] \; ; $$

from this, we set (for a coincident pair):

$$ \begin{aligned} {}^{\theta_3}F^{T} &= {}^{\sigma}F - {}^{\theta_2}F^{T;\infty} \\ {}^{\theta_3}p^{(T)} &= {}^{\theta_2}p^{(T)} \\ {}^{\theta_3}\tau &= {}^{\theta_1}\tau + {}^{\theta_2}\tau \end{aligned} \qquad \text{(IV-16)} $$

and solve for the constants of the approximating curve $\theta_3$. This curve joins $\theta_2$ at point $T$, so that we have a segmented compound similar to the select and ultimate graduation described in Part II. A similar technique may be readily devised for an exclusive pair.

c. Approximation on the "lesser attenuation"

An approximation using a single element which gives a somewhat different distribution of the cost than method (a) is to take for the constant of attenuation the value of the lesser constant of a pair. We then set:

$$ \begin{aligned} {}^{\sigma}F' &= {}^{\theta_1}F' + {}^{\theta_2}F' \\ {}^{\sigma}\tau &= {}^{\theta_1}\tau + {}^{\theta_2}\tau \end{aligned} $$

and by these two equations, with substitution of the constant already known, we derive the other constants.

Method (c) understates bounded $F$ values in the shorter ranges, overstating in the longer ranges. Method (a) does the opposite. Accordingly, improved accuracy can sometimes be obtained by computing the desired bounded values by both methods, and taking the average. Comparison of the two methods is also useful in giving some idea of the maximum error, since the values computed by each method will in many cases (but not always!) lie on each side of the true value.

A variety of other approximations are obviously possible. These will serve as examples. Usually, the investigator will find it possible, either by quick methods or by the more refined technique of compressed curves, to evaluate complex benefits with consistency and acceptable accuracy, and without excessive labor. Thus the basic experience may be assembled in the form of elementary functions and disability rates, and almost any sort of combination evaluated confidently using appropriate techniques and careful assumptions.

### 11. *Numerical Example of Composite Evaluation*

The method of evaluation by compressed curves will become more clear by working out a specific example.

Let us evaluate the claim cost, using the tables of Appendix C, for the following major medical benefit for the year of age 40, male:

Covered medical expenses are defined to include:

1. Hospital room and board up to $25 daily.
2. In-hospital doctor calls up to $10 daily.
3. A California Relative Value Surgery Schedule of 5 units ($500 top fee), which allows in addition to the basic fee up to 15% of the amount of the schedule limit for anesthesiologist's fee, and up to 15% for assistant surgeon's fee.
4. 80% of other medical expenses.

Benefits are payable for covered medical expense in excess of a $250 deductible, subject to a maximum benefit of $7,500.

Symbolically, we may express the desired cost as:

$$\underset{\infty\,:\,\infty\,:\,500:80\%}{25h:10c:s'\;\;:m}\; S^{\,250;7750}_{\,40(m)}$$

To compute this cost, we will require six continuance functions from the tables:

$$h_1 = {}^{a}(19.11, 17.21, 3.8) \qquad (-h)s = {}^{a}(28.99, 26.99, 6.5)$$

$$h_2 = {}^{a}(27.78, 25.88, 1.4) \qquad (h)m = {}^{a}(445.4, 415.0, 5)$$

$$(h)s = {}^{a}(101.2, 95.2, 5) \qquad (-h)m = {}^{a}(94.93, 89.93, 3.68)$$

The computation then involves the following steps.

*Step 1.* Conversion of units

We must first alter the basic functions to agree with the units of coverage in the desired benefit.

a. $h$ functions

The \$10 doctor call benefit we will assume to be equivalent to \$4 daily room. Hence we have \$25 + \$4 = \$29 as a daily equivalent unit. Converting the $h$ functions for \$29 daily units gives:

$$29h_1 = {}^{a}(554, 499, 3.8) \qquad 29h_2 = {}^{a}(806, 751, 1.4)$$

b. $(h)s$ function

We will assume full utilization of the basic benefit and the anesthesiologist benefit and 33% utilization of the assistant surgeon benefit. Thus, for utilization of the complete benefit, we have the factor $1 + .15 + (.33 \times .15) = 1.2$. Multiplying by 5 units, we have a conversion factor of $1.2 \times 5 = 6$, so that we get:

$$(h)s' = {}^{a}(607, 571, 5)\,.$$

c. $(-h)s$ function

For out-of-hospital surgery, we will assume full utilization of the basic benefit, 33% for anesthesiologist, and none for assistant surgeon. This gives a factor of $(1 + .05)$. In order to avoid undue multiplicity of composite combinations, we will assume an additional 20% to cover nonsurgical miscellaneous out-of-hospital expense, giving a complete factor of $(1 + .05) \times 1.2$. Multiplying by 5 units, we have $(1 + .05) \times 1.2 \times 5 = 6.3$, giving

$$(-h)s' = {}^{a}(183, 170, 6.5)\,.$$

d. $(h)m$ function

Let us assume that the basic tabular function is adequate for an area of \$15 prevailing room charges, and that the miscellaneous cost level varies as $\sqrt{x/15}$, where $x$ is the cost level in the area where we expect our plan to be sold. For this, then, we get an "area cost" factor of $\sqrt{25/15}$, or 1.291.

Next, let us assume that this is a reasonable function to modify to account for the cost of private nursing care, assuming, moreover, that such care increases in incidence with increasing size of miscellaneous hospital expenses. We can approximate this by loading the constant of attenuation. Assuming that the average nursing care claim is \$250.00 and that such care occurs on 20% of the hospital claims, we must load the attenuation to increase ${}^{(h)m}F$ by \$50.00.

Next, we must assume some out-of-hospital miscellaneous cost to be coincident with hospital claims. We will assume that an additional 33% loading in the conversion factor will cover this, to avoid complicating the number of exclusive composite sets we have to deal with.

Finally, our benefit provides 80% insurance. Thus, accounting for all but the nurse benefit, we have a factor of $1.291 \times 1.33 \times 0.80 = 1.373$. This gives $(h)m' = {}^{a}(612, 570, 5)$.

${}^{(h)m'}F'$ has the value $612/4 = 153$, so to get the loaded constant $a^*$ accounting for the nurse benefit, we have:

$$a^* = (a-1)\frac{153}{193} + 1 = 4.17\,,$$

where $193 = 153 + (.8 \times 50)$.

Hence our final converted $(h)m$ function is:

$$(h)m' = {}^{a}(612, 570, 4.17)\,.$$

e. $(-h)m$ function

We have already accounted for the coincident occurrence of this benefit along with hospitalization or out-of-hospital surgery. Therefore all that is left is $(-hs)m$, *i.e.*, such expense occurring without coincident hospital or surgical costs. This residue will be evaluated by proper choice of the partial claim rate from Table 1, not by modification of the basic function.

Assuming the same "area cost" factor as for $(h)m$, and 80% insurance, we have the factor $1.291 \times 0.8 = 1.034$, giving

$$(-hs)m = {}^{a}(98.2, 93.0, 3.68)\,.$$

Summarizing, we now have the converted functions

$$\begin{aligned} 29h_1 &= {}^{a}(554, 499, 3.8) & (-h)s' &= {}^{a}(183, 170, 6.5) \\ 29h_2 &= {}^{a}(806, 751, 1.4) & (h)m' &= {}^{a}(612, 570, 4.17) \\ (h)s' &= {}^{a}(607, 571, 5) & (-hs)m &= {}^{a}(98.2, 93.0, 3.68)\,. \end{aligned}$$

*Step 2.* Derivation of Compressed Curves

We must now carry out the compression of curves involved in coincident sets in order to evaluate the composite curve. There are 4 such sets.

a. The set $[(h)m', 29h_1]$

To evaluate $T$ in the compression formula, which will compress the $(h)m'$ function, we have:

$$\left(\frac{612}{570+T}\right)^{3.17} = .01\,,\ \text{giving } T = 2{,}057\,.$$

$${}^{(h)m'}F'^{T} = 193\,(1 - .01) = 191.1, \text{ with } {}^{(h)m}\tau = 42\,.$$

$${}^{29h_1}F'^{\,T} = 198\left[1 - \left(\frac{554}{2556}\right)^{2.8}\right] = 195.2\,,\ \text{with } {}^{29h_1}\tau = 55\,.$$

Hence for the compression formula, we have:

$$^{[(h)m',29h_1]}I \doteq 1.185 \frac{(191.1)^{.31-.039\times 4.17}}{4.17^{.0621}\times 208.2^{.31}} = .4528\,.$$

Assuming an intermediate correlation giving 50% compression, we have

$$^{50\%}I \doteq .5(1 + .4528) = .7264\,,$$

whence

$$^{(I)(h)m'}f'^{l} = .7264 \times .420 = .305, \text{ a Lambda index}\,.$$

Interpolating in Table 7, we have $^{(I)}l = 2.36$. Since $F'$ and $\tau$ are not altered by compression, we have:

$$^{(I)}\lambda = 193 \times 3.36 = 648, \text{ and } ^{(I)}\lambda' = 648 + 42 = 690\,.$$

Thus the compressed function is:

$$^{(I)}(h)m' = {}^{\lambda}(648, 690, 2.36)\,.$$

This function and the $29h_1$ function are then plotted and carefully graphed. The composite $\sigma$ function is then plotted by adding pairs of abscissas with equal ordinates, *i.e.*, values of $p^{(t)}$ (section IV-5), and from this graph the required limits of integration of the elements can be obtained to evaluate the composite function between any desired limits.

b. The set $[(h)s', (h)m', 29h_1]$

Fortunately, the set is compressed in the order $(h)m'$, $(h)s'$, so that we can employ the results of set (a), and go directly to the compression of the last element, $(h)s'$.

To obtain $T$, we have:

$$\left(\frac{607}{571+T}\right)^4 = .01\,, \qquad T = 1{,}350$$

$$^{(h)s'}F'^{T} = 152\,(1 - .01) = 150.5, \qquad \text{with} \qquad ^{(h)s'}\tau = 36\,.$$

From the graph of $[(h)m', 29h_1]$ we obtain:

$$^{[(h)m',\,29h_1]}F'^{T} = {}^{(I)(h)m'}F'^{535} + {}^{29h_1}F'^{815} = 272\,, \quad \text{and} \quad ^{[(h)m',\,29h_1]}\tau = 97\,.$$

Hence the compression formula is:

$$^{(h)s',\,[(h)m',\,29h_1]}I \doteq 1.185 \frac{150.5^{.31-.039\times 5}}{5^{.0621}\times 333^{.31}}.$$

Again assuming 50% compression to result from partial correlation, we have

$$^{50\%}I = .5(1 + .316) = .658\,,$$

whence

$$^{(I)}f'^{l} = .658 \times .410 = .270, \qquad \text{and} \qquad ^{(I)}l = 1.3$$

$$^{(I)}\lambda = 2.3 \times 152 = 349, \qquad ^{(I)}\lambda' = 349 + 36 = 385\,,$$

so that we have

$$^{(I)}(h)s' = {}^{\lambda}(349, 385, 1.3)\,.$$

This can then be plotted on the same graph with $[(h)m', 29h_1]$ to obtain the 3-element composite $[(h)s',(h)m', 29h_1]$.

c. The set $[(h)m', 29h_2]$

As in set (a), $T = 2{,}057$, and $^{(h)m'}F'^{T} = 191.1$.

$$^{29h_2}F'^{\,T} = 2015\left[1 - \left(\frac{806}{751 + 2057}\right)^{.4}\right] = 792\,,$$

and $^{29h_2}T - {}^{(h)m'}T = 13$

giving the compression formula

$$^{[(h)m',\,29h_2]}I \doteqdot .296\,, \qquad ^{50\%}I = .648\,.$$

Hence $^{(I)}f'^{l} = .648 \times .420 = .272, \qquad ^{(I)}l = 1.34$

$$^{(I)}\lambda = 2.34 \times 193 = 452, \qquad ^{(I)}\lambda' = 452 + 42 = 494\,,$$

so that we have, in this case:

$$^{(I)}(h)m' = {}^{\lambda}(452, 494, 1.34)\,.$$

d. The set $[(h)s', (h)m', 29h_2]$

Again, we can use the results of set (c), and need only to determine $^{(I)}(h)s'$.

As in set (b), $T = 1{,}350$, $^{(h)s'}F'^{T} = 150.5$.

From the graph of set (c), we get:

$$^{[(h)m',\,29h_2]}F'^{T} = {}^{(I)(h)m'}F'^{293} + {}^{29h_2}F'^{1057} = 720$$

and $^{[(h)m',\,29h_2]}T - {}^{(h)s'}T = 61$.

Thus the compression index is given by:

$$^{(h)s',\,[(h)m',\,29h_2]}I \doteqdot .2485\,, \qquad ^{50\%}I = .6242\,.$$

Therefore, $^{(I)}f'^{l} = .6242 \times .410 = .256$ and $^{(I)}l = 1.08$

$^{(I)}\lambda = 2.08 \times 152 = 316, \qquad ^{(I)}\lambda' = 316 + 36 = 352$, giving

$^{(I)}(h)s' = {}^{\lambda}(316, 352, 1.08)\,.$

*Step 3.* Evaluation of Bounded Integrals

a. $[(h)m, 29h_1]$

From the graph, the limits of integration for ${}^{[(h)m,\,29h_1]}F^{250;7750}$ are:

$$138 \text{ and } 665 \text{ for } {}^{(I)}(h)m' ,$$

$$112 \text{ and } 7{,}085 \text{ for } 29h_1 .$$

(The actual readings cannot, of course, be so accurate. The readings have been adjusted to equal the desired total limits.)

Evaluating and adding the integrals gives:

$${}^{[(h)m,\,29h_1]}F^{250;7750} = 262.7 .$$

We need a discrete adjustment for the room benefit equal to:

$$-\ .5 \times 29\ (p^{(112)} - p^{(7085)}) = -.5 \times 29 \times .7 = -10.15 ,$$

so that the final value is $262.7 - 10.15 = 252.55$ .

b. $[(h)s', (h)m', 29h_1]$

The limits of integration are:

| | | | |
|---|---|---|---|
| for ${}^{(I)}(h)s'$ | : | 76 and | 378 |
| for ${}^{(I)}(h)m'$ | : | 93 and | 660 |
| for $29h_1$ | : | 81 and | 6,712 |
| Total: | | 250 | 7,750. |

The discrete adjustment for $29h_1$ will be $-.5 \times 29 \times .825 = -11.97$. Evaluating, we obtain:

$${}^{[(h)s',(h)m',\,29h_1]}F^{250;7750} = 422.5 .$$

c. $[(h)m', 29h_2]$

The limits are:

| | | | |
|---|---|---|---|
| ${}^{(I)}(h)m'$ | : | 95 and | 452 |
| $29h_2$ | : | 155 and | 7,298 . |

The discrete adjustment for $29h_2$ is $-11.73$, giving:

$${}^{[(h)m',\,29h_2]}F^{250;7750} = 1{,}251 .$$

d. $[(h)s', (h)m', 29h_2]$

The limits are:

| | | | |
|---|---|---|---|
| ${}^{(I)}(h)s'$ | : | 57 and | 336 |
| ${}^{(I)}(h)m'$ | : | 73 and | 448 |
| $29h_2$ | : | 120 and | 6,966 |
| Total: | | 250 | 7,750 . |

The discrete adjustment on $29h_2$ is $-12.42$. Evaluating, we have:

$$^{[(h)s',\,(h)m',\,29h_2]}F^{250;7750} = 1{,}419\,.$$

e. $(-h)s'$

Evaluating the single element function gives:

$$^{(-h)s}F^{250;7750} = .343, \text{ almost negligible.}$$

f. $(-hs)m$

The single element integration gives:

$$^{(-hs)m}F^{250;7750} = 1.283\,.$$

*Step 4.* Evaluation of $^{\sigma}S$

The remaining step is to evaluate the final composite $S_x$ function. To do this we simply multiply each exclusive $F$ value by its associated partial claim rate from Table 1. Hence:

$$\substack{25h:10c:\,s':m\\ \infty:\infty:500:80\%}S^{250;7750}_{40(m)} = 252.55 \times .045 + 422.5 \times .0354$$

$$+1{,}251 \times .00263 + 1{,}419 \times .00205 + .343 \times .0371$$

$$+ 1.283 \times .251 = \$32.86\,.$$

Thus the derivation is quite complex for a benefit of this type. In practice, the values could be obtained in this manner for decennial ages, and the entire scale by age obtained by any suitable method of interpolation. Any of the simplified approximations of section 10 could be employed, but the results would be much less reliable.

The compression technique can be applied to a wide variety of combination benefits. When "inside maximums" are involved, a slight additional step is needed. For example, if we are to evaluate the hospital claim cost

$$\substack{15h:(h)m\\ 90:150}S_x^{50;\infty}$$

in which room and board is covered up to 90 days, miscellaneous hospital expenses up to \$150, all subject to a \$50 deductible, the steps are all exactly equivalent to those of the preceding example (though much simpler!) except that now truncated elements are involved in the composite. When $(h)m$ is compressed, the truncation point, $k$, of the compressed curve must be chosen so that

$$^{(I)(h)m}F^{k} = {}^{(h)m}F^{150}\,.$$

The necessary value is readily obtained by solving from the equating value $^{(I)}f'^{k}$.

In all other respects, the computation proceeds along exactly parallel lines. Some additional error, however, is involved here. The modified composite curve obtained by adding the abscissas of these artificially "correlated" elements, when these are truncated, will deviate somewhat from the true composite at points beyond either point of truncation, and the error will be greater in the case of continuance points beyond the point of truncation of the uncompressed element. Usually, however, we need only to compute the claim cost *excluded* by the deductible in cases of inside maximums, and if the limits in the elements lie to the left of the truncation points, this error will not be involved.

## V. MISCELLANEOUS CONSIDERATIONS

### 1. *Effect of Boundaries on Utilization*

Before simply accepting the mathematical results of computation for, say, a given bounded $F$ value to be used in rate computation, the actuary should consider the possible need to provide loading for a change in utilization resulting from the elimination period or deductible amount. It is well known that such limitations on the benefit have an effect on utilization. For example, the rate of entrance upon claim for time loss benefits with 90 day elimination periods is apt to be less than is the probability of a 90 day disability on policies with first day coverage. Similar influences may affect the rate of utilization of major medical benefits as compared to first dollar or low deductible coverage. These influences should not be overlooked, and it is especially important to know the nature of the experience from which continuance constants have been determined.

### 2. *Heterogeneity in Combinations*

It is also highly important to weigh the distorting effects which may be introduced through combining basic elementary values into composites, which may in themselves create new variables or conditions not consistent with the elementary functions when considered separately. For example, a coverage providing diagnostic benefits outside of the hospital will very likely affect the claim value of another benefit paying for such services during hospital confinement. Many such situations may develop, and the actuary cannot afford to lose sight of these underneath the mathematical manipulations that represent the costs of these benefits.

### 3. *Simplifications in Deriving Commutation Columns*

Occasionally, if the $f'$ and $\tau$ values are quite similar over a range of ages, the $K_x$ function over the range may be computed for unlimited benefits, and the boundary adjustment for deductibles or maximums computed directly from the $K_x$ function. This simplification is taken advantage of by Bartleson and Olsen in their paper dealing with reserves on hospital and surgical insurance (*TSA* IX). In some cases, considerable labor can be saved in this way.

4. *Methods of Intrinsic Loading*

Frequently it will be desirable to build some loading factor into the basic continuance constants. It is worth while to consider the effect of loading each of the several constants.

a. Claim Rate, $r$. Increase in this constant obviously distributes the loading effect over all ranges of continuance.

b. Attenuation Constants, $a$ or $l$. Decrease in $a$ provides a greater loading the longer the duration of continuance. Decrease in $l$ does the same up to the limit of continuance, $\lambda'$, which is inherent in the Lambda function.

c. Range Constant $\alpha$. Increase in this constant distributes loading over the entire continuance curve except for the very short durations, where its effect is negligible. If $\alpha'$ is unchanged, the "minimum duration," however, will be increased, adding to the short duration loading.

d. Range Constant $\lambda$. Decrease in this constant will increase loading in the middle ranges. The formula for ${}^{\lambda}F^{\infty}$ might suggest the contrary, but the effect works out this way because of the shift in $\tau$ or else in the lower limit of integration with respect to the lower limit used to compute ${}^{\lambda}F^{\infty}$.

e. Range Constant $\alpha'$. Decrease in $\alpha'$ will increase loading slightly in the short durations, by increasing the "minimum duration."

f. Range Constant $\lambda'$. Increase in $\lambda'$ will increase the "minimum claim" and also the intrinsic limit of continuance, thus having an increasing effect for longer durations.

5. *Open Claim Reserves*

Where open claim volume is sufficiently large to justify it, continuance formulas may be employed to evaluate unpaid claim liability on either medical or loss of time benefits.

a. Individual Claim Reserves

It will be evident that the unpaid liability on a claim still pending at continuance $t$ ($t$ being a monetary or a time unit) is given by:

$$V^{t;T_2}=\frac{F^{t;T_2}}{p^{(t)}}, \tag{V-1}$$

without commutation, $T_2$ being the benefit limit, and

$${}^{(i)}V^{t;T_2}=\frac{{}^{(i)}F^{t;T_2}}{{}^{(i)}p^{(t)}}, \tag{V-2}$$

with commutation.

With commuted benefits approximated by an elementary function, it may happen that the error is significant when the duration has become large. In such a case, some revision may become necessary.

b. Aggregate Valuation

Where there is a large volume of claims on similar benefits, aggregate valuation may be practical. Assuming a "stationary population" of claims, the aggregate liability is given by:

$$\frac{N}{p^{(T_1)}} \int_{T_1}^{T_2} p^{(t)} F^{t;T_2} dt, \tag{V-3}$$

$N$ being the constant number just entering upon claim at any given moment. On a per claim basis,

$${}_{\text{ave.}}V^{T_1;T_2} = \frac{\int_{T_1}^{T_2} p^{(t)} F^{t;T_2} dt}{\int_{T_1}^{T_2} p^{(t)} dt}, \tag{V-4}$$

whence

$${}_{\text{agg.}}V = n \times {}_{\text{ave.}}V^{T_1;T_2},$$

$n$ being the total number of claims pending under the benefit.

It is of interest to work out the per claim value for the hypothetical case where the deductible $= \tau$, with no upper limit. We have, using the Alpha function:

$$\int_{\tau}^{\infty} p^{(t)} F^{t;\infty} dt = \int_{\tau}^{\infty} \left(\frac{a}{a'+t}\right)^{a} \frac{a}{a-1} \left(\frac{a}{a'+t}\right)^{a-1} dt$$

$$= \frac{a}{a-1} \int_{\tau}^{\infty} \left(\frac{a}{a'+t}\right)^{2a-1} dt = \frac{a^{2a}}{a-1} \int_{\tau}^{\infty} (a'+t)^{1-2a} dt$$

$$= \frac{a^{2a}}{(a-1)} \left[\frac{(a'+t)^{2-2a}}{2-2a}\right]_{\tau}^{\infty} = \tfrac{1}{2} F'^{2},$$

assuming $a > 1$; also,

$$\int_{\tau}^{\infty} p^{(t)} dt = F',$$

so that ${}_{\text{ave.}}V^{\tau;\infty} = \tfrac{1}{2}F'$, each claim being thus, on the average, halfway settled. The theoretical assumption here is continuous and instantaneous payment of benefits. The normal lag in payment might be accounted for by such an adjustment as:

$$_{\text{ave.}}V^{T_1;T_2} = \frac{\int_{T_1}^{k} p^{(t)} F^{T_1;T_2} dt + \int_{k}^{T_2} p^{(t)} F^{T_1+t-k;T_2} dt}{\int_{T_1}^{T_2} p^{(t)} dt},$$

where the assumption is a lag of $k$ units between incurment and payment. $k$ may be adjusted to reflect both the lag in reporting and in processing of the claim.

## 6. *Aggregate Benefit Limits*

Frequently policies contain aggregate limitations on benefit. For example, a major medical plan may provide up to $5,000 of benefit for each illness prior to age 65, and an aggregate benefit of $5,000 for all illnesses incurred after age 65.

During the aggregate period, there will begin to develop within the insured population a considerable number of lives whose maximums for a current disability will be effectively reduced to some figure moderately below the limit, since these lives will have had small claims. A very small number of lives will be included who have only a very limited residue of benefit remaining.

The effect of this will be to progressively distort the continuance function for each successive year in the direction of a higher attenuation. This effect may be approximately expressed by the following alteration of the formula for the force of termination:

$$_{(\text{agg.})}\pi_x^{(t)} = \frac{k^{(x-x_1)} a}{a' + t},$$

where $x$ is the current age and $x_1$ the age at which the aggregate limitation takes effect. $k$ will be a value slightly greater than 1 determined by judgment or analysis of the probability distributions involved in successive years of tabular or actual experience. The approximation, while rough, provides a simple means of giving some weight to the attrition in the aggregate benefit.

## 7. *Use of Statistical Measures to Determine Constants*

While no methods have been demonstrated in this paper which depend upon statistical measures to obtain values of the constants for an elementary function, it is possible to use these in cases where the entire continuance is to be graduated by a single elementary function. For this purpose, such measurements as the sample mean and variance and the moments of the distribution may be employed to develop equations for solution of the constants. This approach is only mentioned here, and will not be developed further in this introductory study of continuance theory.

## CONCLUSION

This paper has been prepared as an experimental venture into the adaptation of mathematical graduation to disability and medical continuance, in order to achieve simplicity and compactness in the presentation and manipulation of data, and to suggest a basis for standardization of tabular values such as may be used for reserve standards.

Doubtless very much by way of improvement and new technique will occur to others, and the author will feel much rewarded if this theoretical study stimulates further experimentation and if it provides something of practical utility. This study is basically theoretical, and the methods will lend themselves to various practical situations with varying degrees of success. We have found that the methods can be mastered well by clerks who are capable of learning to use the log log slide rule.

The author is extremely grateful to the staff of patient associates who have prepared the several tables in the Appendixes, and in particular to Messrs. James Steinkraus and Stanley Bates, who spent much labor in preliminary research with the functions, and to Mr. William Hoop, who supervised the preparation of the tables.

## APPENDIX A

## SYSTEM OF NOTATION

1. *Basic Symbols*

- $\pi$ Force of termination
- $p$ Probability of continuance of claim
- $q$ Probability of termination of claim
- $z$ Slope of the logarithmic tangent of $p$
- $r$ Annual rate of morbidity
- $r'$ Absolute annual rate of morbidity
- $\rho$ Relative morbidity rate; *i.e.*, ${}^{\theta}\rho$ = fraction of total rate $r$ involving specific benefits $\theta$
- ${}^{\theta}r$ Partial claim rate, *i.e.*, ${}^{\theta}r = r \cdot {}^{\theta}\rho$
- $F$ Average size of claim
- $F'$ Value of complete continuance integral
- $f$ Ratio of incomplete to complete integral
- $f'$ Complement of $f$; *i.e.*, $f' = 1 - f$
- $S$ Annual cost of claim
- $\alpha, \beta, \gamma, \ldots$ Generalized symbols for range constants of Alpha functions. The symbols also represent the generalized benefits represented by the functions.
- $a, b, g, \ldots$ Generalized symbols for attenuation constants of Alpha functions

$\lambda, \mu, \nu, \ldots$ Generalized symbols for range constants of Lambda functions. The symbols also represent the generalized benefits represented by the functions.

$l, m, n, \ldots$ Generalized symbols for attenuation constants of Lambda functions

$\tau$ Minimum duration of claim, *i.e.*, $p^{(t)} = 1.0$ for $t \leq \tau$

$\sigma$ Range constant of an elementary approximation to a composite function, also used to indicate the nature of benefits as composite, or as compound

$s$ Attenuation constant of an elementary approximation to a composite function

$\theta$ General symbol for a benefit or function without reference to its nature as an Alpha or a Lambda function

$R$ Root of a continuance function, *i.e.*, the root of the fraction $f'$. A table of $(R^t)^x$ values may also be used to determine $p^{(t)}$.

$I$ Compression index for uncorrelated coincident benefits

The symbols H, K, P, A, V, N, and D are reserved for use in their customary sense.

2. *Significance of Subscripts and Superscripts*

a. Post-subscript symbols

These are used in their usual sense to mean age. In addition, sex is identified in this position.

(1) $x(m)$ refers to male aged $x$.

(2) $x(f)$ refers to female aged $x$.

b. Post-superscript symbols

These symbols are used in their customary Sickness notation to mean the limits of continuance, with a slight revision.

(1) The symbol $t_1;t_2$ signifies the continuance interval included within the durations $t_1$ and $t_2$. Hence the *period* of benefit $= t_2 - t_1$, and *not* $t_2$, as has been customary. It is the author's opinion that this is a more logical and convenient meaning, since in the formulas these are the constants that are actually employed.

(2) The *absence* of any symbol is to be taken to mean the same as the symbol $0;\infty$, *i.e.*, the limits of the complete function.

c. Pre-superscript symbols

(1) These symbols identify the benefits represented by the function. In generalized discussion, the symbols are $\alpha$, $\beta$, $\gamma$, . . . , for

Alpha functions; $\lambda, \mu, \nu, \ldots$, for Lambda functions; $\theta_1, \theta_2, \ldots$, for unrestricted functions; $\sigma$ for composite or compound functions.

(2) For specified benefits, these are used:
*d* income disability
*h* hospital room and board
*m* miscellaneous expense
*c* doctor calls
*e* diagnostic examinations
*s* surgery
*n* nurse
*p* pregnancy

(3) Primed symbols indicate some special or nonstandard definition or restriction of the benefit.

(4) Where the type of function is to be shown as well as the specified benefit, the symbols $\alpha$ or $\lambda$ are shown as pre-superscripts inside parentheses.

(5) Pre-superscripts are used in condensed notation to identify the numerical values. See section f below.

d. Pre-subscript symbols

These symbols describe any inside limits that control specified benefits and are shown in the exact sequence of the benefit symbols. The symbol $\infty$ indicates "no inside limits" when an unrestricted benefit is in combination with other restricted benefits. This notation will become clear in the examples in section e.

e. Examples of the Notation

(1) ${}^{10h:(h)m:s}_{90:100\,:300}F^{50;\infty}_{40(m)}$ = Average size of claim for a male aged forty for medical benefits with a \$50 deductible outside limit, providing the following:

(a) \$10 per day for daily hospital room and board, up to 90 days
(b) Miscellaneous hospital expense up to \$100
(c) A surgical schedule with top fee of \$300

(2) ${}^{25h:\,s'\,:5c:m}_{\infty:500:\infty:80\%}S^{250;7750}_{30(f)}$ = Annual claim cost for a female aged 30 for major medical benefits with outside limits of \$250 deductible and \$7,500 maximum benefit, and the following inside restrictions:

(a) Covered expense for hospital room and board limited to \$25 daily

(b) Covered surgeons' fees limited by a $500 schedule with special provisions (may be assistant surgeon or anesthesiologist fee provisions)

(c) Covered doctor call expenses limited to $5 daily

(d) Covers 80% of other expense

(3) ${}^{m}_{75\%}p^{(5000)}_{50(m)}$ = Probability that covered expense in a claim on a male aged fifty will reach $5,000 when the coverage is 75% of all medical expenses. (Note that the outside limits of deductible or maximum are not directly relevant to the probability function.)

(4) ${}^{d}p^{(t_1;t_2)}_{40(f)}$ = Probability that a woman aged 40 at date of disablement and still disabled at duration $t_1$, will continue to be disabled at duration $t_2$ from the date of disablement.

(5) $p^{(0;t)} = p^{(t)}$ = Probability that disability will continue to duration $t$ (where $t$ may be in time or monetary units).

These examples illustrate the considerable descriptive power of the system.

f. Condensed Notation for Continuance Definition

It is convenient to employ a condensed shorthand notation to define the continuance data for a certain benefit. For this purpose, pre-superscript symbols are used together with defining values enclosed in parentheses. The enclosed values invariably follow the sequence $(r)$, $(\rho, \alpha, \alpha', a)$ or $(r)$, $(\rho, \lambda, \lambda', l)$.

(1) For example, the notation (.06), ${}^{\alpha}(1.0, 44.3, 39.7, 2.3)$ completely describes the $S$ function defined by a claim rate of .06 and a single Alpha function in which $\alpha = 44.3$, $\alpha' = 39.7$, and $a = 2.3$.

(2) The notation ${}^{d}[(.12), {}^{\alpha}(.9, 2.6, 2.56, 3.0)^{yr.}, {}^{\lambda}(.1, 84.8, 85, 3.4)^{yr.}]$ completely describes the income disability $S$ function which is a compound function composed of an Alpha and a Lambda element with claim rate .12 and constants shown, and in which the continuance time unit is one year.

(3) Where the function is the approximation to a similar function discounted at interest, the pre-superscript symbol $(i)$ is added, thus:

$${}^{(.02)\alpha}(5.7, 5.4, 2.6)\,.$$

This defines an Alpha function which approximates another one discounted at 2%. When only 3 numbers appear, $(\alpha, \alpha', a)$ is indicated and $\rho r$ is not defined.

Where the exact discounted function is meant, the symbol $(i)$ is written over the function instead of as a pre-superscript, thus:

$$ {}^{(.025)a}p^{(t)} \doteq {}^{a}\overset{(.025)}{p}{}^{(t)} = (1.025)^{-t}\left(\frac{a}{a'+t}\right)^{a}. $$

(4) With coincident functions, a brace rather than a bracket is used to distinguish these from compound functions. The difference may also usually be inferred by inspecting the $\rho$ constants of each element. Thus we may have: $\{(.09),\ {}^{\alpha}(1.0, 37.2, 36.8, 2.16),\ {}^{\beta}(1.0, 17.4, 17.3, 3.7)\}$ where the relative rates of both elements evidently overlap.

(5) Pre-superscripts may also be used in connection with combinations of algebraic notational symbols enclosed in parentheses, thus:

$$ {}^{\sigma}F_x = {}^{\theta_1}(\rho F)_x + {}^{\theta_2}(\rho F)_x $$

$$ = {}^{\theta_1}\rho_x \cdot {}^{\theta_1}F_x + {}^{\theta_2}\rho_x \cdot {}^{\theta_2}F_x, $$

so that the $\rho$ and $F$ of each term are different quantities.

# APPENDIX B

## SUMMARY OF BASIC FORMULAS

| Formula | Type of Function | | |
|---|---|---|---|
| | $a$ | $e$ | $\lambda$ |
| 1. $p^{(t)}$, probability of continuance | $\left(\frac{a}{a'+t}\right)^{a}$, $(t \geq a-a')$ | $e^{-(t-a')/F'}$, $(t \geq a')$ | $\left(\frac{\lambda'-t}{\lambda}\right)^{l}$, $(\lambda'-\lambda \leq t \leq \lambda')$ |
| 2. $\pi^{(t)}$, force of termination | $\frac{a}{a'+t}$ | $\frac{1}{F'}$ | $\frac{l}{\lambda'-t}$ |
| 3. "$\pi$" test, to determine type of function from 3 $p^{(t)}$ values | $\frac{(w-v)\log X}{(v-u)\log Y} > 1$ where $X = \frac{p^{(u)}}{p^{(v)}}$ and $Y = \frac{p^{(v)}}{p^{(w)}}$ | $= 1$ | $< 1$ |
| 4. $F'$, the complete continuance integral | $\int_{a-a'}^{\infty} {}^{a}p^{(t)}dt = \frac{a}{a-1}$ | $\int_{a'}^{\infty} {}^{e}p^{(t)}dt = F'$ | $\int_{\lambda'-\lambda}^{\lambda'} {}^{\lambda}p^{(t)}dt = \frac{\lambda}{l+1}$ |
| 5. $F'^{T}$, the incomplete continuance integral | $\int_{a-a'}^{T} {}^{a}p^{(t)}dt = {}^{a}F'\left[1-\left(\frac{a}{a'+T}\right)^{a-1}\right]$ | $\int_{a'}^{T} {}^{e}p^{(t)}dt = F'\left[1-e^{-(T-a')/F'}\right]$ | $\int_{\lambda'-\lambda}^{T} {}^{\lambda}p^{(t)}dt = {}^{\lambda}F'\left[1-\left(\frac{\lambda'-T}{\lambda}\right)^{l+1}\right]$ $(T < \lambda')$ |
| 6. $F^{T}$, the average size of claim within $T$ units | $F'^{T} + a - a' = F'^{T} + \tau$, $(T > \tau)$ | $F'^{T} + a'$, $(T > a')$ | $F'^{T} + \lambda' - \lambda = F'^{T} + \tau$, $(T > \tau)$ |
| 7. Index of Attenuation | $f'^{a} = \left(\frac{a-1}{a}\right)^{a-1}$ | $f'^{e} = .368$ | $f'^{l} = \left(\frac{l}{l+1}\right)^{l+1}$ |
| 8.† ${}^{(\theta_1,\theta_2)}I = \frac{{}^{(I)\theta_1}f'^{a}}{{}^{\theta_1}f'^{a}}$ | $\doteq \frac{1.185\,({}^{\theta_1}F{*}^{T})^{.31-.039a}}{a^{.0621}\,({}^{\theta_2}F{*}^{T})^{.31}}$ | $F$ values may be directly evaluated by integration | No formula developed, but case is rare in practice |

† The index of compression, where $T$ is such that ${}^{\theta_1}f'^{T} = .01$. $a$ = attenuation of $\theta_1$, and * means that the difference between ${}^{\theta_1}\tau$ and ${}^{\theta_2}\tau$ is added to the function of greater $\tau$.

# APPENDIX C

## MEDICAL AND DISABILITY CONTINUANCE TABLES

Seven tables are provided in this Appendix which define the basic continuance functions associated with each basic medical and disability benefit. The medical functions are designed as far as possible for consistency with the 1956 Intercompany Hospital and Surgical Tables. The disability functions are based on the 1952 Disability Study.

TABLE 1

PARTIAL MEDICAL MORBIDITY RATES BASED ON 1956 INTERCOMPANY TABLES

| Age and Sex | Total Medical Morbidity Rate, $r_x$ | ${}^{h_1 s}r_x$ | ${}^{(-s)h_1}r_x$ | ${}^{h_2 s}r_x$ | ${}^{(-s)h_2}r_x$ | ${}^{(-h)s}r_x$ | ${}^{(-hs)m}r_x$ |
|---|---|---|---|---|---|---|---|
| Child | .3878 | .0572 | .0328 | .00000 | .00000 | .0278 | .2700 |
| *Men* | | | | | | | |
| 20 | .3379 | .0423 | .0321 | .00223 | .00168 | .0364 | .2232 |
| 25 | .3316 | .0383 | .0330 | .00211 | .00180 | .0350 | .2214 |
| 30 | .3252 | .0367 | .0337 | .00201 | .00184 | .0354 | .2155 |
| 35 | .3382 | .0357 | .0372 | .00201 | .00209 | .0360 | .2252 |
| 40 | .3732 | .0354 | .0450 | .00205 | .00263 | .0371 | .2510 |
| 45 | .4180 | .0354 | .0546 | .00214 | .00330 | .0388 | .2838 |
| 50 | .4764 | .0367 | .0655 | .00232 | .00413 | .0417 | .3261 |
| 55 | .5505 | .0386 | .0791 | .00255 | .00524 | .0454 | .3796 |
| 60 | .6442 | .0418 | .0953 | .00291 | .00662 | .0505 | .4471 |
| 65 | .7139 | .0427 | .1085 | .00312 | .00796 | .0525 | .4991 |
| 70 | .7536 | .0425 | .1162 | .00329 | .00900 | .0525 | .5301 |
| 75 | .7719 | .0423 | .1189 | .00347 | .00975 | .0525 | .5450 |
| 80 and over | .7812 | .0422 | .1194 | .00367 | .01038 | .0525 | .5531 |
| *Women* | | | | | | | |
| 20 | .3851 | .0575 | .0341 | .00119 | .00071 | .0251 | .2665 |
| 25 | .4467 | .0691 | .0356 | .00159 | .00081 | .0317 | .3079 |
| 30 | .4937 | .0736 | .0412 | .00179 | .00101 | .0351 | .3410 |
| 35 | .5353 | .0724 | .0513 | .00199 | .00141 | .0364 | .3718 |
| 40 | .5734 | .0720 | .0594 | .00224 | .00186 | .0382 | .3997 |
| 45 | .6069 | .0696 | .0683 | .00247 | .00243 | .0393 | .4248 |
| 50 | .6370 | .0648 | .0788 | .00262 | .00318 | .0394 | .4482 |
| 55 | .6653 | .0601 | .0885 | .00275 | .00405 | .0398 | .4701 |
| 60 | .6926 | .0561 | .0968 | .00293 | .00507 | .0410 | .4907 |
| 65 | .7198 | .0522 | .1044 | .00317 | .00633 | .0429 | .5108 |
| 70 | .7440 | .0522 | .1079 | .00353 | .00737 | .0429 | .5301 |
| 75 | .7623 | .0514 | .1104 | .00400 | .00860 | .0429 | .5450 |
| 80 and over | .7716 | .0510 | .1106 | .00442 | .00958 | .0429 | .5531 |

1. *Explanation of the Tables.*

a. Table 1 shows total medical morbidity rates by age and sex, together with the "partial" rates associated with each basic exclusive set of coincident benefits. The symbols have these meanings:

$h_1s$: Surgery coincident with Element 1 of the compound hospitalization table (Table 2).

$(-s)h_1$: Hospitalization under Element 1 without coincident surgery.

$h_2s$: Surgery coincident with Element 2 of the compound hospitalization table.

$(-s)h_2$: Hospitalization under Element 2 without coincident surgery.

$(-h)s$: Nonhospitalized surgery in which no hospitalization occurs on the same disability. Out-of-hospital surgery occurring in the same disability with hospitalization is counted either under $h_1s$ or $h_2s$.

$(-hs)m$: Miscellaneous nonhospital medical expense occurring with neither hospitalization nor surgery on the same disability. Where such expense occurs in the same disability with hospital or surgery expense, it is counted under the other partial rate which is involved.

The partial rates were derived by obtaining values of $\rho$ for each set from sampled claims of Occidental experience, and applying these relative rates to the 1956 Intercompany Hospital and Surgical rates to derive the partials. The "total" rate is the sum of the partials shown, and remains *exclusive* of maternity. Very limited data were available for the $(-hs)m$ benefit.

The fact that the intercompany tables reach level, equal values for men and women at 65 for surgery and at 80 for hospitalization resulted in some inconsistencies between male and female partial rates. Rather than reconcile these, we have "let the chips fall where they may" for the present, so that the male and female partial rates shown do not reach complete equivalence at stipulated ages as in the 1956 Intercompany Tables.

b. Table 2 gives hospital confinement continuance in the form of a 2-element Alpha compound table. The constants of each element are determined to give $\rho^0 = 1$ independently of the other element, the relative incidence of the two elements being given by means of the partial morbidity rates ${}^{h_1}r$ and ${}^{h_2}r$. The unit is one day.

c. Table 3 gives miscellaneous hospital expense in the form of single element Alpha curves. The unit is one dollar.

The table also presents miscellaneous nonhospital medical expense. This includes *all* miscellaneous medical costs not included in hospital, miscellaneous hospital, surgical-anesthesiologist benefits, private nurse, or doctor call benefits. It therefore covers out-of-hospital diagnostic examinations, drugs and supplies, and miscellaneous medical care and services other than surgery, anesthesia, private nurse, or hospital doctor calls. The unit is one dollar.

## TABLE 2

### 1956 Intercompany Table—Hospital Confinement Continuance
(Unit One Day)

| Age and Sex | ${}^{h_1}r_x$ | ${}^{a}h_1$ Function | ${}^{h_2}r_x$ | ${}^{a}h_2$ Function |
|---|---|---|---|---|
| Child..... | .1000 | (3.360, 2.200, 1.98), $\epsilon$ = .906 | 0 | 0 |
| *Men* | | | | |
| 20..... | .07438 | (15.00, 13.10, 4.00) | .003915 | (24.68, 22.78, 1.500) |
| 25..... | .07309 | (14.76, 12.86, 3.95) | .003910 | (25.49, 23.59, 1.475) |
| 30..... | .07045 | (15.05, 13.15, 3.90) | .003855 | (26.32, 24.42, 1.450) |
| 35..... | .07290 | (16.74, 14.84, 3.85) | .004103 | (27.08, 25.18, 1.425) |
| 40..... | .08042 | (19.11, 17.21, 3.80) | .004680 | (27.78, 25.88, 1.400) |
| 45..... | .08996 | (22.33, 20.43, 3.75) | .005441 | (28.50, 26.60, 1.375) |
| 50..... | .1022 | (25.06, 23.16, 3.70) | .006455 | (29.07, 27.17, 1.350) |
| 55..... | .1177 | (26.16, 24.26, 3.65) | .007785 | (29.75, 27.85, 1.325) |
| 60..... | .1371 | (25.81, 23.91, 3.60) | .009529 | (30.29, 28.39, 1.300) |
| 65..... | .1512 | (27.04, 25.14, 3.55) | .01108 | (30.86, 28.96, 1.275) |
| 70..... | .1587 | (30.96, 29.06, 3.50) | .01229 | (31.39, 29.49, 1.250) |
| 75..... | .1612 | (49.18, 47.28, 3.45) | .01322 | (31.88, 29.98, 1.225) |
| 80..... | .1616 | (65.29, 63.39, 3.40) | .01405 | (32.41, 30.51, 1.200) |
| *Women* | | | | |
| 20..... | .09163 | (181.0, 179.6, 32.0) | .001870 | (178.0, 176.6, 5.70) |
| 25..... | .1047 | (183.9, 182.5, 31.5) | .002356 | (178.4, 177.0, 5.66) |
| 30..... | .1148 | (195.0, 193.6, 31.0) | .002822 | (178.9, 177.5, 5.61) |
| 35..... | .1237 | (205.3, 203.9, 30.5) | .003432 | (179.4, 178.0, 5.56) |
| 40..... | .1314 | (215.8, 214.4, 30.0) | .004065 | (180.0, 178.6, 5.50) |
| 45..... | .1379 | (226.6, 225.2, 29.5) | .004855 | (180.7, 179.3, 5.43) |
| 50..... | .1436 | (237.0, 235.6, 29.0) | .005827 | (181.4, 180.0, 5.36) |
| 55..... | .1486 | (249.0, 247.6, 28.5) | .006838 | (182.2, 180.8, 5.28) |
| 60..... | .1529 | (264.0, 262.6, 28.0) | .008045 | (183.0, 181.6, 5.20) |
| 65..... | .1566 | (288.0, 286.6, 27.5) | .009468 | (183.9, 182.5, 5.11) |
| 70..... | .1601 | (329.0, 327.6, 27.0) | .01094 | (184.9, 183.5, 5.01) |
| 75..... | .1618 | (492.0, 490.6, 26.5) | .01256 | (185.9, 184.5, 4.91) |
| 80..... | .1616 | (629.0, 627.6, 26.0) | .01405 | (187.0, 185.6, 4.80) |

## TABLE 3

### Miscellaneous Expense Continuance
(Unit One Dollar)

| Age | ${}^{(h)}m r_x$ | | ${}^{a(h)}m$ Function | ${}^{(-h)}m r_x$ | | ${}^{a(-h)}m$ Function |
|---|---|---|---|---|---|---|
| Child. | .1000 | | (444.0, 440.0, 7.00) | .3730 | | (86.30, 81.30, 5.022) |
| | Men | Women | | Men | Women | |
| 20... | .0783 | .0935 | (489.5, 485.1, 6.00) | .3115 | .3700 | (85.70, 80.70, 4.350) |
| 25... | .0770 | .1071 | (482.3, 471.0, 5.75) | .3084 | .4250 | (89.82, 84.82, 4.182) |
| 30... | .0743 | .1176 | (472.1, 454.3, 5.50) | .2998 | .4686 | (92.47, 87.47, 4.015) |
| 35... | .0770 | .1271 | (460.3, 436.2, 5.25) | .3122 | .5089 | (94.23, 89.23, 3.848) |
| 40... | .0851 | .1355 | (445.4, 415.0, 5.00) | .3461 | .5452 | (94.93, 89.93, 3.680) |
| 45... | .0954 | .1428 | (431.8, 395.8, 4.75) | .3892 | .5776 | (94.94, 89.94, 3.512) |
| 50... | .1087 | .1494 | (415.4, 373.7, 4.50) | .4448 | .6076 | (94.03, 89.03, 3.345) |
| 55... | .1255 | .1554 | (398.2, 351.1, 4.25) | .5151 | .6355 | (92.37, 87.37, 3.178) |
| 60... | .1466 | .1609 | (380.1, 327.7, 4.00) | .6037 | .6616 | (89.98, 84.98, 3.010) |
| 65... | .1623 | .1661 | (361.2, 303.8, 3.75) | .6714 | .6869 | (86.97, 81.97, 2.842) |
| 70... | .1710 | .1710 | (341.0, 270.0, 3.50) | .7111 | .7111 | (86.85, 81.85, 2.675) |
| 75... | .1744 | .1744 | (308.0, 225.0, 3.25) | .7294 | .7294 | (82.87, 77.87, 2.508) |
| 80... | .1756 | .1756 | (312.0, 224.0, 3.00) | .7387 | .7387 | (81.74, 76.74, 2.340) |

d. Table 4 gives surgical expense continuance for a \$100 California Relative Value Schedule. Hospitalized surgery (either $h_1s$, or $h_2s$) is separated from nonhospitalized [$(-h)s$] and each is graduated by a single element Alpha function. This table is therefore based on surgical benefits differing from those of the "standard" schedule of the 1956 Intercompany

TABLE 4

SURGERY CONTINUANCE
\$100 CALIFORNIA RELATIVE VALUE SCHEDULE
(Unit One Dollar)

| Age and Sex | ${}^{(h)s}r_x$ | ${}^{\alpha}(h)s$ Function | ${}^{(-h)s}r_x$ | ${}^{\alpha}(-h)s$ Function |
|---|---|---|---|---|
| Child....... | .0572 | (50.55,44.55,6.00) | .0278 | (19.75,17.75,7.5) |
| *Men* | | | | |
| 20....... | .04455 | (90.75,84.75,5.500) | .03645 | (26.89,24.89,7.000) |
| 25....... | .04038 | (92.95,86.95,5.375) | .03502 | (28.43,26.43,6.875) |
| 30....... | .03872 | (94.38,88.38,5.250) | .03538 | (28.75,26.75,6.750) |
| 35....... | .03763 | (97.33,91.33,5.125) | .03607 | (28.87,26.87,6.625) |
| 40....... | .03715 | (101.2,95.22,5.000) | .03715 | (28.99,26.99,6.500) |
| 45....... | .03743 | (112.3,106.3,4.875) | .03887 | (31.13,29.13,6.375) |
| 50....... | .03894 | (125.2,119.2,4.750) | .04176 | (33.78,31.78,6.250) |
| 55....... | .04119 | (135.5,129.5,4.625) | .04541 | (35.95,33.95,6.125) |
| 60....... | .04474 | (135.5,129.5,4.500) | .05046 | (35.66,33.66,6.000) |
| 65....... | .04577 | (135.0,129.0,4.375) | .05253 | (35.61,33.61,5.875) |
| *Women* | | | | |
| 20....... | .05866 | (87.14,80.14,6.000) | .02514 | (9.410,7.410,2.800) |
| 25....... | .07085 | (101.0,93.98,6.246) | .03165 | (9.060,7.060,2.700) |
| 30....... | .07535 | (120.1,113.1,6.406) | .03515 | (9.500,7.500,2.600) |
| 35....... | .07455 | (139.1,132.1,6.488) | .03645 | (10.10,8.100,2.500) |
| 40....... | .07418 | (155.7,148.7,6.500) | .03822 | (10.67,8.670,2.400) |
| 45....... | .07196 | (159.6,152.6,6.449) | .03934 | (10.56,8.560,2.300) |
| 50....... | .06741 | (152.3,145.3,6.344) | .03949 | (9.870,7.870,2.200) |
| 55....... | .06285 | (145.0,138.0,6.191) | .03985 | (9.310,7.310,2.100) |
| 60....... | .05900 | (140.4,133.4,6.000) | .04100 | (8.930,6.930,2.000) |
| 65....... | .05547 | (135.1,128.1,5.777) | .04283 | (8.450,6.450,1.900) |

TABLE 5

Hospital Confinement Continuance by days:
1952 Group Hospital Study: 31 day plans, 10× Miscellaneous Benefit (*TSA* IV, 87, 89, 93)
The functions graduate Col. 3, excluding 0 days

| | ${}^{h_1}p$ | ${}^{\alpha}h_1$ | ${}^{h_2}p$ | ${}^{\alpha}h_2$ |
|---|---|---|---|---|
| Child................ | 1.0 | (3.36, 2.2, 1.98), $\epsilon = .906$ | 0 | 0 |
| Male................ | .945 | (17.825, 15.925, 3.7846) | .055 | (28.43, 26.53, 1.397) |
| Female (Nonmaternity) | .966 | (191.4, 190, 34) | .034 | (185.4, 184, 5.7) |

TABLE 6a

1952 DISABILITY TABLE—PERIOD 2, BENEFIT 2

(Unit One Month)

| Central Age $x$ | 1,000 ${}^{d_1}r'_{x\,90}$ | $d_1$ Function | 1,000 ${}^{d_2}r'_{x\,90}$ | $\lambda_{d_2}$ Function | 1,000 ${}^{d_3}r'_{x\,90}$ | $\lambda_{d_3}$ Function | 1,000 ${}^{d_4}r'_{x\,90}$ | $\lambda_{d_4}$ Function | 1,000 ${}^{d_5}r'_{x\,90}$ | $d_5$ Function |
|---|---|---|---|---|---|---|---|---|---|---|
| 17.5 | .1566 | (939.2, 942.2, 2.199) | .0112 | (822.5, 825.5, 1.379) | .2406 | (484.0, 487.0, 3.410) | .3806 | (219.4, 222.4, 4.721) | 1.651 | $\alpha$(165.7, 162.7, 17.08) |
| 22.5 | .2516 | (883.6, 886.6, 2.222) | .2272 | (401.7, 404.7, 3.045) | .6844 | (653.4, 656.4, 22.83) | 0 | 0 | 1.277 | $\alpha$(22.99, 19.99, 4.610) |
| 27.5 | .4128 | (862.9, 865.9, 2.705) | .0328 | (667.8, 670.8, 1.649) | .3244 | (396.5, 399.5, 4.324) | 0 | 0 | 2.160 | $\alpha$(287.5, 284.5, 21.48) |
| 32.5 | .5476 | (714.8, 717.8, 1.829) | .0911 | (714.2, 717.2, 3.143) | 0 | 0 | 0 | 0 | 2.513 | $\alpha$(49.62, 46.62, 3.177) |
| 37.5 | .7828 | (686.2, 689.2, 1.859) | .2530 | (558.6, 561.6, 2.304) | .5577 | (263.3, 266.3, 4.134) | 1.071 | (58.67, 61.67, 2.640) | 1.766 | $\lambda$(25.45, 28.45, 5.737) |
| 42.5 | 1.728 | (683.8, 686.8, 2.727) | .1896 | (485.9, 488.9, .6027) | .6585 | (157.3, 160.3, 2.982) | 2.252 | (80.02, 83.02, 5.765) | 1.490 | $\lambda$(7.133, 10.133, 1.926) |
| 47.5 | 3.049 | (605.7, 608.7, 2.616) | .3325 | (427.3, 430.3, .5888) | 2.464 | (777.8, 780.8, 33.40) | 0 | 0 | 3.495 | $\alpha$(35.05, 32.05, 7.372) |
| 52.5 | 5.056 | (549.9, 552.9, 2.600) | .7369 | (372.1, 375.1, .7278) | 1.361 | (154.0, 157.0, 2.955) | 2.999 | (72.47, 75.47, 4.115) | 3.418 | $\lambda$(14.81, 17.81, 3.420) |
| 57.5 | 10.83 | (518.0, 521.0, 3.200) | .7206 | (386.7, 389.7, .8321) | .7617 | (297.3, 300.3, .2874) | 3.562 | (11.61, 14.61, 2.940) | 5.767 | $\alpha$(1661., 1658., 87.15) |

## TABLE 6b

"MODIFIED" 1952 DISABILITY TABLE—PERIOD 2, BENEFIT 2

(Unit One Month)

| Central Age $x$ | 1,000 ${}^{d_1}r'_x$ | ${}^{\lambda}d_1$ Function | 1,000 ${}^{d_2}r'_x$ | ${}^{\lambda}d_2$ Function | 1,000 ${}^{d_3}r_x$ | ${}^{a}d_3$ Function |
|---|---|---|---|---|---|---|
| 17.5 | .1691 | (990.0, 990.0, 2.324) | .2457 | (486.0, 486.0, 3.399) | 2.501 | (217.7, 217.7, 15.00) |
| 22.5 | .2534 | (930.0, 930.0, 2.381) | .2325 | (426.0, 426.0, 3.262) | 2.485 | (186.5, 186.5, 14.60) |
| 27.5 | .4495 | (870.0, 870.0, 2.530) | .2936 | (366.0, 366.0, 3.295) | 2.784 | (176.6, 176.6, 13.80) |
| 32.5 | .6443 | (810.0, 810.0, 2.346) | .4230 | (306.0, 306.0, 3.592) | 3.168 | (148.1, 148.1, 12.70) |
| 37.5 | 1.046 | (750.0, 750.0, 2.354) | .5839 | (246.0, 246.0, 3.736) | 4.218 | (87.71, 87.71, 11.40) |
| 42.5 | 1.878 | (690.0, 690.0, 2.406) | .6985 | (186.0, 186.0, 3.628) | 5.697 | (72.94, 72.94, 10.00) |
| 47.5 | 3.421 | (630.0, 630.0, 2.428) | .7049 | (126.0, 126.0, 4.327) | 8.284 | (59.87, 59.87, 8.600) |
| 52.5 | 6.077 | (570.0, 570.0, 2.481) | 0 | 0 | 11.84 | (50.47, 50.47, 7.300) |
| 57.5 | 12.51 | (510.0, 510.0, 2.753) | 0 | 0 | 12.35 | (67.81, 67.81, 6.200) |
| 62.5 | 24.11 | (450.0, 450.0, 2.844) | 0 | 0 | 13.22 | (104.0, 104.0, 5.400) |
| 67.5 | 39.76 | (390.0, 390.0, 2.480) | 0 | 0 | 13.83 | (130.7, 130.7, 5.000) |

Additional Extension Function for 1st 90 days (All Ages) ${}^{d_4}r' = .5$ $\quad d_4$ Function $= {}^{a}(1.633, 1.633, 6.6)$

Tables, but we decided on this course since our available experience was largely on the California Relative Value basis. The unit is one dollar.

e. Table 5 presents functional graduations of the hospital continuance derived from the 1952 Intercompany Hospital Study compiled by Mr. Gingery. The values shown are derived from the data for 31 day plans with 10× miscellaneous benefits. The unit is one day.

f. Table 6*a* presents an extremely precise functional graduation of the 1952 Disabled Life data (*TSA* 1952 Reports), using Tables 7 and 8 of that report which give 1930–1950 termination rates under Benefit 2. The unit is one month and the constants are adjusted to give $p^{(3)} = 1$ for each element, so that the functions are intended for use with disability rates giving the rate of entering upon the 90th day of disability, such as the Period 2 rates given in Table 2 of the 1952 Reports.

Table 6*b* is a modification of 6*a* providing a simplified set of functions and with a function included for extension of the table to elimination periods of less than 90 days. This table, in relation to the 1952 Tables, thus serves a similar purpose to the Conference Modification of the 1926 Class 3 Table. The constants in Table 6*b* give $p^{(0)} = 1$ and the unit is one month.

g. Table 7 gives values of $f'^a$ for various values of $a$, useful in computing "compression" constants.

TABLE 7

VALUES OF INDEX OF ATTENUATION $f'^a$ OR $f'^l$

| $a$ | $f'^a$ | $a$ | $f'^a$ | $l$ | $f'^l$ | $l$ | $f'^l$ |
|---|---|---|---|---|---|---|---|
| 1.00 | 1.000 | 1.80 | .522 | ∞ | .368 | 1.20 | .264 |
| 1.01 | .955 | 1.90 | .510 | 100.00 | .366 | 1.10 | .257 |
| 1.02 | .924 | 2.00 | .500 | 30.00 | .365 | 1.00 | .250 |
| 1.03 | .899 | 2.50 | .465 | 20.00 | .356 | .90 | .241 |
| 1.04 | .878 | 3.00 | .445 | 10.00 | .350 | .80 | .232 |
| 1.05 | .858 | 3.50 | .432 | 9.00 | .349 | .70 | .221 |
| 1.06 | .841 | 4.00 | .422 | 8.00 | .347 | .60 | .209 |
| 1.07 | .826 | 5.00 | .410 | 7.00 | .344 | .55 | .200 |
| 1.08 | .812 | 6.00 | .402 | 6.00 | .340 | .50 | .192 |
| 1.09 | .798 | 7.00 | .396 | 5.00 | .335 | .45 | .183 |
| 1.10 | .787 | 8.00 | .393 | 4.00 | .328 | .40 | .173 |
| 1.15 | .737 | 9.00 | .390 | 3.50 | .323 | .35 | .162 |
| 1.20 | .698 | 10.00 | .387 | 3.00 | .316 | .30 | .149 |
| 1.30 | .644 | 20.00 | .377 | 2.50 | .308 | .25 | .134 |
| 1.40 | .606 | 30.00 | .374 | 2.00 | .297 | .20 | .117 |
| 1.50 | .577 | 100.00 | .370 | 1.90 | .294 | .15 | .096 |
| 1.60 | .555 | ∞ | .368 | 1.80 | .290 | .10 | .072 |
| 1.70 | .537 | | | 1.70 | .286 | .075 | .057 |
| | | | | 1.60 | .283 | .050 | .041 |
| | | | | 1.50 | .280 | .025 | .022 |
| | | | | 1.40 | .275 | .01 | .0095 |
| | | | | 1.30 | .270 | .00 | .0000 |

## 2. *Method of Derivation of the Tables*

### a. Table 2

Table 2 was obtained by using a preliminary functional graduation of Gingery's 1952 data as a starting point. The Gingery graduation for children was adopted directly; this is a single element curve with an $\epsilon$ value to adjust the values of $F$. We assumed the adult data to be reasonably representative of experience near age 40. Occidental experience was sampled by age groups to get an approximate idea of the shift in the continuance pattern by age. From this we prepared graduated values of the attenuation for both elements, the relative incidence of the elements, and the range constants for the $h_2$ element, at quinquennial ages up to 80. The range constants for the $h_1$ element were then solved from the $F^{90}$ values given in the 1956 Intercompany Hospital Tables, so that the functions of Table 2 are consistent with the 1956 tables in giving equal $F^{90}$ values.

There appears to be some question as to the necessity of the two-element graduation used for adults. The extrapolation of the 1952 data beyond 31 days, which is largely responsible for the introduction of the second element, is obtained from data not fully homogeneous with the first 31 days. We nevertheless retained the compound graduations because the results obtained from the two-element compound function are more consistent with Table B of the paper, "Reserves for Individual Hospital and Surgical Expense Insurance" (*TSA* IX), by Bartleson and Olsen, than is the case when single element graduation is attempted. This question of validity, however, is sufficient to call for caution in any attempt to use the compound graduations given here where extended continuance is involved.

### b. Table 3

Table 3 gave us considerable trouble. We attempted a straight-forward $F$-function solution of the constants by using the 1956 Intercompany Tables directly, which give values of $F^{25}$, $F^{50}$, $F^{100}$, $F^{150}$, and $F^{250}$. The resulting solutions tested out exceptionally well, but with one disturbing defect—there was absolutely no orderly progression by ascending ages in the values of the constants. The constants of each curve appeared to be completely independent of its neighbors, possibly because of the high sensitivity of the constants to small changes in their underlying equations. We finally simply adopted a rather arbitrary scale of attenuation constants graduated by constant first differences only, and solved for the remaining values on $F^{50}$, $F^{100}$, and $F^{150}$, obtaining results that tested out within 9% of any other value in the 1956 tables. These results were adopted, except that we adopted $\tau$ rather than $\epsilon$ for $F$ values.

The out-of-hospital portion of Table 3 is based on extremely limited data, primarily experience with diagnostic benefits alone. We were forced to make very broad comparisons between hospital and nonhospital miscellaneous costs and then to derive the $(-h)m$ functions by appropriate modification of the miscellaneous hospital functions. In general, the modifications assume that the nonhospital morbidity rate is approximately 400% of the hospital, whereas the average size of claim is approximately 25% of the miscellaneous hospital average claim. It was finally assumed that about 75% of such claims are incurred without coincident hospital or surgical expense, in computing ${}^{(-hs)m}r_x$ for Table 1.

c. Table 4

To obtain Table 4, we sampled Occidental experience by hospitalized versus nonhospitalized surgery to obtain the continuance patterns and relative incidence. The resulting pairs of graduation functions were graduated by age to give smoothly progressing constants, and then adjusted to equate to the 1956 table values of $F$ as modified to fit the California Relative Value Schedule.

d. Table 5

The 1952 functions shown in this table are the basic curves underlying Table 2. These curves were obtained by 6-point graphic graduation (2-element compounds) of the 1952 continuance tables presented by Mr. Gingery.

e. Table 6

Table 6*a* is a direct functional graduation of the Benefit 2 termination rates in the 1952 Disability study (*TSA* 1952 Reports), using either 12- or 15-point graduation (*i.e.*, 4 or 5 elements). The resulting functions provide an extremely faithful reproduction of the 1952 Tables except for some deviation at the extreme ends of the disability curves.

Table 6*b* is a modification aimed at reducing the number of elements, for greater computing convenience. The method of derivation was to group the elements at each age in 6*a* according to similarities in the curve values and combine various pairs of elements by substituting for each combined pair a single element producing $r'$ and $F'$ values equal to the sum of those of the pair. The values of $r'$ in the table are adjusted for a duration of zero rather than the 3 months duration of the 1952 rates. The constants of each element produce ${}^{\theta}p^{(0)} = 1$.

Table 6*b* also provides an extension of values for central ages 62.5 and

67.5. In addition, an extra function is supplied to extend the table values back to durations under 3 months. A single function for all ages appears to be sufficient for this purpose. Thus any claim annuity evaluations for eliminations of less than 90 days will involve compound functions of 4 elements through age 47.5 and 3 elements for higher ages.

The functions can be readily discounted for any rate of interest by the methods of section III-4.

### 3. *Limitations in the Tables*

As has been pointed out, the tables are at some points based on extremely limited data. This is particularly true of the graduations of the constants by age, which vary from graduations assuming constant third differences down to graduations that merely assume constant first differences in some cases. Thus, while the medical tables as a whole conform very closely to the 1956 Intercompany Hospital and Surgical Tables, they still contain arbitrary features and should be used with caution, especially in evaluating benefits such as major medical that involve extrapolation of benefits beyond the data on which the tables are based. This paper is primarily theoretical, and the medical tables are not intended to be an authoritative graduation of reliable experience in large volume. The 1952 Disability functions, however, reproduce the 1952 Tables quite faithfully.

## DISCUSSION OF PRECEDING PAPER

JOHN H. MILLER:

Mr. Barnhart is to be congratulated on an excellent paper presenting techniques for constructing and graduating continuance tables for which there has been a very definite need.

It would be very desirable if, in the discussion of this paper, some member of the Society should present a historical account of the development of the continuance table. Doubtless Mr. Cammack's classic paper in Volume VII of the *Proceedings* of the Casualty Actuarial Society, in which he presented a modification of the old Manchester Unity Sickness Tables, would stand out as an important landmark in this development.

### (AUTHOR'S REVIEW OF DISCUSSION)

E. PAUL BARNHART:

I greatly appreciate Mr. Miller's kind remarks concerning the paper, and I hope that the techniques described will prove as useful to him and others as they have to me in my own work with continuance data.

Two questions were asked of me repeatedly by individuals discussing the paper with me and, while they have not been raised through written discussion, I will nevertheless attempt to provide some answer to these inquiries:

1. Do I have available any recent disability continuance data graduated by the techniques described?
2. How practical are the continuance functions when it comes to constructing claim costs by electronic computer?

In answer to the first, we conducted at Occidental a study of the experience under commercial disability policies during the years 1953 and 1954. Because of limitations in the exposure size of various subgroups, all occupational classes were thrown together, so that the results were an average of all occupational classes accepted by the Company.

These two years proved to be extremely favorable experience years for the Company, and it was therefore decided that a minimum safe level for a table representing expected rates and continuance of commercial disability claims should be about 150% of this 1953–54 experience.

The study was limited to the first year of claim duration, and beyond the first year the graduation was very slowly merged into the Benefit 2, Period 2, data of the 1952 disability study by altering the functions in Table 6*b* of the paper, which presents the "modified" 1952 Table, to make these functions reproduce approximately 150% of the first year experience in the study. We found that it was possible to accomplish this by using two elementary functions, one of them the long term $d_1$ function from the Modified Table, the other an adjustment function to fit the total continuance to the first year study data.

The following table resulted as a composite of experience under elimination periods ranging from zero to ninety days. Because of this composite nature of the table, it appeared adequate for elimination periods of 30 days or longer, but not necessarily adequate for shorter eliminations.

BASIC COMMERCIAL DISABILITY TABLE

(Unit One Month)

| Central Age $x$ | 1,000 $_{d_1}r'_x$ | $\lambda d_1$ Function | 1,000 $^{d_1}S^\infty_x$ | 1,000 $_{d_2}r'_x$ | $\alpha d_2$ *Function* | 1,000 $^{d_2}S^\infty_x$ |
|---|---|---|---|---|---|---|
| 17.5........ | .1691 | (990.0, 990.0, 2.324) | 50.36 | 500.0 | ( .327, .327, 2.0) | 163.5 |
| 22.5........ | .2534 | (930.0, 930.0, 2.381) | 69.70 | 500.0 | ( .388, .388, 2.2) | 161.7 |
| 27.5........ | .4270 | (870.0, 870.0, 2.530) | 105.2 | 500.0 | ( .448, .448, 2.4) | 160.0 |
| 32.5........ | .5799 | (810.0, 810.0, 2.346) | 140.4 | 500.0 | ( .509, .509, 2.6) | 159.0 |
| 37.5........ | .8368 | (750.0, 750.0, 2.354) | 187.1 | 500.0 | ( .808, .808, 3.5) | 161.6 |
| 42.5........ | 1.333 | (690.0, 690.0, 2.406) | 270.0 | 550.0 | (1.020, 1.020, 4.4) | 165.0 |
| 47.5........ | 2.292 | (630.0, 630.0, 2.428) | 421.2 | 580.0 | (1.458, 1.458, 6.0) | 169.1 |
| 52.5........ | 3.889 | (570.0, 570.0, 2.481) | 636.8 | 620.0 | (1.633, 1.633, 6.6) | 180.8 |
| 57.5........ | 9.132 | (510.0, 510.0, 2.753) | 1,241.0 | 650.0 | (1.633, 1.633, 6.6) | 189.5 |
| 62.5........ | 19.53 | (450.0, 450.0, 2.844) | 2,286.3 | 675.0 | (1.633, 1.633, 6.6) | 196.8 |
| 67.5........ | 35.78 | (390.0, 390.0, 2.480) | 4,009.8 | 700.0 | (1.633, 1.633, 6.6) | 204.1 |

The exposure was also limited by the fact that none of it was older than the fifth policy year. Exposure in the first policy year was excluded. Consequently, we concluded that the experience was hardly ultimate, and that it was reasonable to assume that it would understate an ultimate experience to an increasing extent with advancing age. Accordingly, the $d_2$ adjustment function was modified by progressively increasing the value of $^{d_2}S^\infty_x$ up to the oldest central age of 67.5, using a completely arbitrary ratio scale with constant third differences. The attenuations of each function were then lowered to spread more of the continuance beyond the first two or three months, since the original $d_2$ function of the table attenuates extremely rapidly into insignificant values. Finally, the ranges were modified so as to preserve the $S^\infty$ values already determined.

The final table, which we named the "Occidental 1956 Commercial Disability Table," is as follows. The table includes values of $\rho_x$ from which accident only values may be computed by applying the $\rho_x$ ratios to the desired values from the main table.

OCCIDENTAL 1956 COMMERCIAL DISABILITY TABLE
(Unit One Month)

| Central Age $x$ | $1{,}000\ {}^{d_1}r'_x$ | $\lambda_{d_1}$ Function | $1{,}000\ {}^{d_1}S^{\infty}_x$ | $1{,}000\ {}^{d_2}r'_x$ | $a_{d_2}$ Function | $1{,}000\ {}^{d_2}S^{\infty}_x$ |
|---|---|---|---|---|---|---|
| 17.5 | .1691 | (990.0, 990.0, 2.324) | 50.36 | 500.0 | ( .327 , .327 , 2.000) | 163.5 |
| 22.5 | .2534 | (930.0, 930.0, 2.381) | 69.70 | 500.0 | ( .3588, .3588, 2.100) | 163.1 |
| 27.5 | .4270 | (870.0, 870.0, 2.530) | 105.2 | 500.0 | ( .4592, .4592, 2.400) | 164.0 |
| 32.5 | .5799 | (810.0, 810.0, 2.346) | 140.4 | 500.0 | ( .6112, .6112, 2.835) | 166.5 |
| 37.5 | .8368 | (750.0, 750.0, 2.354) | 187.1 | 525.0 | ( .7862, .7862, 3.340) | 176.4 |
| 42.5 | 1.333 | (690.0, 690.0, 2.406) | 270.0 | 550.0 | ( .9690, .9690, 3.850) | 187.0 |
| 47.5 | 2.292 | (630.0, 630.0, 2.428) | 421.2 | 580.0 | (1.142 , 1.142 , 4.300) | 200.7 |
| 52.5 | 3.889 | (570.0, 570.0, 2.481) | 636.8 | 620.0 | (1.334 , 1.334 , 4.625) | 228.2 |
| 57.5 | 9.132 | (510.0, 510.0, 2.753) | 1241.0 | 650.0 | (1.491 , 1.491 , 4.760) | 257.8 |
| 62.5 | 19.53 | (450.0, 450.0, 2.844) | 2286.3 | 675.0 | (1.571 , 1.571 , 4.640) | 291.3 |
| 67.5 | 35.78 | (390.0, 390.0, 2.480) | 4009.8 | 700.0 | (1.516 , 1.516 , 4.20 ) | 331.6 |

Values of (acc.) $\rho'_x$, *i.e.*, the fraction of $r'_x$ assumed to arise from accident claims

| Central Age | (acc.) ${}^{d_1}\rho'_x$ | (acc.) ${}^{d_2}\rho'_x$ |
|---|---|---|
| 17.5 | .300 | .200 |
| 22.5 | .270 | .195 |
| 27.5 | .243 | .190 |
| 32.5 | .219 | .185 |
| 37.5 | .198 | .180 |
| 42.5 | .180 | .175 |
| 47.5 | .165 | .170 |
| 52.5 | .153 | .165 |
| 57.5 | .144 | .160 |
| 62.5 | .138 | .155 |
| 67.5 | .135 | .150 |

Since the table is not based on known ultimate disability experience, it is obviously not to be relied upon as a recent table suitable for either reserves or gross premium computation. Nevertheless it may have considerable value to other actuaries as a comparison against their own experience data or assumptions.

In answer to the second question, the functions are nicely adapted to computer use. In fact, convenient computer application was one of the reasons for the attempt to develop a mathematical basis for continuance graduation.

To perform computer calculations with the functions, the basic problem is to find a suitable technique for digital approximation of the exponential expressions involved. Any of several methods may be employed, but the most efficient is probably one of these two:

1. A storage table of 4 or 5 place logarithms to 3 place arguments, thus requiring storage of 1,000 table values.

2. Approximation using the binomial expansion.

*Method 1*

This technique consumes very little computer time and is sufficiently accurate for most purposes. Since the roots requiring evaluation are quantities between 0 and 1, it is ordinarily preferable to store the logarithms in their negative form.

*Method 2*

This approach is more accurate than Method 1, since the computations may be carried out to any desired degree of accuracy. It also releases much of the storage capacity tied up by the log table in Method 1. The computations, however, require more computer time.

The technique is to evaluate any required root, $R^T$, and then consider the desired value of $f'^T$ in the form $(1 - s)^a$, where $s = 1 - R$.

By the binomial expansion we then have the series

$$f' = 1 - a\,s + \frac{a\,(a-1)}{2!}\,s^2 - \frac{a\,(a-1)\,(a-2)}{3!}\,s^3 + \ldots,$$

wherein any term

$$T_{n+1} = -\,sT_n \cdot \frac{a - n + 1}{n}.$$

Thus the series may be conveniently generated by obtaining each successive term from the preceding one. Moreover, a very simple test is available whereby the limit of the maximum error involved by stopping at any given term may be computed.

By performing the ratio test for a convergent series, we find:

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = s,$$

so that the maximum value of the series beyond any given stopping term, $T_n = k$, is of the form:

$$k\,(s + s^2 + s^3 + \ldots)$$

or

$$\frac{k\,s}{1 - s}.$$

Thus the value of $s/(1 - s)$ may be computed and each value of $T_n$ readily tested to determine whether sufficient terms have been computed for the desired degree of accuracy. Usually 12 to 18 terms will develop results accurate within .00001, which is more than sufficient. The method is the suggestion of Mr. William Hoop.