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# DISCUSSION OF PRECEDING PAPER

PAUL M. KAHN:

Mr. Feay's paper, "Introduction to Nonproportional Reinsurance," provides a most welcome discussion of the topic and seems indicative of a growing interest in nonproportional reinsurance evinced by American actuaries. European actuaries have for some time been working in this field and have developed a considerable body of mathematical literature devoted to an analysis and solution of the problems attendant upon the use of stop-loss reinsurance, in particular upon the calculation of premiums. Much of this work is concerned with the collective theory of risk developed principally by Scandinavian and Swiss actuaries. The purpose of this discussion is to illustrate briefly the application of risk theory to the problem described by Mr. Feay, that of calculating stop-loss reinsurance premiums.

In Mr. Feay's expression for the nonproportional stop-loss reinsurance premium

$$_{R}\mathrm{P}_{L}^{H}=\frac{R}{M}\int_{t=L}^{t=H}(t-L)\,f(t)\,dt\,,$$

he assumes that the amount, $t$, in dollars of death claims for $S$ dollars of insurance for $N$ lives is normally distributed with mean $M$ and standard deviation $\sigma_M$. Let us take $R=100\%$ and $H=\infty$. In collective risk theory, one considers the gain from the risk business of the company as a whole, rather than the gains arising from individual policies or risks. Hence $f(t)$ is a function only of the expected number of claims and of the distribution of the size of individual claims. Using the formulas given in Ammeter's paper in "Non-Proportional Reinsurance" edited by Stefan Vajda (number 1 of Feay's bibliography), one can make calculations parallel to those in Tables 3 and 4 of Mr. Feay's paper.

For large retention limits the differences are somewhat more significant; this circumstance is to be expected, for Ammeter remarks that the exact distribution of claims is remarkably skew, "so that in fact heavy deviations occur more frequently than under the normal distribution. Evidently this skewness is significant for comparatively large values of $u$ (the maximum retention limit); only if $u$ approaches to unity does the normal distribution lead to sufficiently accurate premium values." (*Op. cit.*, pp. 91-92.) The differences between these figures and those of Table 3 in Mr. Feay's paper are small, but for Table 4 the risk theoretic approach gives values which are greater than Mr. Feay's, especially for large retention limits. These calculations—admittedly only spot-checked—were based upon the expected number of deaths and the distribution of individual claims given in Mr. Rosenthal's paper, to which Mr. Feay makes reference. The case of $100,000 maximum insurance on one life involves a division by size of all the policies into twelve classes; the average size policy

TABLE 3'

NONPROPORTIONAL REINSURANCE PREMIUMS FOR LIFE INSURANCE GROUP

| Maximum Retention Limit | Premiums as Percentage of Expected Claim Costs |
|---|---|
| 100% | 5.23% |
| 113 | 1.15 |
| 120 | 0.41 |
| 125 | 0.17 |
| 135 | 0.02 |

TABLE 4'

NONPROPORTIONAL REINSURANCE PREMIUMS FOR A LIFE INSURANCE COMPANY FOR AMOUNTS OF INSURANCE AT RISK

PREMIUMS AS PERCENTAGES OF TRUE ONE YEAR TERM COSTS FOR AMOUNTS AT RISK

Maximum Amounts of Insurance at Risk of One Life and Number of Lives as Specified

| Maximum Retention Limit | Maximum Insurance on One Life | |
|---|---|---|
| | $25,000 | $100,000 |
| A. 10,000 Lives | | |
| 100% | 7.5099% | 10.4529% |
| 120 | 1.6024 | 3.9685 |
| 135 | 0.3564 | 1.6066 |
| B. 50,000 Lives | | |
| 100% | 3.3585% | 4.6747% |
| 120 | 0.0359 | 0.2846 |
| 135 | 0.0001 | 0.0144 |
| C. 100,000 Lives | | |
| 100% | 2.3748% | 3.3055% |
| 120 | 0.0012 | 0.0348 |
| 135 | 0. | 0.0001 |

per class is considered the size of each policy in the group. One may suspect that this situation would be as difficult as most arising in practice and that the risk theoretic methods could be applied to these also. Although the calculations involved are not so simple as those suggested by our author, the differences in the figures may be considered sufficient to warrant a more precise solution to the problem.

As another application, let us consider a population of 3,000 lives, each insured for one dollar, with their ages distributed according to Mr. Feay's Table 1 and with mortality assumed to follow the $X_{17}$ Table. The stop-loss reinsurance premiums, as a percentage of the expected claims or net premiums $M$, are calculated for maximum retention limits of 100%, 110%, 120%, and 130% by four different methods. The first method is that suggested by the author where the mean and variance are calculated

PREMIUM FOR STOP-LOSS REINSURANCE EXPRESSED AS PERCENTAGE OF EXPECTED CLAIMS

| Maximum Retention Limit | Feay | Individual Risk Theory | Collective Risk Theory | |
|---|---|---|---|---|
| | | | Exact | Esscher Approximation |
| 100%...... | 7.552% | 7.528% | 7.594% | 7.584% |
| 110........ | 3.582 | 3.561 | 3.731 | 3.716 |
| 120........ | 1.415 | 1.401 | 1.563 | 1.567 |
| 130........ | 0.456 | 0.449 | 0.580 | 0.573 |

using $q$, the over-all mortality rate. The second is that of the classical or individual risk theory (see, *e.g.*, "A Statistical Treatment of Actuarial Functions" by Walter O. Menge, *RAIA*, 1937) in which the amount of total claims is considered to be the sum of the claims on the individual policies and the mean and standard deviation are computed using $q_x$, the mortality rate for the $n_x$ people aged $x$. The formula for the premiums under this method is essentially the same as the author's, the difference being in the calculation of the variance $\sigma_M^2$.

The third method uses the exact formula derived from the collective theory of risk; the ability to use the "Tables of the Incomplete Gamma Function" by Karl Pearson in this case prompted the choice of a population size of 3,000. The fourth method applies the approximation due to Esscher to the formula of method three; in practice, this would generally be the formula used, as it was in the above tables. Both of these methods are described in Ammeter (*op. cit.*).

The values produced by the four methods are close in the case of 100% maximum retention limit, but the differences between the collective risk theoretic methods and the others are greater for larger retention limits. If $M$ (the expected claims) were large in these cases, the difference in net premiums could be quite significant. For gross premiums, Ammeter suggests a loading proportional to the standard deviation of the total amount of claims (*op. cit.*, p. 89).

For retention limits near 100% of expected claims, Mr. Feay's method seems practical and accurate. But for higher limits, his method produces values less than those produced by collective risk theory; this situation can lead to significant understatement of the nonproportional reinsurance net premiums.

DANIEL J. LYONS:

In his paper Mr. Feay discusses ordinary insurance, health and accident, and security benefits. I want to comment briefly respecting ordinary insurance only.

Mr. Feay has concluded that the reinsurance business of life companies is a field for nonproportional reinsurance. That is a major conclusion of the paper.

While this may be true, I suggest that it is not a proper conclusion because it has not been established in the paper. There is a section in the paper captioned "Purpose," but it refers only to what Mr. Feay says may be "one of the principal purposes, namely to eliminate excessive fluctuation in insurance costs." There is nothing else in the paper to support the conclusion. A much more adequate statement of the purpose of reinsurance is necessary, if an intelligent decision is to be made on the form of the reinsurance. This is a complex subject and requires almost company by company consideration.

A new company gets help in organization, administration and supervision from its reinsurer. It gets much needed assistance in the selection of risks. It is protected against mortality hazards which it could not possibly undertake itself. As a matter of fact, reinsurance for a new company has a great many other values that we cannot take time to go into here.

In a medium size company the reasons are different and fewer. For the giant companies the reasons are much more complex and I sometimes think they are quite beyond the scope of actuarial evaluation.

As in everything else the cost of reinsurance in relation to its benefits must be considered and here it seems to me that our profession has a real obligation to make a searching analysis. No other profession is competent to tackle it.

I believe that reinsurance needs to be reconsidered by all companies from first principles. What does a company expect from reinsurance and how much is it willing to pay for it? This is a fertile field for mature thinking all the way through. We cannot base a conclusion such as Mr. Feay has given us on the mathematical theory that he has developed for nonproportional reinsurance.

In conclusion, I suggest that Mr. Feay deserves the thanks of all of us for taking what may prove to be a first step toward some really constructive work in this field of reinsurance. I think he has given us an excellent paper.

J. STANLEY HILL:

I should like to express my appreciation to Mr. Feay for introducing us to the subject of nonproportional reinsurance.

It is evident that the premiums obtained are to some degree a function of the statistical methods employed. As the nature of the benefit or the nature of the insured group becomes less homogeneous, the choice of statistical methods becomes more perplexing and less satisfying. In such situations, Monte Carlo techniques can be of assistance, but they tend to use large amounts of computer time when relied on solely.

We have obtained a practical solution by combining the two techniques, using the Monte Carlo technique to spot-check results obtained by the more classic statistical methods employed. Once we become satisfied through Monte Carlo experiments that a particular statistical hypothesis is producing valid results for a given subject and a given type of benefit formula, we then feel more confident in proceeding under the classic statistical methods.

IRVING ROSENTHAL:

The members of the Society are indebted to Mr. Feay for his courage, talent and industry in invading this most difficult area of inquiry. Nonproportional insurance or reinsurance, or stop-loss insurance as it is usually called, is a branch of collective risk theory, which until now has been pretty much forbidden territory to North American actuaries. As Mr. Feay's bibliography suggests, collective risk theory is something of a specialty with the actuaries of the countries of the North of Europe, but European actuaries in general seem to be quite at home in it, judging by the many papers in the *Transactions of the International Congress of Actuaries*. The main field of application of the theory has been fire and casualty insurance, but life insurance and health insurance are by no means excluded. The theoretical treatment of the subject by our European colleagues has been so generalized as to make it applicable to any and every form of insurance.

In this Introduction, Mr. Feay renders us all a very great service by leading us gently by the hand, so to speak. He starts off with a classification of reinsurance methods which divides the individual risk methods from the collective risk methods. He outlines the scope of the various methods and suggests many areas of application, including areas where two or more methods can be fruitfully combined. He then undertakes the arduous task of instructing us in the development of net premiums for stop-loss insurance for both life insurance and health insurance.

I have nothing but praise for that portion of Mr. Feay's paper which deals with the net premium development for stop-loss life insurance and I accept the calculations presented in Table 3 and Table 4 without question. However, when it comes to his development of stop-loss premiums for health insurance, I have some doubts. I think that in his strong desire to avoid mathematical complexities he has introduced simplifications which may not be warranted.

The theoretical problems of stop-loss premium computation apparently involve three distinct aspects. I have a few comments to make on each of these.

1. *The proper assumption as to the mathematical form of the frequency or probability distributions of collective risks*

Here a typical question is whether the probability distribution, which sets forth the series of probabilities for the various possible collections of aggregate claims, should be the normal distribution, the Poisson distribution, the Pearson Type III, or one of the many other probability distributions which have been devised or discovered. Mr. Feay is convinced that the normal distribution will do for most forms of insurance, including health insurance.

This is not the opinion of some other investigators. Hans Ammeter in his paper in the Vajda collection listed as item (1) in Mr. Feay's bibliography concludes, if I understand him correctly, that the normal distribution is not suitable, regardless of the number of unit risks exposed, for forms of insurance where (*a*) the amount of claim on the individual risk is a variable, and not a predetermined amount as it is in life insurance, and (*b*) where there is an element of "contagion" or lack of independence among the contingencies affecting the individual risks.

2. *The proper techniques for computing the parameters or constants required for the probability functions referred to in* (1)

All too frequently this involves serious practical difficulties because of the lack of proper experience or empirical data to provide a basis for estimating the parameters. The usual way out is to beat the parameters out of scanty and reluctant empirical data with the aid of an assortment of plausible theoretical assumptions. This is tricky work as I will attempt to show in discussing Mr. Feay's ingenious method of calculation of the standard deviations in the unnumbered table preceding his Table 8.

3. *Provision for safety margins for stop-loss premiums*

When the probability function described in (1) has been selected and the parameters referred to in (2) have been developed, by hook or crook, you have finally a probability model which you can use to compute net premiums. You can never be quite sure that this model adequately represents the underlying probability situation even as far as the past is concerned. Furthermore, since underlying conditions may change, the probability model whose parameters were estimated from last year's statistics, or the statistics of the last decade, may be a very poor model indeed for this year, or the next decade.

The problem of safety margins arises, of course, in all forms of insurance. In ordinary individual life insurance it is not a serious problem because the long range or secular trend of the whole structure of mortality rates is downward. Because most of us are conditioned by our life insurance training and experience, we tend to slur over the difference between a chance deviation which occurs within a fixed probability model and a basic change in the models themselves.

For example, in my own company, the Guardian Life, the 1958 ratio of actual to expected net risk mortality was 47½%. In 1957 it was about 42½%. Now a swing of this kind in a company of our size could not possibly be due to chance fluctuation in the theoretical probability sense. What happened was that there were more black balls in the probability urn in 1958. Yet we, in the company, thought of the swing as only a chance fluctuation. Had our business in force consisted entirely of stop-loss reinsurance of various groups of risks, the claim fluctuation between the two years (measured as a proportion of stop-loss premiums calculated on the 1957 probability model) would have been very much greater than the degree indicated by a shift from 42.5% to 47.5%. Unfortunately, it is not immediately obvious from a study of the premium formulas that a change in the basic underlying probability models has a much more powerful effect on stop-loss reinsurance experience than on regular full risk insurance.

These considerations suggest that the safety or premium contingency margins required for stop-loss collective insurance are proportionately very much larger than for individual insurance premiums. In the Ammeter paper of the Vajda collection, to which I referred previously, there are illustrations of gross premiums which in some cases are more than three times the theoretical net premiums.

Mr. Feay does not treat this important question of contingency or safety margins in stop-loss premiums in his paper. However, he refers to the apparently excessive stop-loss premium quotations of European reinsurers for certain group insurance cases. He judges them to be excessive because they are not lower than the group company retention charges for risk sharing and expenses, including in such expenses commissions and taxes on group insurance premiums which are many times as large as a stop-loss premium would be.

As the last part of my discussion I should like to analyze the computation of the standard deviation figures in Mr. Feay's unnumbered table preceding his Table 8. These figures are intended to serve as the parameters of the probability models which Mr. Feay employs to calculate the probabilities of Table 8 and the health insurance stop-loss premiums of Table 9.

I gather from his analysis that the standard deviations refer to the aggregate dollar amount of claims after dividing such aggregates by the average claim cost ($237.17 in the case of employees). The set of standard deviations under the heading "Average Claim" are based on calculations which ignore the distribution of claim amounts by size, *i.e.*, treat all claims as being equal in amount. This results in a set of standard deviations of the total number of claims. Thus the figure 12.58 in the table is simply $\sqrt{Nr}$, *i.e.*, $\sqrt{158.3}$.

I gather that the standard deviations under the heading "Claim Distribution" represent modifications of the figures in the "Average Claim" column to allow for the effect of variability in claim amounts. Apparently Mr. Feay's technique requires him to translate this effect into an enlargement of the variability of claim frequency.

In calculating these standard deviations, as an example take 16.86, Mr. Feay makes what is, to me, a questionable simplification. He has in his probability model 1,000 employees exposed for a year with expected annual claim frequency of .1583. Thus the expected annual number of claims is 158.3. The expected average claim per employee is $237.17. According to Table 6 the amount of any claim can vary from below $75 to over $1,275. (For convenience we will assume the range of variability runs from $75 to $1,325.) In this model the number of claims in a year can vary from 0 to 1,000, if we ignore the possibility of more than one claim a year on one life, and the aggregate volume of claims can vary from $0 to $1,325,000.

Note that if we allowed for the possibility of repeated claims on each life there would be no finite maximum possible number and volume of claims. The expected annual volume of claims is $37,544 (158.3 × 237.17). Mr. Feay calculates the standard deviation of the annual volume of claims for this model as 16.86 multiplied by $237.17 or $3,999.

The theoretically correct mathematical procedure for calculating the standard deviation of total annual claims in a probability model of this sort, which involves variability in the individual claim amounts, is very complex. It is dealt with in the European literature but I frankly admit that the demonstrations are beyond the limits of my mathematical competence. I don't know how to go about calculating the standard deviation. But it does seem to me that Mr. Feay's procedure is questionable. He avoids the whole problem of variability of individual claims by assuming (see his Table 6) that he has 1,000 employees exposed with an expected annual claim frequency of .033014 and a fixed claim amount of $75, another independent 1,000 exposed with an expected annual claim frequency of .021456 and a fixed claim amount of $125, a third independent 1,000 exposed with expected annual claim frequency of .020918 and a fixed claim amount of $175, and so forth.

Mr. Feay's modification of the probability model leaves the expected number and expected annual dollar volume of claims unchanged, but he has altered the mathematical nature of the model and the range of variation within the model. The number of claims can now vary from 0 to 26,000 (since he has 26 intervals in his Table 6) and the annual dollar volume of claims can vary from $0 to $18,200,000. This assumes that there cannot be more than one claim each year per employee in each amount class. It seems impossible to tell how far the standard deviation of the altered second model differs from the standard deviation of the theoretically correct model and whether it is larger or smaller. It may well be that Mr. Feay's figures are close to the theoretically correct ones or represent a conservative approximation. However, we have no criteria for making such a judgment.

My only reason for going into this analysis was to illustrate the sort of difficulties which arise in determining parameters for the required probability models. It was not to criticize Mr. Feay for making a commendable attempt at radical simplification. Perhaps, stimulated by his example, other members of our Society will interest themselves in solving the intriguing problems of stop-loss premiums for health insurance, as well as other problems in the field of collective risk theory.

PAUL THOMSON:

Mr. Feay is to be congratulated for his interesting and informative paper on this relatively new subject in our actuarial literature. As he points out, and his extensive bibliography demonstrates, the concept has had more attention abroad than at home.

A short time before the proofs of the paper arrived in our office we had been considering the possibilities of some form of stop-loss reinsurance for group A. D. & D. benefits involving a relatively large amount schedule. Since the claim frequency is very small, about .0005 say, the normal curve is not applicable except possibly for very large groups. Also, the amount distribution is expected to be highly skewed toward the lower amounts. In this situation, the technique of modifying the standard deviation for the variance in amounts and hence obtaining probabilities from tables of areas under the normal curve does not seem to be applicable. Since the paper deals with cases where the normal curve basis is appropriate, it may be of interest to report on an approach using the Poisson exponential.

Since the Poisson exponential is a discrete function giving probabilities only for integral numbers of claims, summations must be used rather than integrals and the resulting formulas are not particularly elegant. The notation used follows Mr. Feay's with a few exceptions and additions. It is assumed that the claim rate, $q$, is constant for all ages and amounts, and that reinsurance is unlimited, *i.e.*, the symbols $h$, $H$, and $R$ are not involved. In addition, the following are used:

$s_u$ = average amount of insurance in amount class $u$.
$p_u$ = proportionate number of lives in class $u$.

$$\sum_1^k p_u = 1 \quad \text{and} \quad \sum_1^k p_u s_u = \bar{s} = \text{average amount of insurance.}$$

$N_q = m$ = expected number of claims.
$M = m\bar{s}$ = expected amount of claims.
$y_r$ = probability of exactly $r$ claims by Poisson's formula, $m^r e^{-m}/\lfloor r$, which has the following properties:

$$\sum_0^\omega y_r \doteq 1\,, \qquad \sum_0^\omega r y_r \doteq m\,, \qquad r y_r = m\, y_{r-1}\,.$$

The amount of the net reinsurance premium for claims in excess of $L$, the retention limit, is $M\mathrm{P}_L$ or $m\bar{s}\mathrm{P}_L$ which can be stated as follows:

$$m\,\bar{s}\mathrm{P}_L = \sum_{u=1}^k p_u \sum_{r=j_u+1}^\omega (r\,s_u - L)\, y_r\,,$$

where $j_u$ is largest value of $r$ for which $rs_u \not> L$.

By rearrangement and use of the above properties of the Poisson formula, this can be expressed as:

$$m\,\bar{s}\mathrm{P}_L = m\,\bar{s} - L(1-y_0) + m\sum_{u=1}^{k} p_u \sum_{r=1}^{j_u}\left(\frac{L}{r} - s_u\right) y_{r-1}$$

or

$$\mathrm{P}_L = 1 - \frac{L}{M}(1-y_0) + \frac{1}{\bar{s}}\sum_{u=1}^{k} p_u \sum_{r=1}^{j_u}\left(\frac{L}{r} - s_u\right) y_{r-1}\,.$$

The utility of such a formula is dubious for rapid calculations where a large number of amount classes are involved. To illustrate the results, values of $\mathrm{P}_L$ were calculated for the following simple situation:

$$N = 2000 \qquad q = .0005 \qquad N_q = m = 1$$

| | | | | |
|---|---|---|---|---|
| $u$: | 1 | 2 | 3 | 4 |
| $s_u$: | 1 | 2 | 3 | 4 |
| $p_u$: | .4 | .3 | .2 | .1 |

$$\bar{s} = \sum_{1}^{4} s_u p_u = 2 \qquad M = m\,\bar{s} = 2$$

The values of $s_u$ may represent units of insurance such as $1,000, $5,000, etc., the values of $\mathrm{P}_L$ being independent of the absolute amount. This is, of course, a purely hypothetical distribution used only to simplify the illustration. Values of $y_r$ are:

| $r$ | $y_r$ | $r$ | $y_r$ |
|---|---|---|---|
| 0 | .368 | 4 | .015 |
| 1 | .368 | 5 | .003 |
| 2 | .184 | 6 | .001 |
| 3 | .061 | | |

The values of $\mathrm{P}_L$ are shown in the tabulation on page 61 in comparison with those that would result if a level amount of insurance is assumed on each life equal to the average amount $\bar{s}$.

This comparison indicates the degree to which reinsurance premiums can be understated by a level amount assumption. The much greater values of $\mathrm{P}_L$ in comparison with those obtained by Mr. Feay for the same retention limits show the combined effect of skewed distributions in both claim frequencies and amounts.

GEORGE F. M. MAYO:

The problem of nonproportional reinsurance is a complicated one and this very complexity has, to my mind, confounded confusion further. I feel that some very clear statement of the objectives in mind is necessary, and the fact that this is lacking is the only serious criticism of Mr. Feay's paper. In an attempt to fill the gap, may I agree with Mr. Feay that the purpose of such reinsurance is to eliminate excess claims where the definition of "excess" will vary with the circumstances. It is important, however, to realize that excess claims can result from either or both of two causes, namely:

(i) statistical fluctuations, but with underlying rates of mortality being those assumed;
(ii) variations in the underlying rates of mortality, the most obvious causes being catastrophic in nature, such as war, pestilence, earthquake or other natural hazard.

Precisely whether the actuary will consider both of these or one only (usually the first one) depends upon circumstances. Thus for a group con-

| Maximum Retention Limit ($l$) | Values of $P_L$ | |
|---|---|---|
| | Amount Distribution | Level Amount |
| 100% | 44.2% | 36.8% |
| 113 | 40.3 | 33.4 |
| 120 | 38.2 | 31.5 |
| 125 | 36.6 | 30.2 |
| 150 | 29.1 | 23.6 |

tract rerating, the actuary will normally be concerned only about the first, thereby placing the insurance company on risk for inaccurate guesses as to mortality rates but making employers in general bear among themselves the brunt of random fluctuations. Similarly in setting retention limits, the actuary will frequently feel that the first cause only need be considered. In accepting nonproportional reinsurance from another company, however, the actuary must allow for both causes of excess and, since his profit is limited and his loss almost unlimited, he will require considerable margins, thereby producing the effect noted by the author that the premiums charged for nonproportional reinsurance seem on the high side. In the one case where I was concerned in the quotation of such a premium, we asked a gross premium approximately twice what I had calculated the net premium required to be. Our quotation was not accepted.

The problems to which nonproportional reinsurance may be applied are, as stated by the author, premium charges for use in group insurance retentions, setting of retention limits, setting of premiums for reinsuring excess claims on self-insured group plans, and reinsurance of an office's own business against catastrophic hazard. The paper seems to me to suggest that, as the problem is the same one, the same technique may be applied. I venture to disagree. In the first place, as stated already, consideration of the different aspects of excess claims will lead to different answers in different situations, while, in the second, the large numbers of lives involved for an office's total business will permit certain techniques (such as the Normal Curve approximation) which would not be available for use in the case of group plans, particularly when life insurance with low claim rates is under consideration. Thus the author gives premium rates for groups of 6,319, 10,000, 50,000 and 100,000 lives. An insured group with 6,319 lives is large and one with 100,000 phenomenal. The majority of cases we are interested in are much, much smaller.

I feel that, for group retentions, the proper solution is to use the technique of calculating the risk from first principles, using electronic equipment to offset the formidable calculation load.

While I have had several hopeful thoughts, these have not yet materialized into a workable computer program.

JAMES B. ROSS:

In his brief oral comments Mr. Hill alluded to the use of Monte Carlo methods to construct an approximation to the underlying risk distribution. I would like to elaborate a little on this approach via computer-generated pseudo-random numbers, and have prepared a few concrete examples.

Under this approach, and given a set of specific census data showing age (or mortality rate) and amount of insurance for each life in the subject group, *no* attempt is made to find an analytic formula which will adequately represent the distribution of losses that might arise, say, in a year's time. Instead the problem is attacked directly by storing the data in a computer and setting up a pseudo-random number generator.

The technique is similar to that employed by Boermeester (*TSA* VIII, 1) in dealing with the dispersion of annuity values. One year's "experience" is collected by proceeding systematically through the individual lives, stopping at each to compare a freshly generated pseudo-random number of appropriate magnitude with the mortality rate for the life in question. If the comparison shows the pseudo-random number less than the mortality rate, one has a "death loss." These "death losses" are summed as the data are traversed, the total being one year's "experience." A considerable body of such experience years will form a frequency table that gives an explicit numerical approximation to the "true" frequency distribution.

Nonproportional reinsurance premiums for schemes such as Mr. Feay discusses can, of course, be calculated by direct truncation of this approximate frequency distribution.

Using the IBM 650 and an overflow type of pseudo-random number generator we made several tests on fairly small groups of differing sizes, and with differing age and amount distributions. Characteristics of each test case are given in Table 1; the distributions of the results of 10,000 experience years run on each set of data are shown in Table 2.

Mr. Hill points out (and from our experience he is eminently correct) that straight Monte Carlo is expensive, and that the technique is perhaps better used to validate a particular mathematical formula as representative of the underlying distribution. Perhaps we would be closer to a *practical* solution to the representation of risk distributions if some of the impressive intellectual effort detailed in Mr. Feay's next-to-last section were diverted to the problem of the inexpensive generation of random numbers!

TABLE 1

| Case No. | Number of Lives | Case Characteristics | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Amount | | | Age | | | |
| | | Minimum | Maximum | Average | Minimum | Maximum | Average | Weighted Average |
| A...... | 10 | $2,000 | $10,000 | $3,400 | 21 | 63 | 47 | 47 |
| B...... | 31 | 1,000 | 1,000 | 1,000 | 20 | 66 | 40 | 40 |
| C...... | 31 | 3,000 | 10,000 | 5,000 | 20 | 66 | 40 | 41 |
| D...... | 100 | 1,000 | 10,000 | 5,360 | 20 | 84 | 46 | 48 |

This method assumes the lives are independent, hence does not evaluate the catastrophe risk (as is present in group insurance). On the other hand it is perfectly general as to ages or amounts, and does not lean on any "reinsurance of unusually large amounts" to simplify the distribution.

I very much enjoyed Mr. Feay's paper, and want to thank him particularly for having laid down a working terminology and classification scheme for further discussion of nonproportional reinsurance problems. Assembling published papers on this topic is difficult work, and we are indebted to Mr. Feay for his bibliography.

(AUTHOR'S REVIEW OF DISCUSSION)

HERBERT L. FEAY:

The writer is pleased by the interesting and thoughtful discussions of the paper and is flattered by the complimentary remarks. The main pur-

## TABLE 2

DISTRIBUTION OF AGGREGATE DEATH LOSSES
10,000 EXPERIENCE YEARS

| AGGREGATE DEATH LOSSES (000'S) IN EXPERIENCE YEAR | CASE | | | |
|---|---|---|---|---|
| | D | C | B | A |
| $ 0 | 3,780 | 8,462 | 8,462 | 9,138 |
| 1 | 35 | 0 | 1,375 | 524 |
| 2 | 1,283 | 0 | 150 | 252 |
| 3 | 384 | 970 | 12 | 23 |
| 4 | 501 | 0 | 1 | 0 |
| 5 | 406 | 60 | | 53 |
| 6 | 428 | 61 | | 7 |
| 7 | 220 | 0 | | 3 |
| 8 | 178 | 12 | | |
| 9 | 180 | 4 | | |
| 10 | 1,017 | 345 | | |
| 11 | 92 | 0 | | |
| 12 | 373 | 0 | | |
| 13 | 141 | 66 | | |
| 14 | 146 | 0 | | |
| 15 | 151 | 6 | | |
| 16 | 130 | 5 | | |
| 17 | 63 | 0 | | |
| 18 | 50 | 0 | | |
| 19 | 56 | 0 | | |
| 20 | 145 | 5 | | |
| 21 | 23 | 0 | | |
| 22 | 45 | 0 | | |
| 23 | 37 | 2 | | |
| 24 | 42 | 0 | | |
| 25 | 22 | 0 | | |
| 26 | 13 | 0 | | |
| 27 | 7 | 0 | | |
| 28 | 9 | 1 | | |
| 29 | 7 | 0 | | |
| 30 | 11 | 1 | | |
| 31 | 1 | | | |
| 32 | 6 | | | |
| 33 | 3 | | | |
| 34 | 4 | | | |
| 35 | 5 | | | |
| 36 | 3 | | | |
| 37 | 1 | | | |
| 38 | 0 | | | |
| 39 | 0 | | | |
| 40 | 1 | | | |
| 41 | 1 | | | |
| Totals.... | 10,000 | 10,000 | 10,000 | 10,000 |

pose of the paper was to introduce the subject of nonproportional reinsurance and to develop an interest and a desire for further investigation and study by American actuaries. It is pleasing to know that Mr. Kahn plans to conduct a research study at the University of Michigan that will include nonproportional reinsurance. Let us hope that the Society will be receiving the benefits of his work in a future paper.

Mr. Hill refers to the use of Monte Carlo techniques and Mr. Mayo mentions the use of computer procedures for calculating nonproportional reinsurance premiums from first principles. It would be helpful if these ideas were developed further and the results submitted to the Society. The calculations can provide the criteria which Mr. Rosenthal indicates are needed for judging the standard deviations for health insurance.

Mr. Rosenthal questions the standard deviations developed for health insurance by use of the Poisson distribution formula. In his discussion of Mr. Dougherty's paper (reference (5) of the bibliography), Mr. Rosenthal uses the standard deviation of the Lexis distribution. The writer suggests that an investigation be made of the use of this distribution for insurance claim costs and insurance claim cost rates. Two tests for the applicability of the Lexis distribution are the Lexis ratio and the Charlier coefficient of disturbancy. The Lexis distribution can be substituted for the compound Poisson (negative binomial) distribution used by Mr. Ammeter in his development. (See reference (1), formula (6), page 84.) The results of the suggested investigation, the writer believes, would be of interest to the Society.

The Lexis distribution requires an estimate of the mean and of the standard deviation of the future claim cost rates for the basic generalized group (the universe) from which the experience of the smaller case is assumed to be a sample. This appeals to the writer as being a better statistical procedure than the direct estimate of an arbitrary $h$ factor as seems to be required by Mr. Ammeter's formula.

I agree with Mr. Lyons that reinsurance is a complex subject and the purpose of reinsurance deserves adequate study. Actuarial literature contains many excellent and extensive discussions on the subject and full length books have been published on reinsurance. In the United States, these discussions have been primarily for proportional reinsurance. In the bibliography, I listed two papers by actuaries, namely Irving Rosenthal's paper (4) and Edward Dougherty's paper (5). The writer did not want to repeat the excellent discussions in these papers, but a more definite reference to them may have been advisable.

The subtitle "Purpose of Reinsurance" is not the best title for the subdivision of the paper that follows that title. A better subtitle is "Elimination of Chance Variations." Mr. Mayo indicates in his discussion that this subsection is not a clear statement of the objectives of nonproportional reinsurance. The writer agrees with this, but can flatter himself with the comment that the primary purpose of the paper was to introduce a scientific discussion of nonproportional reinsurance and not to emphasize the practical applications of the results. Incidently, the practical objectives of this reinsurance are discussed in the book edited by S. Vajda (1), but this discussion does not particularly apply to the conditions and benefits for United States life insurance companies.

The writer also agrees in general with Mr. Mayo's comments on the catastrophe hazard. Three of the references deal with this subject. These are Edward Green's paper (7), Harold Crawford's paper (3), and William Fee's paper (20). The writer recommends that, if the catastrophe hazard is substantial, it be evaluated separately and not included with random fluctuations in the premium calculations for nonproportional reinsurance.

Mr. Rosenthal points out that another subject not developed in the paper is the contingency loading for the premiums. One of the preliminary drafts of the paper contained comments on this and other problems, but these were eliminated in order to keep the length of the paper within a reasonable number of pages. Other problems include:

(1) Loading for expenses.
(2) Dividend problems.
(3) Contingency reserves to be accumulated.
(4) Control of underwriting of the original insurer.
(5) Application of insurance laws.
(6) Effect on present types of business.
(7) Terms of the reinsurance contract.
(8) Fluctuating exposures in a group or classification of risks.
(9) Rerating problems.

Mr. Thomson has given us a demonstration of the use of the Poisson distribution to the A. D. & D. benefits which have a very low claim rate on an annual basis. This is the distribution suggested by Mr. Ammeter on page 82 of reference (1) for use with constant basic-probabilities. As explained in the paper in connection with the discussion of health insurance, it is not necessary to have a low claim rate on an annual basis in order to have the Poisson distribution apply. The rate for one year can be the sum of a large number of very short period average rates. Each of the short period rates has a Poisson distribution and the sum of these distributions for one year gives a Poisson distribution for the total rate for one year. Proof of this is given on page 104 of *The Elements of Probability Theory* by Harald Cramér.

In fact, the Poisson distribution could be better for life insurance than the binomial distribution. The life insurance claim cost rate given by the ratio of claim payments to initial exposure can be changed to ratio of claim payments to average continuous exposure for the year.

The calculations for life insurance as used in the paper have not given consideration to withdrawals from and additions to the cases. For a short period of one year, it is reasonable to assume that a collection of risks will have a constant exposure with new risks replacing terminations. This assumption leads to the Poisson distribution as illustrated by the development in Arthur Bailey's paper (11).

As pointed out by Mr. Thomson, the Poisson distribution is a discrete distribution and involves a considerable amount of calculation work. When $n$ is sufficiently large, the Poisson distribution can be approximated by the normal distribution. Several authorities suggest that with a mean of 25 or more this approximation is satisfactory. The paper recommends using the more conservative rule of 35 for the mean as followed for the 1951 Impairment Study.

The paper suggests that when the number of claims is less than 35, a study be made of the use of a Pearson Type III distribution and of a Gram-Charlier distribution using a Type III distribution. The results of such a study should be of interest to the Society. As pointed out by Mr. Mayo, large cases were used for the illustrative purposes of the paper in order to have the normal distribution as a satisfactory approximation for calculation purposes. The use of the Type III distribution can meet Mr. Mayo's requirements for small groups.

Mr. Kahn and Mr. Rosenthal raise some serious questions regarding the results given in the paper and especially regarding the use of the normal distribution. After further study, the writer has doubts as to the applicability of these criticisms.

Mr. Kahn questions the adequacy of the premiums developed in the paper for complete nonproportional reinsurance for life insurance on the basis of premiums calculated with certain formulas developed by Mr. Ammeter in reference (1). There are several formulas in Mr. Ammeter's paper and Mr. Kahn does not state which ones he used. The writer is assuming that Mr. Kahn used formula (23) on page 94 of reference (1) for his calculations.

The writer suggests that there are five reasons why Mr. Ammeter's formulas give different results than those included in Tables 3 and 4 of the paper.

One important reason is probably the differences in the distribution of claims by amounts of insurance. The writer used the exact distribution of amounts of insurance as given in Table 3 of Mr. Rosenthal's paper (4). Mr. Ammeter's formula (23) of his paper (1) depends on using a mathematical formula for this distribution. The writer doubts that this formula as given on page 92 of reference (1) properly reflects the actual distributions of Mr. Rosenthal's paper (4). The claim distribution does have an important effect on the size of the nonproportional reinsurance premium. This is indicated by the increase in these premiums with increase in retention limit. Mr. Thomson brings out this point in his comment on why his premiums are higher than those given in the paper.

Mr. Ammeter's formula for the claim distribution by amount of claim involves an *e* exponential function. Judging from the formulas used by R. E. Beard (16) and C. O. Segerdahl (17), more complex formulas are needed for accurate duplication of actual claim distributions. It is very doubtful if Mr. Ammeter's formula can be used satisfactorily for the distributions of insurance claims as used for the calculations in the writer's paper.

Another probable reason for a difference between the premiums by the writer's procedure and by Mr. Ammeter's procedure is that the writer uses assumed constant basic-probabilities, whereas Mr. Ammeter uses assumed fluctuating basic-probabilities as stated on page 83 of reference (1). The writer assumes that the true rate of mortality (or morbidity) for the infinitely large generalized group will remain constant for the period concerned and that variations for the particular case (a limited sample from the generalized group) will be the result of fortuitous fluctuations. Mr. Ammeter assumes that this rate for the infinitely large generalized group is also a variable so as to produce sets of values from time to time in the future. This is the assumption used to develop the Lexis distribution. A reference on this point is page 85 of *Handbook of Mathematical Statistics* edited by H. L. Rietz.

If the period under investigation is a long time in the future, such as is the case for the ultimate ruin problem, the writer can see the need for considering fluctuations in the basic assumptions for the large generalized group. He does not believe that this is necessary or practical for a short period in advance such as for one year as assumed in the development of premiums for his paper. The ultimate ruin problem can be expressed as how long it will be before a company becomes insolvent or as how much surplus is needed to give a 95% degree of safety in the future to an insurance company.

Mr. Ammeter gives two illustrations for his fluctuating probabilities. On page 80 of reference (1), he states that uncertainty of the basic data and of the parameters changes the mathematical model. He also states here that in many branches it is not even possible to estimate the claim rates in a sufficiently accurate manner. The writer believes that the establishing of proportional reinsurance premiums in such cases is enough of a gamble without assuming the additional difficulties involved in complete nonproportional reinsurance.

Mr. Ammeter's second illustration for fluctuating probabilities is on page 85 of reference (1). He changes the number of black balls in his urn from time to time and thus changes his basic probabilities. This is the classic procedure for arriving at the Lexis distribution as mentioned previously.

In his development, Mr. Ammeter uses a so-called compound Poisson distribution or negative binomial distribution which includes an $h$ factor. When $h$ becomes infinite, this distribution becomes the ordinary Poisson distribution. The $h$ factor is Mr. Ammeter's measure of the fluctuation in the sets of basic probabilities. He indicates that this factor cannot be established satisfactorily by mathematical or statistical procedures and must be estimated.

Mr. Kahn does not indicate what value he used for $h$. If he used infinity, he will have assumed that the number of claims has a Poisson distribution, so that the differences in premiums as shown by his comparisons are due to other reasons.

The writer believes that there is adequate information available in the home offices of the life insurance companies of the United States and Canada for accurately determining a conservative true claim cost rate that can be expected to be reasonably stable for a short period of years in the future. The true expected claim cost rate can include estimated changes (such as the projections for future annuity mortality) and can include a small increase to cover uncertainties following the usual procedures in projecting mortality for nonparticipating life insurance and annuity policies.

One demonstration of the reliability of the experience available is the comparatively small amount of variation in the aggregate mortality and morbidity ratios on a year-to-year basis as included in the Society's Reports of Mortality and Morbidity Experience. The average deviation for the percentages for the five years from 1953 to 1957 is less than 3% for hospital benefits and less than 2% for weekly indemnity insurance, ordinary life insurance, and group life insurance. The hospital increase is higher probably because of the more substantial effect of inflation. The effect of inflation can be offset by determining the annual average claim cost rate as the ratio of total claim costs to the total number of units of exposure times the average claim cost for the year. Exact consideration of age, industry, sex and other such measurable causes of variation probably would also reduce the amount of the average variability. The variability would also be decreased for the group insurance if all group benefits, including life insurance, were combined.

The third reason for a difference between the premiums by the two procedures (the writer's and Mr. Ammeter's) is that both the expected mean claim rate and the standard deviation must be changed when a change is made from the binomial distribution procedure to the Poisson distribution procedure. As explained in the paper, the claim rate used for mortality is the ratio of the number of claims to the number included in the initial exposure. This permits the use of the binomial $(p + q)^n$ with the $n$ representing the initial exposure. If the Poisson distribution is to be used for life insurance, the claim rate must be changed to the ratio of number of claims to the average number included in the exposure. This rate is referred to in text books as the central death rate. Assuming an even distribution of claims in the year, the approximate central death rate, $m_x$, for the Poisson distribution is taken as $q_x \div (1 - \frac{1}{2}q_x)$.

The fourth reason for a difference is the assumption regarding exposure. With the binomial distribution, the exposure decreases as deaths occur, but with the Poisson distribution the exposure is assumed to remain constant. When a death occurs, a new person is immediately added to the group. In order to be equivalent, either the initial exposure must be increased for the binomial distribution or the average exposure must be decreased for the Poisson distribution.

The fifth reason is the errors inherent in the limitations of the tables available. The writer used *Tables of Applied Mathematics* by J. W. Glover. The significant figures available in the tables decrease as the number of standard deviations increase. The tables are also for multiples of the standard deviation to the nearest second decimal. These conditions combine to reduce the significant figures for the $A$ and $B$ parts of formula (2) so that even less significant figures are available for the difference of these two values.

The writer has recalculated the nonproportional reinsurance premiums of Table 3 for 6,319 persons in a life insurance group using the following:

*a*) Standard deviations and mean rates to six significant figures.

*b*) Interpolations on the values in Glover's Tables to secure approximate values for multiples of the standard deviation to four decimal places.

*c*) Standard deviation and mean value for the Poisson distribution, assuming a constant exposure with a central death rate equal to .00932.

*d*) Standard deviation and mean value for the binomial distribution, assuming an initial exposure equal to the average exposure of (*c*) and with that initial exposure reducing as deaths occur with a mortality rate equal to .00932.

*e*) The continuous method in accordance with formula (2), because the number of claims is sufficiently large to use the normal distribution as a proper approximation of both the Poisson and the binomial distributions.

The results are given in Table 10 which is included in this discussion.

Mr. Rosenthal makes some rather strong statements questioning the reliability of mean claim cost rates for health insurance and of the standard deviations for these rates based on the experience available. He refers to beating out parameters from scanty, reluctant, and empirical data with

TABLE 10

NONPROPORTIONAL REINSURANCE
PREMIUMS FOR LIFE INSURANCE GROUP
Continuous Method of Calculation

| Maximum Retention Limit | Poisson Distribution | Binomial Distribution | |
|---|---|---|---|
| | | Recalculated | Table 3 |
| 100% | 5.198% | 5.174% | 5.175% |
| 113 | 1.090 | 1.076 | 1.081 |
| 120 | 0.351 | 0.345 | 0.345 |
| 125 | 0.137 | 0.133 | 0.134 |
| 135 | 0.015 | 0.014 | 0.012 |

the aid of an assortment of plausible theoretical assumptions and to the use of a probability model that you can never be sure adequately represents the underlying probability situation. If it were as difficult to forecast future mortality and morbidity costs as Mr. Rosenthal's comments would seem to indicate, the entire basis for guaranteed cost insurance for a period of years would be questionable.

The fluctuations in the basic-probabilities are much greater for some of the other lines of business than for life insurance. Mr. Ammeter's comments on page 83 of reference (1) indicate agreement with this. He states that for "life insurance, the Poisson distribution is at least a useful and sufficient approximation to the real distribution."

The fact that the Guardian had a difference of 5 percentage points in mortality ratios between 1957 and 1958 is not sufficient proof that accurate forecasts cannot be made of mortality and morbidity claim cost rates. Once upon a time a bridge player was dealt a bridge hand with 13 spades. If there were more black balls in Mr. Rosenthal's urn (or more spades in the deck of cards), there has been a change in the exposure. An analysis of the insurance in force (or of the urn or of the deck of cards) will show whether or not there has been such a change.

Mr. Kahn seems to indicate that there is a difference between the risk theory used in the paper and that used by Mr. Ammeter that justifies separate names of individual risk theory and collective risk theory. Actually there is no difference in fundamental procedure, but rather a difference in emphasis in arriving at the total claim costs for one year. Formula (1) of the paper is the same as Mr. Ammeter's formula (17a) on page 91 of reference (1). Both formulas are for the total sum to be paid out for claims during the period of exposure and this sum is the variable.

The paper emphasizes the use of the number of lives, the amounts of individual insurance, and the claim rates in securing the total sum to be paid out in claims, but these totals are inherent in Mr. Ammeter's development. His $r$ for the number of claims is the number of lives times the claim rate. His $p(z)$ is a distribution of individual claims by amount of payment, just as are Tables 6 and 7 of the paper and Table 3 of Mr. Rosenthal's paper (4).

Another difference in emphasis is that the writer's discussion uses claim cost rates and claim rates, whereas the existence of such rates can only be inferred from Mr. Ammeter's formulas. The claim cost rate is the ratio of the dollar value of claims to the dollar value of exposure, and the claim rate is the ratio of the number of claims to the number of risks in the exposure. For the calculations for Mr. Rosenthal's paper (4), the claim cost rate and the claim rate are assumed to be the same, but the standard deviation for these two rates obviously differs with differences in distributions of exposure by amounts of insurance.

Mr. Ammeter's development after his formula (17a) differs from that in the paper. He first establishes a mathematical formula for the claim distribution by amount of claim which, as indicated previously, probably does not satisfactorily represent the distributions used for the calculations in the paper.

Mr. Ammeter uses a Gram-Charlier Type A distribution formula with two terms in his development for cases for which the normal distribution as included in his formula (21) is not satisfactory. The use of this distribution to represent a skewed distribution is questioned by several authorities. References on this point are *Mathematical Statistics* by H. L. Rietz, *Advanced Theory of Statistics* by M. G. Kendall, and Louis R. Salvosa's book which is reference number (21) of the paper. Mr. Kahn might find that a satisfactory field of investigation is to the use of Dr. Salvosa's results for the Pearson Type III distributions in the calculations required for nonproportional reinsurance and for collective risk theory in general.

Mr. Kahn quotes the comments of Mr. Ammeter on the skewness of the claim distribution by amounts. The writer did not state that the claim amounts were distributed in accordance with the normal distribution. In fact it is obvious from Table 3 of Mr. Rosenthal's paper (4) and from Tables 6 and 7 of the paper that the distribution of claims by amounts is skewed. This skewness has an effect on the premium rates, but this does not indicate that $f(t)$ of formula (1) must be skewed. This is easy to demonstrate for the claim distributions used in the paper. Each amount class is a separate probability distribution for the number of claims and for the total claim costs equal to the number of claims times the class average. The usual statistical measure of skewness can be calculated for the probability distribution of each amount class and for the combined group for all classes. The skewness of the combined probability distribution will be less than the average skewness of the individual classes. In fact, under the central limit theorem, the combined distribution is much closer to the normal distribution than is any one of the class distributions if the contributions of each class to the total are not significantly different. An explanation of the addition of variables is given on pages 82 to 113 of *The Elements of Probability Theory* by Harald Cramér.

This agrees with the statement in Walter Menge's paper in *The Record* referred to by Mr. Kahn. The following is quoted from Mr. Menge's paper:

> It has been shown that the distribution of the sum of a number of independent variables approaches the so-called "normal curve of error" as a limit when the number of variables is increased without bound. Thus, in a relatively large group, the question of skewness is largely theoretical; and for practical purposes the distribution of the random fluctuations in mortality, as they are reflected in present values, may be assumed to be normal.

In the paper, the writer recommends the continued use of regular nonproportional reinsurance methods for large amounts of insurance on a comparatively few lives and for risks for which the ratings are very uncertain. One reason for this is to be sure of the application of the central limit theorem and of the normal distribution to the combined probability distribution for the amount and rating classes.

Mr. Kahn's reference to Mr. Menge's paper is a valuable addition to the list of references in American actuarial publications for the application of probability to reinsurance. The title of the paper, "A Statistical Treatment of Actuarial Functions" does not indicate this application.

Mr. Kahn's comment seems to question the division of the insurance into 12 classes with an average amount used to represent each class. This can be tested by the use of a larger number of classes.

The use of class intervals for amounts of insurance is justified. Assuming the binomial distribution or the Poisson distribution to apply to the number of claims, it is true that each claim will not be for the same amount but can vary within the class limits. When this variation is reasonable, the effect is to change a discrete distribution of claim costs to a continuous distribution. The classic description of this is Karl Pearson's explanation of the approach of a discrete asymmetrical binomial distribution to his Type III continuous distribution without becoming the symmetrical normal distribution. A good discussion on this point is given in Chapter IV of *Sampling Statistics and Applications* by J. G. Smith and A. J. Duncan of Princeton University.

The final table in Mr. Kahn's discussion compares premiums as calculated with four formulas for 3,000 lives, each insured for the same amount. It is interesting to note from this table that the procedure used by the writer gives slightly higher net premiums than the procedure used by Mr. Menge in his paper. Part of this difference is due to the fact that Mr. Menge's procedure uses interest functions, whereas interest was ignored for the purposes of the paper. The differences for the "Collective Risk Theory" formulas as shown in the table must be due to differences in basic probability assumptions. If no $h$ factor or other allowance for variations in basic probabilities is included in these formulas, the differences may be due to other reasons discussed previously.

Mr. Rosenthal accepts the writer's procedure for determining the complete nonproportional reinsurance premiums for life insurance but questions the procedure for health insurance. Actually, if the Poisson distribution is used, if the basic claim cost rate is assumed to remain constant, and if the exposure is assumed to remain constant, the two procedures give the same values for the mean and the standard deviation for the combined classes.

If the health insurance procedure were applied to Mr. Rosenthal's life insurance company, there would first be prepared a distribution of claims by size similar to Tables 6 and 7 of the paper. The standard deviation is then calculated by the procedures described in the paper for health insurance. A second calculation of the standard deviation is made, using the third formula in Mr. Rosenthal's paper (4) except that the Poisson variance is substituted for the binomial variance. The results agree.

This equivalent can be demonstrated mathematically. Using the Poisson distribution, the variance for the number of claims for Class S by the life insurance procedure is:

$$a)\quad \sigma_s^2 = A_s^2 \times r \times n_s\,.$$

Under the procedure for health insurance, this value is:

$$b)\quad \sigma_s^2 = A_s^2 \times r_s \times N\,.$$

Under the basic assumptions, $r$ and $r_s$ are related as follows:

$$c)\quad r_s = r(n_s \div N)\ .$$

By substituting the right-hand term of ($c$) for the $r_s$ in expression ($b$), expression ($a$) is secured.

This indicates to the writer that he has not changed the mathematical nature of the model and the range of variation with the model as Mr. Rosenthal suggests.

Mr. Rosenthal is correct in stating that the writer assumed that the expected number of claims and the expected dollar volume of claims will remain constant. This is another way of saying that the writer assumes that the basic probabilities will not change significantly for the one year period. The basic factors for the generalized population are assumed to remain constant. The particular collection of risks is a sample from this larger group and both the claim rates and claim cost rates for the sample can vary from the corresponding assumed stable ratios for the generalized group.

Mr. Kahn has pointed out personally to the writer that the $B$ factor in formula (2) of the paper may not be entirely clear since it is not in the form for the area usually used for the standardized normal distribution. The area for this $B$ factor is from $(L - M)/\sigma$ to infinity. This area is 1 minus the value of the area in Table 1 on page 273 of *The Elements of Probability Theory* by Harald Cramér and is .5 minus the value in the Table of Areas and Ordinates of the Normal Curve of Error on pages 394 to 413 of *Tables of Applied Mathematics* by James W. Glover.

In the paper, the writer emphasizes the need for an accurately determined expected mean claim cost rate so that the causes of variation can be limited to a large number of comparatively small causes. With an accurately determined expected mean claim cost rate for one year, the writer suggests that it is satisfactory to use the normal distribution if the number of claims for that year is of sufficient size.

Mr. Ross's discussion includes illustrations of a computer-generated pseudo-random number procedure for securing a distribution of claim costs. His procedure assumes an underlying probability function in that each risk is subject to probability rates of $q$ and $p$ and these probability rates are assumed to remain constant for a year. Mr. Ross determines the true frequency distribution of the total claim costs from 10,000 sample distributions without any direct assumption as to the nature of distribution of the universe.

The determination of a frequency distribution of a universe from a number of sample distributions involves the question of how many trials are needed to give a reasonable degree of accuracy. Suppose we have ten perfect prisms, each with three sides marked with an "H" and one side with a "T." How many times must these ten perfect cubes be cast in an unbiased manner on an unbiased surface to give an average distribution that is a satisfactory representative distribution of the number of "H's" in each throw? From probability theory we know that the distribution is the asymmetrical binomial distribution of $(\frac{3}{4} + \frac{1}{4})^{10}$. If the average distribution based on the samples differs from this theoretical distribution, how can we determine if the differences are due to chance or if they are due to other factors such as the prisms not being perfect?

Sampling theory can be applied to this problem. Each experience year can be viewed as a sample of the universe. These sample years can be combined to give larger samples. The means and standard deviations of the samples will vary from the true values of the universe, but the amount of variation will decrease as the sizes of the samples increase. If the distribution for the universe is a normal distribution, the means and standard deviations of the samples can be related to those of the universe and the probable errors determined for these averages for the samples. If the universe is not normal, this problem becomes very involved. Apparently there are no completely satisfactory methods other than some conversion procedures that will change the distribution of the universe to an approximate normal distribution.

S. Vajda in his paper (14) of the bibliography discusses the problem of determining the distribution of a universe from a limited number of samples. He used random numbers in his investigation. Mr. Vajda's findings show that a limited number of sample distributions does not give as satisfactory a basis for determining claim cost distributions as a mathematical formula. Mr. Vajda's studies are based on 100 and 1,000 sets of five items and 100 sets of fifty items, so that he uses a considerably smaller sample than Mr. Ross's sample of 10,000 experience years.

Table 2 of Mr. Ross's discussion gives a distribution of total claim costs for his 10,000 experience years for each of his four cases. Obviously, because of the small exposures involved, the distributions are skewed. It is also noted that only Case B gives a smooth distribution.

Mr. Ross assumes the same exposure for each of his experience years, so that the Poisson distribution can be assumed to apply. Case B has the same amount of insurance on each life, so that the total exposure and the total claims for the case can be determined and the average claim rate secured. This rate for Case B is .005517. This is a mean rate based on a sample of 1,715 claims for a continuous exposure of 310,000 risks for one year. On the assumption that the distribution of the mean rate for samples of this size will be the normal distribution, the standard deviation for this mean rate is .000133. The area included in the normal distribution from two standard deviations less than the mean to two standard deviations more than the mean represents 95.45% of total area. Using two standard deviations as the limit, the minimum and maximum claim rates for Case B are .005251 and .005783, respectively. Using the mean for the sample and these two limits, three Poisson distributions have been calculated for Case B. The percentages for these distributions together with those from Mr. Ross's Table 2 are given in Table 11.

The agreement for none, one and two deaths is close but Mr. Ross's results are somewhat higher for three and four deaths. The differences for three and four deaths can be due to the small number of experience years which have death claims in excess of two claims.

TABLE 11

| Number of Death Losses | Mr. Ross's Case B | Poisson Distribution | | |
|---|---|---|---|---|
| | | .005251 | .005517 | .005783 |
| 0 | 84.62% | 84.98% | 84.29% | 83.59% |
| 1 | 13.75 | 13.83 | 14.41 | 14.99 |
| 2 | 1.50 | 1.13 | 1.23 | 1.34 |
| 3 | 0.12 | 0.06 | 0.07 | 0.08 |
| 4 | 0.01 | .00 | .00 | .00 |
| | 100.00% | 100.00% | 100.00% | 100.00% |

One method of testing the accuracy of the Poisson distribution is to fit this distribution to the actual claim distribution for Case B by use of a curve-fitting procedure and then check the accuracy of the fit by use of the Chi-square distribution.

Similar calculations cannot be made for the other cases without additional information. If the cases with varying amounts of insurance are subdivided into reasonable amount size classes, distributions can be determined for each class and the class distributions combined to secure the claim cost distribution for the case.

As pointed out by both Mr. Ross and Mr. Hill, the development of a dependable distribution of claim costs for a particular case involves a considerable amount of machine work, and about the same amount of work will be needed for each case. The actual distribution so secured for each case will be a unique one because of differences for ages, sexes, amounts of insurance and occupations among the groups. Because of this, the calculations for most cases will need to be based on a mathematical procedure as indicated in the paper. As pointed out by Mr. Ross, his procedure can be used for a selected number of cases as an independent check of the results for mathematical formulas.

Mr. Ross's method will need extension if basic probabilities are not to be assumed to remain constant and either the multiple Poisson distribution of Mr. Ammeter's procedure or the Lexis distribution is used for estimating the probable future annual claim cost distribution for a case over a period of several years in the future. As stated previously, the writer believes that it is satisfactory to assume constant basic probabilities for a few years in advance if a satisfactory mean rate is developed for the case.

The results of the procedure used by Mr. Ross can be used to develop a mean claim cost rate for a case as indicated previously. The reliability of this claim cost rate as well as the reliability of the distribution of claim amounts should be determined before the averages developed are accepted as satisfactory for nonproportional reinsurance calculations.

The writer again wants to express his thanks to those who have discussed the paper. Let us hope that the stimulus of this discussion will lead to further investigation and to additional reports to the Society.

The writer also wants to thank those who helped in the development of the final paper from preliminary drafts. The development of the premium formulas is not original with the writer. The premium formula was first given to him by Rolf Eckert. The writer also secured assistance from Ralph Tang in understanding the development of the formula. Both of these men are mathematicians and are working in actuarial departments of life insurance companies.

Two men who were especially helpful in reviewing preliminary drafts of the paper are Dr. Hans Bühlman of Zurich, Switzerland, and Irving Rosenthal. Others who assisted by reviewing preliminary drafts include Clifford Woodley, John Wooddy, Paul Jackson and Allen Mayerson. The writer greatly appreciated the help, encouragement and kindly criticism of these men. These men, of course, assume no responsibility for any comment included in the paper or in this reply to the discussion.