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A LAYMAN'S EXPLANATION OF THE EXPECTANCY ANNUITY

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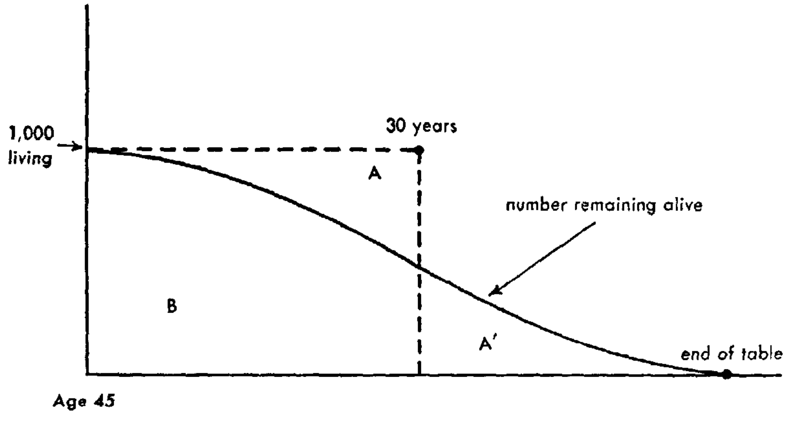
ONE of the major tasks of some actuaries is to substitute actuarial demonstrations for the impressions of laymen. The layman normally has an impression that an annuity for the expectation of life is the same as the average value of the life annuity. The ubiquity of this misconception is shown by the fact that several actuarial papers have been written discussing the relationship and the apparent impossibility of explaining the misconception to the layman; and by the fact that the misconception is mentioned in various text books, both in actuarial text books and in books on life insurance. The misconception is mentioned in the *Encyclopaedia Britannica* under the topic of Annuities, and it has come up many times in our courts and in hearings before congressional committees.

The first step for us to take in explaining the matter to the layman is to state that the annuity for the expectation of life, if there is no interest, is equal to the life annuity, but that interest is the complicating factor. The next step is to state that the difference is not very great, usually in the neigborhood of 6% or 8%, an overstatement on the part of the annuity for the expectation of life as against the average value of a life annuity. In many law cases 6% or 8% is not of any great significance since the question of "all or nothing" is of prime importance. The following explanation has been found to be understandable by lawyers and by accountants when delivered orally.

> If we are obligated to pay 1,000 persons each \$100 a year for so long as each may live, and if we invest enough to provide 1,000 payments of \$100 for a definite number of years, say thirty years, equal to their life expectancy (which is their average future lifetime), then, when the first person dies, we are not obligated to pay the \$100 which we had anticipated would be paid to him at the end of that year. We will be obligated to pay the \$100 later, to someone who lives beyond the thirty year average expectation of life; but we will never need the interest which will be earned on the \$100 in the meantime; consequently we will have invested more than we needed when we invested enough to pay the 1,000 expectancy annuities—the 1,000 thirty year annuities.

The foregoing is just another way of saying that the investment needed to pay an annuity-certain for the expectation of life to 1,000 persons is more than the amount which needs to be invested to pay 1,000 annuities for life only; *e.g.*, the amount which must be invested to pay an annuity-certain for the "thirty year" average expectation of life is greater than the average amount which must be invested to pay an annuity for life when the expectation of life is thirty years.

This can also be seen from the accompanying diagram for age 45 with



a thirty year expectation of life. The area under the curve of the number remaining alive represents the total payments (no interest) of a continuous life annuity for 1,000 persons in the mortality table at age 45, while the area of the rectangle represents the corresponding total payments (no interest) of a continuous annuity-certain for the thirty year expectation of life, and is equal to the area under the curve of the number remaining alive out of the 1,000 living at age 45.

The area marked B is common to both the rectangle and the area under the curve of the number remaining alive.

The area marked A is therefore equal to the area marked A', each being equal to the corresponding total area minus B.

When interest is involved, in calculating the value of the annuity-certain for the expectation of life as an approximation to the average value of a life annuity, the payments represented by the area marked A obviously are not discounted for enough years, since they should be discounted for the years corresponding to A'.

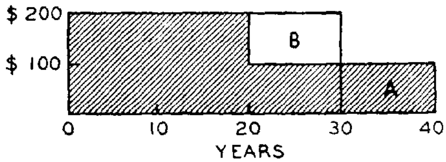
Hence the value of an annuity-certain for the expectation of life is greater than the value of a life annuity.

CHARLES L. TROWBRIDGE:

Mr. Sarason's explanation of the expectancy annuity in layman's terms is ingenious. His oral explanation is rather convincing, and his diagrammatic explanation says the same thing another way.

For what it may be worth, let me present an oversimplified modification of Mr. Sarason's diagrammatic approach. Suppose, Mr. Layman, we had only two lives, and our crystal ball (the mortality table) told us one would live 20 years, the other 40 years. The average expectation of life is then clearly 30 years.

If each is to receive an annuity of $100 per year, the payments under the two life annuities are represented by the shaded area in the figure. To



represent two annuities for the 30 year expectancy, one drops out the 10 years of $100 payments represented by area A and replaces them by 10 years of $100 payments in area B. Although A and B both represent $1,000 of payments, those in area B are more valuable than those in A because they are made 10 years sooner. Hence the annuity for the life expectancy is of greater value than the true life annuity.

An entirely different point of some interest is suggested by Mr. Sarason's comment that the overstatement is usually in the neighborhood of 6% to 8%. Since the overstatement is due to the action of interest, and since there is no overstatement on a 0% interest rate, one might come to the quick conclusion that (for the same mortality table) the higher the interest rate the larger the overstatement.

I was rather surprised to find that, for the oversimplified mortality table behind my diagram above [$q_{x+19} = \frac{1}{2}$, $q_{x+39} = 1$, all other q's zero] the overstatement reaches a peak at about 6% interest, and falls off toward zero as higher rates of interest are assumed. I believe it can be shown that for any mortality table the overstatement is zero for both $i = 0$ and $i = \infty$, with a maximum occurring at some finite interest rate. Both the interest rate where the maximum overstatement occurs, and the absolute value of such maximum overstatement, are dependent upon the mortality table.

DOUGLAS R. BUTT:

Mr. Sarason has given us a concise description of the difference between the costs for an annuity-certain for the expectation of life and a straight life annuity. It, and the accompanying graphical illustration, should be very clearly understood by any layman who has a reasonable understanding of insurance principles. I feel, however, that it would not be a sufficient explanation for a layman who has no knowledge of such things as no-interest annuities, the definition of life expectancy, the significance of areas under curves, or why a group, and not an individual, was used in the argument.

One quite often meets laymen who believe that the life expectancy is the crux of all insurance calculations, and who, at the same time, haven't the slightest idea of how the expectancy is derived. To these people I have, in the past, successfully explained the problem as follows:

(For convenience here, let us call the annuity-certain Plan A, and the life annuity Plan B.)

1. All insurance calculations are based on the average results of a group's experience; therefore we will consider the experience of a group of people at the same age.
2. The expectation of life is equal to the average number of years remaining in the lifetime of a member of the group. This is equal to the total number of years to be lived by members of the group, divided by the number of members in the group at the present time. For this discussion, let us assume that the expectation equals 30 years.
3. Under Plan A the total number of annuity payments to be made is 30 times the number of members in the group. By definition, this is equal to the total number of years to be lived by the group.
4. Under Plan B the total number of payments made is obviously equal to the total number of years to be lived by the group.
5. Thus the same total number of payments is to be made under each plan.
6. All payments made during the first 30 years to living members are common to both plans. The remaining payments under Plan A are made after the deaths of members but only during the 30 year period. The remaining payments under Plan B are made after the 30 year period, to members still alive. The difference in cost of these "remaining" payments represents the difference in cost of the two plans.
7. Because each plan has the same total number of payments, the number of "remaining" payments is the same under each plan.

   (Mr. Sarason's description assumes that the reader understands every-thing up to this point. In dealing with the layman, even more explanation may be necessary under #6.)

8. Every "remaining" payment under Plan B is therefore paid at a later date than every "remaining" payment under Plan A. Depending on the layman's knowledge of interest accumulation, the task of explaining the financial effect of discounting the payments to the beginning of the 30 year period will be an easy, or arduous one.

LAURENCE E. COWARD:

Mr. Sarason's diagram affords an easy way of demonstrating another useful actuarial rule: that the cost of guaranteeing a life annuity for a number of years certain varies approximately as the square of the number of years.

In his diagram the payments of an annuity guaranteed for 30 years are represented by the areas A plus B, plus A′. The 30 year guarantee is represented by the roughly triangular area A. If this area were exactly triangular the payments after death, under a continuous annuity with a 10 year guarantee and 20 year guarantee, would be exactly 1/9 and 4/9 of the payments after death under a 30 year guarantee.

Normally the rule that the cost of a guarantee varies as the square of the term of years would not be used for guarantees as long as 30 years, but it is of practical value in comparing annuities with guarantee terms under 10 years. Further, at ages around normal retirement age d_x is increasing and hence the discount for interest improves the accuracy of the rule.

Example: Ga–1951 Table, male aged 65, 4% interest, annuity monthly in arrears

Annuity for life only	10.162
Annuity guaranteed 10 years	11.316

By the above rule an annuity guaranteed six years would be

$10.162 + 36\% \times 1.154 = 10.578$

True value $= 10.588$.

BYRON STRAIGHT:

Mr. Sarason's explanation is useful in that it starts from first principles, with no technical terms, and with no prerequisites except for the understandings the layman already has in his misconceptions. Although it is doubtful that this explanation would be understood in a courtroom in the bare form presented by Mr. Sarason, particularly where one is dealing with a jury, it should be helpful in dealing with attorneys outside. Mr. Sarason uses the accumulative method and usually this is easier to understand and explain than the present value method. His explanation also shows that the question of simple versus compound interest is not relevant.

(AUTHOR'S REVIEW OF DISCUSSION)

HARRY M. SARASON:

In Mr. Butt's explanation and in my own verbal explanation actuaries now have the means of dispelling the expectancy misconception from the minds of laymen who know enough even to have the misconception. I have not been successful in using the diagram, but the presentation by Mr. Trowbridge may also be helpful.

Mr. Coward's discussion indicates some possible uses of the diagrammatic approach. I have tried, unsuccessfully, to use my diagram as a means of getting a closer approximation to the life annuity value from the expectancy annuity value. I got the diagram from an old actuarial course given by Columbia University. The diagram fits in very well with the explanation in the encyclopedia, that the life annuity has the same number of payments as the expectancy annuity but the life annuity payments are spread over a longer time.

Byron Straight's comments are simple general rules for explaining to laymen, and I have been wondering why it took me so long to develop an explanation of the fact that the value of the expectancy annuity is greater than the average value of the life annuity. The explanation was developed over a period of a year and a half, after I had failed in an attempt to explain the relationship to two attorneys.

If I were a teacher, I would follow the method of an inspiring teacher of English literature who had his students enthusiastically try to improve Gray's "Elegy in a Country Churchyard," a very rewarding effort even though the youngsters could not improve on Gray's five years of work. Every few weeks I would find occasion to ask my students: (1) How can we make the textbook explanation clearer? (2) How can we explain it to the man in the street? (3) How can we improve the explanation of any student? (4) How can we say it more concisely? (5) Under what circumstances is the statement untrue? (6) Do the words mean what we want them to mean? (7) Can we apply anything we have learned to a broader field? (8) Have you tried your "man-in-the-street explanation" on the man-in-the-street? (9) Are there any other questions we should ask ourselves? And my students would learn, even better than my teachers taught me, that you don't really understand anything until you can explain it to others.

In answer to my question (6) about the meaning of words: there is a distinction between the annuity, that is, the actual payments; the annuity value, which is the present value of future actual payments to an individual; the average value of the annuity, which is the value we actuaries use in our calculations; and the value of the average annuity, which, generally, would mean the value of the expectancy annuity. These distinctions can be applied to most actuarial concepts.

The conclusions about annuities are untrue under various circumstances: the inequality is reversed if we have negative interest or if we pay an annuity based on a cost of living index which increases at a greater rate than the interest rate at which our investments increase. The whole calculation is based upon assumptions—mortality assumptions and interest assumptions. The mortality table was never intended to be an assumption for an individual, merely an assumption for a group of individuals. The terminal age of many mortality tables is not even an assumption, merely a convenience. Mathematical tables and formulas should be aids to thinking, not substitutes for thinking. We should think about the way things will be, not about the way we make our calculations.

Another example of how a human mind works, or fails to work, can be seen in my initial incredulous reaction to Mr. Trowbridge's statement that the expectancy annuity–life annuity percentage differential first increases and then decreases as the rate of interest increases. I hadn't even thought about it; but I would have been perfectly willing to assume that the percentage differential simply increases as the rate of interest increases. However, a simple mental evaluation of the payments represented by the three areas either in his diagram or in my diagram confirms Mr. Trowbridge's conclusion.

As Mr. Butt states, all insurance calculations are based on the average results of a group's experience, but when actuaries publish combined experiences they almost invariably warn against the dangers of relying too closely on the average results of a group's experience. Typical warnings against the dangers of misusing an average experience are in many reports of intercompany investigations and in Mr. Thaler's pioneering paper on major medical insurance. In the annuity field, our Congress appropriated pension funds for the last few Union veterans of the Civil War on the assumption that they would all live to the end of a table "based on the average result of a group's experience." The assumption that these few would all live several years to the end of the table seems conservative, but it actually was not conservative enough. This is no criticism of Mr. Butt's statement, which is absolutely true; but this whole discussion is on a very elementary topic, and I want to warn students of the very serious danger involved in relying too much upon the average results of a group's experience. Overreliance on average results of past experience is probably the greatest pitfall trap into which actuaries can fall.